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Research Article

Positive Periodic Solutions of Nonlinear First-Order Functional Difference Equations

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We consider the existence, multiplicity, and nonexistence of positive T-periodic solutions for the difference equations $\Delta x(n)=a(n)g(x(n))x(n)-\lambda b(n)f(x(n-\tau(n)))$, and $\Delta x(n)+a(n)g(x(n))x(n)=\lambda b(n)f(x(n-\tau(n)))$, where $a,b:\mathbb{Z}\to[0,\infty)$ are T-periodic, $\tau:\mathbb{Z}\to\mathbb{Z}$ is T-periodic.

1. Introduction

In the recent years, there has been considerable interest in the existence of periodic solutions of the following equation:

$$x'(t)=\widetilde{a}(t)\widetilde{g}(x(t))x(t)-\lambda\widetilde{b}(t)\widetilde{f}(x(t-\tau(t))), \tag{1.1}$$

where $\widetilde{a},\widetilde{b}\in C(\mathbb{R},[0,\infty))$ are ω-periodic functions, $\int_0^{\omega}\widetilde{a}(t)dt>0$, $\int_0^{\omega}\widetilde{b}(t)dt>0$, τ is a continuous ω-periodic function. Equation (1.1) has been proposed as a model for a variety of physiological processes and conditions including production of blood cells, respiration, and cardiac arrhythmias. See, for example, [1–13] and the references therein.

Let $\mathbb{Z}$ be the set of all integers. In the present paper, we study the existence of positive T-periodic solutions of discrete analogues to (1.1) of the form

$$\Delta x(n)=a(n)g(x(n))x(n)-\lambda b(n)f(x(n-\tau(n))),\quad n\in\mathbb{Z}, \tag{1.2}$$

and the difference equation

$$\Delta x(n)+a(n)g(x(n))x(n)=\lambda b(n)f(x(n-\tau(n))),\quad n\in\mathbb{Z}, \tag{1.3}$$

where $\mathbb{Z}$ is the set of integer numbers, $T \in \mathbb{N}$ is a fixed integer, $a, b : \mathbb{Z} \to [0, +\infty)$ are T-periodic, $a(n) \not\equiv 0$, $b(n) \not\equiv 0$ on $[0, T-1] =: \{0, 1, \ldots, T-1\}$, $\tau : \mathbb{Z} \to \mathbb{Z}$ is T-periodic, and $f, g \in C([0, +\infty), [0, +\infty))$, $\lambda > 0$ is a parameter.

So far, relatively little is known about the existence of positive periodic solutions of (1.2) and (1.3). To our best knowledge, only Raffoul [14] dealt with the special equations of (1.2) and (1.3) of the form

$$\Delta x(n) = \alpha(n)x(n) - \lambda b(n) f(x(n - \tau(n))), \tag{1.4}$$

with $\alpha(n) = a(n) - 1 > 0$, and

$$\Delta x(n) + \alpha(n)x(n) = \lambda b(n) f(x(n - \tau(n))), \tag{1.5}$$

with $\alpha(n) = 1 - a(n) > 0$, and determining values of λ, for which there exist positive T-periodic solutions of (1.4) and (1.5), respectively.

It is the purpose of this paper to study more general equations (1.2) and (1.3) and generalize the main results of Raffoul [14]. We establish some existence, multiplicity, and nonexistence results of positive periodic solutions for (1.2) and (1.3), respectively. The main tools we will use are fixed point theorem in cones and fixed point index theory [15, 16]. Throughout this paper, we denote the product of $y(n)$ from $n = a$ to $n = b$ by $\prod_{n=a}^{b} y(n)$ with the understanding that $\prod_{n=a}^{b} y(n) = 1$ for all $a > b$.

The rest of the paper is arranged as follows: in Section 2, we give some preliminary results. Section 3 is devoted to generalize the main results of Raffoul [14]. Finally, in Section 4, we state and prove some existence, multiplicity, and nonexistence results of positive periodic solutions for (1.2) and (1.3). For related results on the associated differential equations, see Wang [11].

2. Preliminaries

In this paper, we make the following assumptions.

(H1) $a, b : \mathbb{Z} \to [0, +\infty)$ are T-periodic, and $a(n) \not\equiv 0$, $b(n) \not\equiv 0$ on $n \in [0, T-1]$, $\tau : \mathbb{Z} \to \mathbb{Z}$ is T-periodic.

