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Research Article

Parameter Identification of a Class of Economical Models

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In mathematical modeling of economic problems, it is common to assume that economic laws influence the process and analyze the type of solution obtained. Sometimes these laws have unknown parameters and they worth studying if these parameters can be determined. In this paper, a structured system is considered and the identifiability is analyzed. In case the parameters can be uniquely identified, an algorithm for obtaining the model parameters is presented and, finally, the existence of a balanced growth solution is studied.

1. Introduction

Linear ordinary differential or difference equations are frequently used in modeling many engineering, economic, and social processes. These equations describe the system behavior with a relative degree of accuracy and yet are simple enough to be solved. It is well known that any differential or difference equation of order n can be written as a system of n first-order differential or difference equations. This process works for any higher-order equation, linear or not, and it consists of expressing the top derivative or difference as a function of the lower ones. An nth order equation gives rise to a first order system in n variables whose coefficient matrix is called companion matrix. The characteristic polynomial of an nth order linear equation is the same as the characteristic polynomial of the companion matrix. Then, a companion matrix

$$\mathcal{C}(\boldsymbol{\alpha}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} & \alpha_{n-1} \end{pmatrix} \tag{1.1}$$

is associated with the monomial characteristic polynomial

$$p(\zeta) = -\sum_{i=0}^{n} \alpha_i \zeta^i, \quad \text{with } \alpha_n = -1. \tag{1.2}$$

In the last years, the use of dynamical models in economic theory is quite usual, in stark contrast, with the classical static economic models. Many of these models are mathematically formulated by higher-order difference equations (see [1]), which derive to a structured representation, where the companion matrix can be used.

As already mentioned, economic problems are usually modeled by structured systems, that is, systems where the matrices have a number of fixed zero entries while the rest of the entries are not known. This kind of system is widely considered in the literature, than in control theory (see [2, 3]), as in computing methods to solving it. When the size of the system is large, for solving these structured systems, fast algorithms are used. In [4] an exhaustive analysis on numerical properties of several fast algorithms was shown. In [5] an algorithm for solving dynamic economic models developing first-order and second-order iterations was studied.

In structured systems, the analysis of the unknown entries plays an important role. For example, the economic models shown in [1] have a parameterized structure, where the parameters are given information on the economic process. A first step for the validation of the model is to identify the parameters that appear in the system. Before identifying the parameters, it should be examined whether the parameters can be determined from the data. The identification of parameters allows solving the model. And finally, it is necessary to analyze the obtained solution. In particular, in economic models, the existence of a balanced growth solution is interesting. In the economical sense, balanced growth refers to simultaneous coordinated expansion of several sectors.

The aim of this paper is to study the identifiability problem and the balanced growth solution of an economic problem. The structure of the paper is as follows. In Section 2 we treat the identification problem of a companion structured system. In Section 3, we show a structured Leontief model associated to a determinate economical process and give some results about the identifiability of its parameters. This analysis leads us to construct an algorithm to obtain the parameters of the model. Finally, in the last section we study the balanced growth solution of an economic model.

2. Identification Problem of a Structured System

In the first part of this paper, we discuss a problem of identifiability for a linear differential or difference system

$$x'(t) = \mathcal{C}(\mathbf{p})x(t) + b(\mathbf{p})u(t) \tag{2.1}$$

or

$$x(k+1) = C(\mathbf{p})x(k) + b(\mathbf{p})u(k), \tag{2.2}$$

where the coefficient matrices have a fixed structure, $C(\mathbf{p})$ is a companion matrix and $b(\mathbf{p})$ is a monomial vector given by

$$b(\mathbf{p}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & p_n^b \end{pmatrix}^T, \tag{2.3}$$

with the parameter vector $\mathbf{p} = \begin{pmatrix} \mathbf{p}_{n1} & \cdots & \mathbf{p}_{nn} & \mathbf{p}_n^b \end{pmatrix}^T$ belonging to a subset $\mathcal{P} \subseteq \mathbb{R}^r$. Note that, the matrix $C(\mathbf{p})$ is a companion matrix where the last row is the vector $\begin{pmatrix} \mathbf{p}_{n1} & \cdots & \mathbf{p}_{nn} \end{pmatrix}^T$.