(H2) $f, g : [0, +\infty) \to [0, +\infty)$ are continuous, $0 < l \le g(x) \le L < \infty$ with l, L are positive constants, and $f(x) > 0$, for $x > 0$. Also,

$$\sigma_l = \prod_{s=0}^{T-1} (1 + a(s)l), \qquad \sigma_L = \prod_{s=0}^{T-1} (1 + a(s)L), \tag{2.1}$$

it is clear that $1 < \sigma_l \le \sigma_L$. Let

$$M(r) = \max\{f(x) \mid 0 \le x \le r\}, \qquad m(r) = \min\left\{f(x) \mid \frac{\sigma_l - 1}{(\sigma_L - 1)\sigma_L} \cdot r \le x \le r\right\}. \tag{2.2}$$

The following well-known result of fixed point index is crucial to our arguments.

Lemma 2.1 (see [15, 16]). *Let E be a Banach space, and let K be a cone in E. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tu \neq u$ for $u \in \partial K_r = \{u \in K : \|u\| = r\}$.*

(i) *If $\|Tu\| \geq \|u\|$ for $u \in \partial K_r$, then*

$$i(T, K_r, K) = 0. \tag{2.3}$$

(ii) *If $\|Tu\| \leq \|u\|$ for $u \in \partial K_r$, then*

$$i(T, K_r, K) = 1. \tag{2.4}$$

Let X be the set of all real T-periodic sequences. This set, endowed with the maximum norm $\|x\| = \max_{n\in[0,T-1]}|x(n)|$, is a Banach space. The next lemma is essential in obtaining our results.

Lemma 2.2. *Assume that (H1)-(H2) hold. Then, $x \in X$ is a solution of (1.2) if and only if*

$$x(n) = \lambda \sum_{u=n}^{n+T-1} G_x(n,u)b(u)f(x(u-\tau(u))), \quad n \in \mathbb{Z}, \tag{2.5}$$

where

$$G_x(n,u) = \frac{\prod_{s=u+1}^{n+T-1}(1 + a(s)g(x(s)))}{\prod_{s=0}^{T-1}(1 + a(s)g(x(s))) - 1}, \quad u \in [n, n+T-1]. \tag{2.6}$$

Proof. If $x \in X$ and satisfies (2.5), then

$$\begin{aligned}
x(n+1) &= \lambda \sum_{u=n+1}^{n+T} G_x(n+1,u)b(u)f(x(u-\tau(u))) \\
&= \lambda \sum_{u=n+1}^{n+T-1} G_x(n+1,u)b(u)f(x(u-\tau(u))) \\
&\quad + \lambda G_x(n+1,n+T)b(n+T)f(x(n+T-\tau(n+T))) \\
&= \lambda \sum_{u=n+1}^{n+T-1} G_x(n+1,u)b(u)f(x(u-\tau(u))) \\
&\quad + \lambda G_x(n+1,n+T)b(n)f(x(n-\tau(n))),
\end{aligned}$$

$$\begin{aligned}[1+a(n)g(x(n))]x(n) &= \lambda\sum_{u=n+1}^{n+T-1}[1+a(n)g(x(n))]G_x(n,u)b(u)f(x(u-\tau(u)))\\ &\quad+\lambda[1+a(n)g(x(n))]G_x(n,n)b(n)f(x(n-\tau(n))).\end{aligned} \tag{2.7}$$

From (2.2), we know that

$$\begin{aligned}G_x(n+1,n+T) &= \frac{\prod_{s=n+T+1}^{n+T}(1+a(s)g(x(s)))}{\prod_{s=0}^{T-1}(1+a(s)g(x(s)))-1}\\ &= \frac{1}{\prod_{s=0}^{T-1}(1+a(s)g(x(s)))-1},\\ [1+a(n)g(x(n))]G_x(n,n) &= \frac{\prod_{s=n}^{n+T-1}(1+a(s)g(x(s)))}{\prod_{s=0}^{T-1}(1+a(s)g(x(s)))-1}\\ &= \frac{\prod_{s=0}^{T-1}(1+a(s)g(x(s)))}{\prod_{s=0}^{T-1}(1+a(s)g(x(s)))-1},\\ G_x(n+1,u) &= \frac{\prod_{s=u+1}^{n+T}(1+a(s)g(x(s)))}{\prod_{s=0}^{T-1}(1+a(s)g(x(s)))-1}\\ &= \frac{\prod_{s=u+1}^{n+T-1}(1+a(s)g(x(s)))(1+a(n+T)g(x(n+T)))}{\prod_{s=0}^{T-1}(1+a(s)g(x(s)))-1}\\ &= [1+a(n)g(x(n))]G_x(n,u).\end{aligned} \tag{2.8}$$

So, by (2.7)-(2.8), we can conclude that

$$x(n+1)-[1+a(n)g(x(n))]x(n) = -\lambda b(n)f(x(n-\tau(n))), \tag{2.9}$$

thus, x is a T-periodic solutions of (1.2). On the other hand, if $x\in X$ and satisfies (1.2), then (1.2) is equivalent to

$$\Delta\left[\prod_{s=-\infty}^{n-1}(1+a(s)g(x(s)))^{-1}x(n)\right] = -\lambda b(n)f(x(n-\tau(n)))\prod_{s=-\infty}^{n}[1+a(s)g(x(s))]^{-1}, \tag{2.10}$$

by summing the above equation from $u=n$ to $u=n+T-1$, we obtain (2.5). □

Note that since $a(n)\not\equiv 0$ for $n\in[0,T-1]$ and $0<l\le g(x)\le L<\infty$, we have

$$\frac{1}{\sigma_L-1}\le G_x(n,u)\le\frac{\sigma_L}{\sigma_l-1},\quad n\le u\le n+T-1, \tag{2.11}$$

and $0<(\sigma_l-1)/(\sigma_L-1)\sigma_L<1$.

Define K as a cone in X by

$$K = \left\{ x \in X : x(n) \geq 0, n \in \mathbb{Z}, x(n) \geq \frac{\sigma_l - 1}{(\sigma_L - 1)\sigma_L} \|x\| \right\}. \tag{2.12}$$

Also, define, for r a positive number, Ω_r by

$$\Omega_r = \{x \in K : \|x\| < r\}, \tag{2.13}$$

note that $\partial\Omega_r = \{x \in K : \|x\| = r\}$.

Define $T_\lambda : X \to X$ by

$$T_\lambda x(n) = \lambda \sum_{u=n}^{n+T-1} G_x(n,u) b(u) f(x(u - \tau(u))), \quad n \in \mathbb{Z}, \tag{2.14}$$

where $G_x(n,u)$ is given by (2.6).

Lemma 2.3. *Assume that (H1)-(H2) hold. Then, $T_\lambda(K) \subset K$ and $T_\lambda : K \to K$ is compact and continuous.*

Proof. In the view of the definition of K, for $x \in K$, we have

$$\begin{aligned}(T_\lambda x)(n+T) &= \lambda \sum_{u=n+T}^{n+2T-1} G_x(n+T,u) b(u) f(x(u-\tau(u))) \\ &= \lambda \sum_{s=n}^{n+T-1} G_x(n+T,s+T) b(s+T) f(x(s+T-\tau(s+T))) \\ &= \lambda \sum_{u=n}^{n+T-1} G_x(n+T,u+T) b(u) f(x(u-\tau(u))) \\ &= (T_\lambda x)(n).\end{aligned} \tag{2.15}$$

In fact,

$$\begin{aligned}G_x(n+T,u+T) &= \frac{\prod_{s=u+T+1}^{n+2T-1}(1+a(s)g(x(s)))}{\prod_{s=0}^{T-1}(1+a(s)g(x(s)))-1} \\ &= \frac{\prod_{s=u+1}^{n+T-1}(1+a(s)g(x(s)))}{\prod_{s=0}^{T-1}(1+a(s)g(x(s)))-1} \\ &= G_x(n,u).\end{aligned} \tag{2.16}$$

So, $T_\lambda x \in X$. One can show that for $x \in K$ and $n \in [0, T-1]$,

$$
\begin{aligned}
(T_\lambda x)(n) &\geq \frac{\lambda}{\sigma_L - 1} \sum_{u=n}^{n+T-1} b(u) f(x(u-\tau(u))) \\
&= \frac{\sigma_l - 1}{(\sigma_L - 1)\sigma_L} \cdot \frac{\sigma_L}{\sigma_l - 1} \cdot \lambda \sum_{u=n}^{n+T-1} b(u) f(x(u-\tau(u))) \\
&\geq \frac{\sigma_l - 1}{(\sigma_L - 1)\sigma_L} \|T_\lambda x\|.
\end{aligned}
\tag{2.17}
$$

Thus, $T_\lambda(K) \subset K$, and it is easy to show that $T_\lambda : K \to K$ is compact and continuous. □

Consequently, $x \in K$ is a solution of (1.2) if and only if x is a fixed point of T_λ in K.