In general, an interpretation of a structured linear system is a family of linear systems, with the same structure, together a set of parameters. In this work, this kind of systems is denoted by

$$\mathcal{S}_P(A,B) = \{S(\mathbf{p}) = (A(\mathbf{p}), B(\mathbf{p})), \mathbf{p} \in \mathcal{P}\}. \tag{2.4}$$

A formal definition of structured linear dynamic systems can be found in [6]. Furthermore, $io(\cdot)$ denotes the input-output behavior of the system $S(\cdot) \in \mathcal{S}_P(A,B)$. Note that, $io(\mathbf{p})$ of the system $S(\mathbf{p})$ can completely be characterized by the Markov parameters, defined as $V(j,\mathbf{p}) = A^j(\mathbf{p})B(\mathbf{p}),\ j \geq 0$.

Usually, in the analysis of parametric models, studied two problems can be: identifiability and estimability. The difference between estimability and identifiability is that in the identifiability all possible options of the input-output signal are evaluated, while estimability is the ability to estimate accurately the parameters of a given data set.

A model is identifiable if and only if, for every parameter set there is a unique input-output behavior. Several papers deal with identification of models describing behavior of biomedical, chemical, or other systems, [7, 8].

The problem of the structural identifiability of the model consists of the determination of all parameter sets which give the same input-output structure. The structural identifiability of $\mathcal{S}_P(A,B)$ depends on the number of solutions of $io(\mathbf{p}) = io(\mathbf{q})$. If the parameters can be determined uniquely from the data, that is, the equation $io(\mathbf{p}) = io(\mathbf{q})$ has only the solution $\mathbf{p} = \mathbf{q}$, then the system $\mathcal{S}_P(A,B)$ is *structurally globally identifiable*. On the other hand, if there exists a finite or contable number of solutions for the parameters, the system $\mathcal{S}_P(A,B)$ is *structurally locally identifiable*. Otherwise, the system $\mathcal{S}_P(A,B)$ is *structurally unidentifiable* (see [9, 10]).

Now, we consider the system $\mathcal{S}_P(C,b)$, with $C(\cdot)$ and $b(\cdot)$ given by (1.1) and (2.3). In order to analyze the identification problem of this system, we use the Markov parameters to know its input-output behavior $io(\mathbf{p})$. By the structure of the companion matrix $C(\cdot)$ and by a technical process, we have the following result.

Proposition 2.1. *Consider the structured system $\mathcal{S}_P(\mathcal{C}, b)$. The Markov parameters $V(k, \mathbf{p}) = (v_i^{(k)}(\mathbf{p}))_{i=1,\dots,n}$, $k \geq 0$, are given by*

$$i = 1, \dots, n-1, \quad v_i^{(k)}(\mathbf{p}) = \begin{cases} 0, & k = 0, \\ v_{i+1}^{(k-1)}(\mathbf{p}), & k = 1, \dots, n-1, \end{cases}$$
$$i = n, \quad v_n^{(k)}(\mathbf{p}) = \begin{cases} p_n^b, & k = 0, \\ \sum_{j=n-k+1}^{n} p_{n,j} v_j^{(k-1)}(\mathbf{p}), & k = 1, \dots, n-1. \end{cases} \tag{2.5}$$

Directly from the above result, we obtain the following proposition which gives a condition to solve the identification problem for the system $\mathcal{S}_P(\mathcal{C}, b)$.

Proposition 2.2. *The structured system $\mathcal{S}_P(\mathcal{C}, b)$ is globally identifiable.*

In the next section, we analyze the identification problem for economic models.

3. Economic Dynamic Model

First, we consider the nonlinear discrete-time dynamic model given in [1]

$$y(t) = \alpha_1 + (\alpha_2 - \alpha_3)y(t-1) + \alpha_3 y(t-2) - \alpha_4 \big(y(t-1) - y(t-2)\big)^3, \tag{3.1}$$

where $y(t)$, $t \in \mathbb{N}$, is the income at time t. Parameters used in this model have the following economic interpretation: the parameter α_1 represents the autonomous expenditures, the parameter $\alpha_2 \in (0,1)$, and represents the consumer's reaction against the increase or decrease of his income. Finally, the investment is described using parameters α_3 and α_4, where α_3 represents the difference between the control and the investment function.

This model is interesting because for different values of parameters we obtain well-known linear dynamic models as Hick-Samuelson, Keynes, and the nonlinear dynamic Puu model (see [11]).

In particular, for $\alpha_1 = \alpha_4 = 0$, $\alpha_2 = 1 - \beta$, with $\beta \in (0,1)$ and $\alpha_3 = -\gamma$, $\gamma > 0$, we obtain the Hick-Samuelson model

$$y(t) = \big(1 + \gamma - \beta\big)y(t-1) - \gamma y(t-2), \tag{3.2}$$

which matrix expression is given by the companion matrix

$$\begin{pmatrix} 0 & 1 \\ -\gamma & 1 + \gamma - \beta \end{pmatrix}. \tag{3.3}$$

If we do not have investment, that is, $\alpha_3 = \beta \in [0,1]$, we obtain the Keynes model

$$y(t) = \beta\big(y(t-2) - y(t-1)\big) + \alpha_1 + \alpha_2 y(t-1), \tag{3.4}$$

which has the following coefficient matrix

$$\begin{pmatrix} 0 & 1 \\ \beta & \alpha_2 - \beta \end{pmatrix}. \tag{3.5}$$

Note that these models are structured systems of the kind $\mathcal{S}_P(C, b)$ where the coefficient matrix depends on a vector of parameters which could be identified to analyze its solution. Then, by Proposition 2.2 these models are globally identifiable.

Now, we consider the Leontief model. In the Leontief economic process is assumed that there is a market for different goods in which each industry has only one production process. Assuming that $x(k)$ is the production level vector and $u(k)$ is the demand level vector, the control process (see, e.g., [12, 13]), the system has the structured representation given by

$$Cx(k+1) = (I - P(\mathbf{p}) + C)x(k) - D(\mathbf{p})u(k), \tag{3.6}$$

where C is the capital coefficient matrix, $P(\mathbf{p})$ is the technological coefficient matrix, and $D(\mathbf{p})$ is the demand coefficient matrix (excluding investment). The (i, j)-entry of the matrices $P(\mathbf{p})$ and C represent the amounts of material input and capital of the ith good necessary for the production of one unit of the jth industry, respectively. Since the Leontief model is an economic model, $C \geq 0$, $P(\mathbf{p}) \geq 0$ and $D(\mathbf{p}) \geq 0$. This kind of systems is a dynamic generalized system because the capital coefficient matrix C can be or not be singular. The singularity of this matrix arises because no output from one sector is used in the production of some products.

In this paper, the system (3.6) is a structured linear dynamic system where the capital matrix C is a nonsingular diagonal matrix, $C = \operatorname{diag}(c_1, c_2, \ldots, c_n)$, and the matrix $P(\mathbf{p})$ has a companion matrix structure. By the nature of the technological matrix $P(\mathbf{p}), 0 < p_{nj}^p < 1$, and

$$\sum_{j=1}^{n} p_{nj}^p < 1, \tag{3.7}$$

where the unknown demand $D(\mathbf{p})$ with almost one column of type (2.3)exists.

Since capital matrix C is invertible, the structured generalized system $(C, I - P(\mathbf{p}) + C, D(\mathbf{p}))$ can be transformed into a structured standard system

$$x(k+1) = A(\mathbf{p})x(k) + B(\mathbf{p})u(k), \tag{3.8}$$

where

$$A(\mathbf{p}) = I + C^{-1}(I - P(\mathbf{p})) = \begin{pmatrix} p_{11}^a & p_{12}^a & 0 & \cdots & 0 \\ 0 & p_{22}^a & p_{23}^a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n1}^a & p_{n2}^a & p_{n3}^a & \cdots & p_{nn}^a \end{pmatrix}, \tag{3.9}$$

$$B(\mathbf{p}) = -C^{-1}D(\mathbf{p}) \tag{3.10}$$

is such that it has almost one monomial column of kind (2.3), with $P = \{\mathbf{p} \in \mathbb{R}^{3n-1},\ \mathbf{p} \neq \mathbf{0})\}$.

An important point to note here is the structure of matrix $A(\mathbf{p})$. In the literature, this kind of matrix is known as the modified Leslie (Lefkovitch) matrix, which appears in different areas such as the study of recruitment, survival and population growth rate (see, e.g., [14, 15]). For example, in [16] it is used to describe a discrete age-structured population model where no emigration during harvesting is considered.