Lemma 2.4. *Assume that (H1)-(H2) hold. Let $\eta > 0$, if $x \in K$ and $f(x(n)) \geq x(n)\eta$ for $n \in [0, T-1]$, then*

$$
\|T_\lambda x\| \geq \lambda \frac{\eta(\sigma_l - 1)}{(\sigma_L - 1)^2 \sigma_L} \cdot \sum_{u=0}^{T-1} b(u) \cdot \|x\|. \tag{2.18}
$$

Proof. Since $x \in K$ and $f(x(n)) \geq x(n)\eta$ for $n \in [0, T-1]$, we have

$$
\begin{aligned}
(T_\lambda x)(n) &\geq \frac{\lambda}{\sigma_L - 1} \sum_{u=n}^{n+T-1} b(u) f(x(u-\tau(u))) \\
&= \frac{\lambda\eta}{\sigma_L - 1} \cdot \frac{\sigma_l - 1}{(\sigma_L - 1)\sigma_L} \sum_{u=0}^{T-1} b(u) \cdot \|x\| \\
&= \lambda \frac{\eta(\sigma_l - 1)}{(\sigma_L - 1)^2 \sigma_L} \cdot \sum_{u=0}^{T-1} b(u) \cdot \|x\|.
\end{aligned}
\tag{2.19}
$$

Thus,

$$
\|T_\lambda x\| \geq \lambda \frac{\eta(\sigma_l - 1)}{(\sigma_L - 1)^2 \sigma_L} \cdot \sum_{u=0}^{T-1} b(u) \cdot \|x\|. \tag{2.20}
$$

□

Lemma 2.5. *Assume that (H1)-(H2) hold. Let $r > 0$, if $x \in \partial\Omega_r$, and there exists an $\varepsilon > 0$ such that $f(x(n)) \leq x(n)\varepsilon$ for $n \in [0, T-1]$, then*

$$
\|T_\lambda x\| \leq \lambda\varepsilon \frac{\sigma_L}{\sigma_l - 1} \cdot \sum_{u=0}^{T-1} b(u) \cdot \|x\|. \tag{2.21}
$$

Proof. From the definition of T_λ, for $x \in \partial\Omega_r$, we have

$$\begin{aligned} \|T_\lambda x\| &\le \lambda \frac{\sigma_L}{\sigma_l - 1} \cdot \sum_{u=0}^{T-1} b(u) f(x(u-\tau(u))) \\ &\le \lambda\varepsilon \frac{\sigma_L}{\sigma_l - 1} \cdot \sum_{u=0}^{T-1} b(u) \cdot \|x\|. \end{aligned} \tag{2.22}$$

□

The following two lemmas are weak forms of Lemmas 2.4 and 2.5.

Lemma 2.6. *Assume that (H1)-(H2) hold. If $x \in \partial\Omega_r$, $r > 0$, then*

$$\|T_\lambda x\| \ge \lambda \frac{\sum_{u=0}^{T-1} b(u)}{\sigma_L - 1} \cdot m(r). \tag{2.23}$$

Proof. Since $f(x(n)) \ge m(r)$ for $n \in [0, T-1]$, it is easy to see that this lemma can be achieved in a similar manner as in Lemma 2.4. □

Lemma 2.7. *Assume that (H1)-(H2) hold. If $x \in \partial\Omega_r$, $r > 0$, then*

$$\|T_\lambda x\| \le \lambda \frac{\sigma_L M(r)}{\sigma_l - 1} \cdot \sum_{u=0}^{T-1} b(u). \tag{2.24}$$

Proof. Since $f(x(n)) \le M(r)$ for $n \in [0, T-1]$, it is easy to see that this lemma can be obtained in a similar manner as in Lemma 2.5. □

3. Generalization of the Main Results in [14]

Let

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{x}, \qquad f_\infty = \lim_{x \to \infty} \frac{f(x)}{x}. \tag{3.1}$$

In this Section, we make the following assumptions.

(L1) $f_0 = \infty$.

(L2) $f_\infty = \infty$.

(L3) $f_0 = 0$.

(L4) $f_\infty = 0$.

(L5) $f_0 = m$ with $0 < m < \infty$.