Some properties of the spectrum of $A(\mathbf{p})$ are given in the following result, which are useful to study the stability of the system $\mathcal{S}_P(A, B)$.

Proposition 3.1. *Consider the matrix $A(\mathbf{p})$ given in* (3.9).

(i) *If $p_{i_0,i_0+1}^a = 0$ or $p_{n1}^a = \cdots = p_{ni_0}^a = 0$, with $i_0 > n-1$, then*

$$\left\{p_{11}^a, p_{22}^a, \ldots, p_{i_0,i_0}^a\right\} \subseteq \sigma(A(\mathbf{p})). \tag{3.11}$$

(ii) *If $p_{n-1,n}^a = 0$ or $p_{n1}^a = \cdots = p_{n,n-1}^a = 0$, then*

$$\{p_{11}^a, p_{22}^a, \ldots, p_{n,n}^a\} \subseteq \sigma(A). \tag{3.12}$$

Proof. If we structure $A(\mathbf{p})$ as $\begin{pmatrix} A_1(\mathbf{p}) & A_2(\mathbf{p}) \\ A_3(\mathbf{p}) & A_4(\mathbf{p}) \end{pmatrix}$, with $A_1(\mathbf{p}) \in \mathbb{R}^{n-1\times n-1}$, $A_2(\mathbf{p}) \in \mathbb{R}^{n-1\times 1}$, $A_3(\mathbf{p}) \in \mathbb{R}^{1\times n-1}$, and $A_4(\mathbf{p}) \in \mathbb{R}$, then

$$\begin{aligned} |\lambda I - A(\mathbf{p})| &= |A_1(\mathbf{p})| \left| A_4(\mathbf{p}) - A_3(\mathbf{p}) A_1^{-1}(\mathbf{p}) A_2(\mathbf{p}) \right| \\ &= |A_1(\mathbf{p})| \left| \lambda - p_{nn}^a - \left(p_{n1}^a \frac{p_{12}^a \cdots p_{n-1,n}^a}{|A_1|} + \cdots + p_{n,n-2}^a \frac{p_{n-2,n-1}^a p_{n-1,n}^a}{\left(\lambda - p^a x_{n-2,n-2}\right)\left(\lambda - p_{n-1,n-1}^a\right)} \right.\right. \\ &\qquad \left.\left. + p_{n,n-1}^a \frac{p_{n-1,n}^a}{\lambda - p_{n-1,n-1}^a} \right) \right|. \end{aligned} \tag{3.13}$$

(i) If $p^a_{i_0,i_0+1} = 0$ or $p^a_{n1} = \cdots = p^a_{ni_0} = 0$, with $i_0 > n-1$, then we simplify

$$|A(\mathbf{p})| = \left(\lambda - p^a_{11}\right)\cdots\left(\lambda - p^a_{i_0,i_0}\right)q_{n-i_0}(\lambda), \tag{3.14}$$

where $q_{n-i_0}(\lambda)$ is a polynomial of degree $n - i_0$.

Hence, $\{p^a_{11}, p^a_{22}, \ldots, p^a_{i_0,i_0}\} \subseteq \sigma(A(\mathbf{p}))$.

(ii) By (i) it is straightforward. □

In order to solve the identifiability problem of a family of structured systems of kind (3.6), we show some results on structured standard system $\mathcal{S}_P(A,B)$.

The system (3.9)-(3.10) belongs to a kind of structured systems. In particular, the identifiability of the parameters of a standard system $\mathcal{S}_P(A,B)$ is concerned with the determination of them from the external behavior of the system. That is, to determine the input-output behavior (io) of a model $\mathcal{S}_P(A,B)$, we can use the Markov parameters associated to the system $\mathcal{S}_P(A,B)$.

The parameter identification process for the structured system is followed from the structure of the vectors obtained in the following proposition.