(L6) $f_0 = M$ with $0 < M < \infty$. And let

$$A = \max_{0 \le n \le T-1} \sum_{u=0}^{T-1} G_x(n,u) b(u),$$
$$B = \min_{0 \le n \le T-1} \sum_{u=0}^{T-1} G_x(n,u) b(u). \tag{3.2}$$

Theorem 3.1. *Assume that (H1), (H2), (L5), and (L6) hold. Then, for each λ satisfying*

$$\frac{\sigma_L(\sigma_L - 1)}{\sigma_l - 1} \cdot \frac{1}{BM} < \lambda < \frac{1}{Am}, \tag{3.3}$$

or

$$\frac{\sigma_L(\sigma_L - 1)}{\sigma_l - 1} \cdot \frac{1}{Bm} < \lambda < \frac{1}{AM}, \tag{3.4}$$

equation (1.2) *has at least one positive T-periodic solutions.*

Proof. Using the same method to prove[14, Theorem 2.3] with obvious changes, we can prove Theorem 3.1. The process of the proof is omitted. □

Theorem 3.2. *Assume that (H1) and (H2) hold. If (L1) and (L4) hold, or (L2) and (L3) hold, then* (1.2) *has at least one positive T-periodic solution for any $\lambda > 0$.*

Proof. The proof is similar to arguments to prove [14, Theorem 2.4]. □

The next two corollaries are consequences of the previous two theorems.

Corollary 3.3. *Assume that (H1) and (H2) hold. If (L1) and (L6) hold, or (L2) and (L5) hold, then* (1.2) *has at least one positive T-periodic solution if λ satisfies either $0 < \lambda < 1/AM$ or $0 < \lambda < 1/Am$.*

Corollary 3.4. *Assume that (H1) and (H2) hold. Also, if (L3) and (L6) hold, or (L4) and (L5) hold, then* (1.2) *has at least one positive T-periodic solution if λ satisfies either $\sigma_L(\sigma_L - 1)/(\sigma_l - 1) \cdot (1/BM) < \lambda < \infty$ or $\sigma_L(\sigma_L - 1)/(\sigma_l - 1) \cdot (1/Bm) < \lambda < \infty$.*

4. Existence, Multiplicity, and Nonexistence Results of Positive Periodic Solution(s) for (1.2) and (1.3)

In this section, we use the notations

$$i_0 = \text{number of zeros in the set } \{f_0, f_\infty\},$$
$$i_\infty = \text{number of infinities in the set } \{f_0, f_\infty\}. \tag{4.1}$$

It is clear that $i_0, i_\infty = 0, 1$ or 2. Then, we will show that (1.2) has i_0 or i_∞ positive T-periodic solution(s) for sufficiently large or small λ.

Theorem 4.1. *Assume that (H1)-(H2) hold.*

(a) *If $i_0 = 1$ or 2, then* (1.2) *has i_0 positive T-periodic solutions for $\lambda > (\sigma_L - 1)/m(1)\sum_{u=0}^{T-1} b(u) > 0$.*

(b) *If $i_\infty = 1$ or 2, then* (1.2) *has i_∞ positive T-periodic solutions for $0 < \lambda < (\sigma_l - 1)/\sigma_L M(1)\sum_{u=0}^{T-1} b(u)$.*

(c) *If $i_0 = 0$ or $i_\infty = 0$, then* (1.2) *has no positive T-periodic solution for sufficiently large or small $\lambda > 0$, respectively.*

Proof. (a) Choose $r_1 = 1$. By Lemma 2.6, we infer that there exists a $\lambda_0 = (\sigma_L - 1)/m(r_1)\sum_{u=0}^{T-1} b(u) > 0$, such that

$$\|T_\lambda x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_1},\ \lambda > \lambda_0. \tag{4.2}$$

If $f_0 = 0$, we can choose $0 < r_2 < r_1$ so that $f(x) \le \varepsilon x$ for $0 \le x \le r_2$, where the constant $\varepsilon > 0$ satisfies

$$\lambda\varepsilon\frac{\sigma_L \sum_{u=0}^{T-1} b(u)}{\sigma_l - 1} < 1. \tag{4.3}$$

Thus, $f(x(n)) \le \varepsilon x(n)$ for $x \in \partial\Omega_{r_2}$ and $n \in [0, T-1]$. We have by Lemma 2.5 that

$$\|T_\lambda x\| \le \lambda\varepsilon\frac{\sigma_L \sum_{u=0}^{T-1} b(u)}{\sigma_l - 1}\|x\| < \|x\| \quad \text{for } x \in \partial\Omega_{r_2}. \tag{4.4}$$

It follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 0, \qquad i(T_\lambda, \Omega_{r_2}, K) = 1. \tag{4.5}$$

Thus, $i(T_\lambda, \Omega_{r_1} \setminus \overline{\Omega}_{r_2}, K) = -1$ and T_λ has a fixed point in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$, which is a positive T-periodic solution of (1.2) for $\lambda > \lambda_0$.