Proposition 3.2. *Consider the structured system $\mathcal{S}_P(A,B)$. The Markov parameters $V(k,\mathbf{p}) = (v^{(k)}_i(\mathbf{p}))_{i=1,\ldots,n}$, $k \geq 0$ are given by*

$$k = 0, \quad v^{(0)}_n(\mathbf{p}) = p^b_n,$$

$$k = 1,\ldots,n-1, \quad v^{(k)}_i(\mathbf{p}) = \begin{cases} 0, & i = 1,\ldots,n-k-1, \\ \displaystyle\sum_{j=0}^{1} p^a_{i,i+j} v^{(k-1)}_{i+j}(\mathbf{p}), & i = n-k,\ldots,n-1, \\ \displaystyle\sum_{j=1}^{n} p^a_{i,i-k+j} v^{(k-1)}_{i-k+j}(\mathbf{p}), & i = n. \end{cases} \tag{3.15}$$

We solve the identification problem for the system $\mathcal{S}_P(A,B)$ in the next result.

Proposition 3.3. *The structured system $\mathcal{S}_P(A,B)$ where $A(\mathbf{p})$ and $B(\mathbf{p})$ are defined by* (3.9) *and* (3.10) *with $\mathbf{p} \in \mathcal{P}$ is globally identifiable.*

Proof. We consider two structured systems $S(\mathbf{p})$ and $S(\mathbf{q})$ defined by (3.9)-(3.10) with $\mathbf{p}$ input-output behavior (io) $V(k,p) = V(k,\mathbf{q})$, $k \geq 0$ and we shall prove that $\mathbf{p} = \mathbf{q}$. By the structure of the Markov parameters obtained in Proposition 3.2, for $i = 1,\ldots,n$, it is show that

$$A(\mathbf{p})^k B(\mathbf{p}) = A(\mathbf{q})^k B(\mathbf{q}) \Longrightarrow p^a_{k,k} = q^a_{k,k}, \quad p^a_{k,k+1} = q^a_{k,k+1}. \tag{3.16}$$

Moreover, the rest of unknown entries of $B(\mathbf{p})$ are identified from the first equality $B(\mathbf{p}) = B(\mathbf{q})$. Hence, $\mathbf{p} = \mathbf{q}$. □

Consider the associated standard Leontief model (3.8), in this case given by

$$A(\mathbf{p}) = I + C^{-1}(I - P(\mathbf{p})) = \begin{pmatrix} 1 + \dfrac{1-p_{11}^p}{c_1} & -\dfrac{p_{12}^p}{c_1} & 0 & \cdots & 0 \\ 0 & 1 + \dfrac{1-p_{22}^p}{c_2} & -\dfrac{p_{23}^p}{c_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{-p_{n1}^p}{c_n} & \dfrac{-p_{n2}^p}{c_n} & \dfrac{-p_{n3}^p}{c_n} & \cdots & 1 + \dfrac{1-p_{nn}^p}{c_n} \end{pmatrix}, \tag{3.17}$$

and $B(\mathbf{p}) = -C^{-1}D(\mathbf{p})$ has the column

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \dfrac{-p_n^d}{c_n} \end{pmatrix}^T. \tag{3.18}$$

By Proposition 2.2 the entries of these matrices are identified and hence we can also identify the initial parameters of the vector $\mathbf{p}$. Thus, the initial Leontief model is globally identifiable.

Proposition 3.4. *The Leontief economic model given by* (3.6) *where C is a non-singular diagonal matrix, $C = \operatorname{diag}(c_1, c_2, \dots, c_n)$ and $P(\mathbf{p})$ is a companion matrix, is globally identifiable.*

The following algorithm allow us determine the parameters $\mathbf{p}$ of the Leontief model.

Algorithm

Step 1. Introduce the size of the state vector: n. Introduce the capital matrix C. Introduce the matrices $\{V(k,\mathbf{p}), k = 0,\dots,n\}$ that determine the known external behavior of the system $\mathcal{S}_P(A,B)$ with the structure (3.9)-(3.10). And introduce the position of the monomial column of matrix $V(0,\mathbf{p})$: j, for $k = 0,\dots,n-1$.

Step 2. Choose the jth column of $V(k,\mathbf{p})$ and denote it as $v^{(k)}$.

Step 3. Introduce $v_{n+1}^{(k-1)} = 0$.

Step 4. Construct the following system:

$$v_n^{(0)} = p_n^b,$$

$$v_i^{(k)}(\mathbf{p}) = \begin{cases} \displaystyle\sum_{j=0}^{1} p_{i,i+j}^a v_{i+j}^{(k-1)}(\mathbf{p}), & i = n-k,\dots,n-1, \\ \displaystyle\sum_{j=1}^{n} p_{i,i-k+j}^a v_{i-k+j}^{(k-1)}(\mathbf{p}), & i = n. \end{cases} \tag{3.19}$$

Step 5. Solve the above system and obtain the parameters $p^a_{n,n-k+j}$ for $j = 1,\ldots,n$ and p^b_n and $p^a_{i,i+j}$, for $j = 0, 1$, and $i = n-k,\ldots,n$.

Step 6. Use the parameters obtained in Step 5 and construct the matrices (3.17)-(3.18) and from them obtain p^d_n and $p^p_{i,i+j}$, $j = 0, 1$, and $i = 1,\ldots,n$.

With this process, we have determinate all parameters of the system.

Step 7. Finally, construct the matrices of the system (3.6).

Note that, this algorithm can also be applied when the Hick-Samuelson and the Keynes models are considered, that is, when $p^a_{ii} = 0$ and $p^a_{i,i+1} = 1$, $i = 1,\ldots,n-1$.

Next we present an academic example to clarify the above algorithm. The model considers a number of consumers such that at any given time t each consumer holds quantities of different services or activities. We assume that this kind of market can be modeled by a structured Leontief model. Using the data, the problem is to assure the uniqueness in the parameter of the dynamical model. In this example to determine these parameters we use the above algorithm.

Example 3.5. Tourism demand in a country involves different activities. In this example the external behavior of the foreign demand for tourist services during a period of years is known and we want to apply the algorithm in order to identify the structured model.

The external behavior of the foreign demand for tourist services during a period of four years is given by

$$V(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.4000 \end{pmatrix}, \quad V(1) = \begin{pmatrix} 0 \\ 0 \\ -0.0400 \\ 0.4214 \end{pmatrix}, \quad V(2) = \begin{pmatrix} 0 \\ 0.0049 \\ -0.0861 \\ 0.4445 \end{pmatrix},$$
$$V(3) = \begin{pmatrix} -0.0013 \\ 0.0160 \\ -0.1392 \\ 0.4695 \end{pmatrix}, \quad V(4) = \begin{pmatrix} -0.0058 \\ 0.0349 \\ -0.2001 \\ 0.4964 \end{pmatrix} \tag{3.20}$$

and consider that the capital coefficient matrix (given in thousands)

$$C = \operatorname{diag}(3.8, 8.2, 10, 12.5). \tag{3.21}$$

Then we apply the above algorithm.

Step 1. Introduce $n = 4$, the matrices C, $\{V(k), k = 0,\ldots,4\}$ and $j = 1$, for $k = 0,\ldots,3$.

Steps 2-3

Denote $V(k)$ as $v^{(k)}$ and introduce $v^{(k-1)}_5 = 0$.

Steps 4-5

Solving the system

$$v_4^{(0)} = p_4^b,$$
$$v_i^{(k)}(\mathbf{p}) = \begin{cases} \sum_{j=0}^{1} p_{i,i+j}^a v_{i+j}^{(k-1)}(\mathbf{p}), & i = n-k, \dots, n-1, \\ \sum_{j=1}^{n} p_{i,i-k+j}^a v_{i-k+j}^{(k-1)}(\mathbf{p}), & i = n, \end{cases} \tag{3.22}$$

obtain

$$\begin{gathered} p_4^b = 0.4000, \quad p_{34}^a = -0.1000, \quad p_{44}^a = 1.1000, \quad p_{23}^a = -0.1220, \\ p_{33}^a = 1.1000, \quad p_{43}^a = -0.0139, \quad p_{12}^a = -0.2632, \quad p_{22}^a = 1.1220, \\ p_{42}^a = -0.0114, \quad p_{11}^a = 1.2632, \quad p_{42}^a = 0.1430, \quad p_{43}^a = 0.1740, \\ p_{41}^a = -0.0249, \quad p_{14}^a = 0, \quad p_{24}^a = 0, \quad p_{13}^a = 0, \\ p_{32}^a = 0, \quad p_{21}^a = 0, \quad p_{31}^a = 0. \end{gathered} \tag{3.23}$$

Steps 6-7

From (3.17)-(3.18) the system is identifiable and its coefficient matrices are

$$C = \operatorname{diag}(3.8, 8.2, 10, 12.5),$$
$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.3111 & 0.1430 & 0.1740 & 0.3310 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \end{pmatrix}. \tag{3.24}$$

An interpretation of these results can be, for example the following. It seems reasonable that the fact that capital matrix is diagonal implies that to obtain one unit of production in each industry, an investor is only necessary. On the other hand, the technological matrix is determined as each service depends on one material input except the last service, which needs a part of all material input involved in the process. Finally, the structure of the demand matrix indicates that only the last service has consumers, given in thousands. This service includes all others.

4. Balanced Growth Solution

In economic problems it is important to assure the existence of a solution which output of each sector increases by a constant percentage per unit of time. This kind of solution is called balanced growth solution, and it satisfies

$$x(k) = (1+\delta)^k x_0 \gg 0 \tag{4.1}$$

with the balanced growth rate $\delta > 0$.

The balanced growth problem is solved in [17] for autonomous systems. The interest in obtaining balanced growth solutions for some dynamic model has to do the following question: is it possible to obtain a feedback such that the closed-loop system has a balanced growth solution? A first approximation of the solution to this problem has been given in [18]. In this work a characterization to the existence of a balanced growth solution has been obtained. This characterization is based on the existence of a feedback such that the closed-loop system has a positive eigenvalue with a positive eigenvector.

To analyze the balanced growth solution of the system (3.6), we consider the associated system $\mathcal{S}_P(A, B)$ given by

$$x(k+1) = A(\mathbf{p})x(k) + B(\mathbf{p})u(k). \tag{4.2}$$

The balanced growth problem for Leontief structured model is treated in the following result.

Proposition 4.1. *The balanced growth problem for Leontief structured model* (3.6) *is solvable.*

Proof. Note that technological matrix $P(\mathbf{p})$ is an irreducible matrix, that is, its associated directed graph is strongly connected, which follows by the companion matrix structure of $P(\mathbf{p})$. Moreover, as $0 < p_{nj}^p < 1$, and $\sum_{j=1}^n p_{nj}^p < 1$, we can assure that $\rho(P(\mathbf{p})) < 1$. These conditions on $P(\mathbf{p})$ lead us to the fact that matrix $(I - P(\mathbf{p}))^{-1}$ is positive. If we construct a feedback $u(k) = \delta F x(k)$ with $F > O$, then $(C + D(\mathbf{p})F) \geq 0$. Therefore, the coefficient matrix $(I - P(\mathbf{p}))^{-1}(C - D(\mathbf{p})F)$ of the closed-loop system is positive. Hence, there exists a positive eigenvalue λ with a positive eigenvector x_0, that is,

$$\left(\lambda I - (I - P(\mathbf{p}))^{-1}(C + D(\mathbf{p})F)\right)x_0 = 0, \tag{4.3}$$

and taking $\delta = 1/\lambda$ we have

$$\begin{gathered}
((I - P(\mathbf{p})) - \delta(C + D(\mathbf{p})F))x_0 = 0, \\
\left(-\delta I + C^{-1}(I - P(\mathbf{p})) - \delta C^{-1}(D(\mathbf{p})F)\right)x_0 = 0, \\
(1+\delta)x_0 = \left(I + C^{-1}(I - P(\mathbf{p})) - \delta C^{-1}(D(\mathbf{p})F)\right)x_0, \\
(1+\delta)x_0 = (A(\mathbf{p}) - \delta B(\mathbf{p})F)x_0, \\
(1+\delta)^{k+1}x_0 = (A(\mathbf{p}) - \delta B(\mathbf{p})F)(1+\delta)^k x_0.
\end{gathered} \tag{4.4}$$

Considering the state-feedback $u(k) = -\delta F x(k)$, we show that $x(k) = (1+\delta)^k x_0$ is a solution of the system (3.6). Thus, the balanced growth problem for Leontief structured model (3.6) is solvable. □

The economic meaning of the existence of a positive eigenvector can be given, for instance, when there is one economy where each sector of this economy depends on all others directly or indirectly for either its intermediate product or its capital. We refer to [17] for more details for it and in the case when the economy can be divided into several sub-conomies.

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