If $f_\infty = 0$, there is an $\widehat{H} > 0$ such that $f(x) \le \varepsilon x$ for $x \ge \widehat{H}$, where $\varepsilon > 0$ satisfies (4.3). Let $r_3 = \max\{2r_1, \widehat{H}/((\sigma_l - 1)/\sigma_L(\sigma_L - 1))\}$, and it follows that $x(n) \ge ((\sigma_l - 1)/\sigma_L(\sigma_L - 1))\|x\| \ge \widehat{H}$ for $x \in \partial\Omega_{r_3}$ and $n \in [0, T-1]$. Thus, $f(x(n)) \le \varepsilon x(n)$ for $x \in \partial\Omega_{r_3}$ and $n \in [0, T-1]$. In view of Lemma 2.5, we have

$$\|T_\lambda x\| \le \lambda\varepsilon\frac{\sigma_L \sum_{u=0}^{T-1} b(u)}{\sigma_l - 1}\|x\| < \|x\| \quad \text{for } x \in \partial\Omega_{r_3}. \tag{4.6}$$

Again, it follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 0, \qquad i(T_\lambda, \Omega_{r_3}, K) = 1. \tag{4.7}$$

Thus, $i(T_\lambda, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K) = 1$ and (1.2) has a positive T-periodic solution for $\lambda > \lambda_0$.

If $f_0 = f_\infty = 0$, it is easy to see from the above proof that T_λ has a fixed point x_1 in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ and a fixed point x_2 in $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ such that

$$r_2 < \|x_1\| < r_1 < \|x_2\| < r_3. \tag{4.8}$$

Consequently, (1.2) has two positive T-periodic solutions for $\lambda > \lambda_0$ if $f_0 = f_\infty = 0$.

(b) Choose $r_1 = 1$. By Lemma 2.7, we infer that there exists

$$\lambda_0 = \frac{\sigma_l - 1}{\sigma_L M(r_1) \sum_{u=0}^{T-1} b(u)} > 0, \tag{4.9}$$

such that

$$\|T_\lambda x\| < \|x\| \quad \text{for } x \in \partial\Omega_{r_1},\ 0 < \lambda < \lambda_0. \tag{4.10}$$

If $f_0 = \infty$, there is a positive number $r_2 < r_1$ such that $f(x) \geq \eta x$ for $0 \leq x \leq r_2$, where $\eta > 0$ is chosen so that

$$\lambda \frac{\eta(\sigma_l - 1)}{(\sigma_L - 1)^2 \sigma_L} \sum_{u=0}^{T-1} b(u) > 1. \tag{4.11}$$

Then,

$$f(x(n)) \geq \eta x(n) \quad \text{for } x \in \partial\Omega_{r_2},\ n \in [0, T-1]. \tag{4.12}$$

By Lemma 2.4, we have that

$$\|T_\lambda x\| \geq \lambda\eta \frac{(\sigma_l - 1)}{(\sigma_L - 1)^2 \sigma_L} \sum_{u=0}^{T-1} b(u) \cdot \|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_2}. \tag{4.13}$$

It follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 1, \qquad i(T_\lambda, \Omega_{r_2}, K) = 0. \tag{4.14}$$

Thus, $i(T_\lambda, \Omega_{r_1} \setminus \overline{\Omega}_{r_2}, K) = 1$ and T_λ has a fixed point in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ for $0 < \lambda < \lambda_0$, which is a positive T-periodic solution of (1.2).

If $f_\infty = \infty$, there is an $\widehat{H} > 0$ such that $f(x) \geq \eta x$ for $x \geq \widehat{H}$, where $\eta > 0$ satisfies (4.11). Let $r_3 = \max\{2r_1, \widehat{H}/((\sigma_l - 1)/\sigma_L(\sigma_L - 1))\}$. If $x \in \partial\Omega_{r_3}$, then

$$x(n) \geq \frac{\sigma_l - 1}{\sigma_L(\sigma_L - 1)} \|x\| \geq \widehat{H}, \tag{4.15}$$

and hence,

$$f(x(n)) \geq \eta x(n) \quad \text{for } n \in [0, T-1]. \tag{4.16}$$

Again, it follows from Lemma 2.4 that

$$\|T_\lambda x\| \geq \lambda \frac{\eta(\sigma_l - 1)}{(\sigma_L - 1)^2 \sigma_L} \sum_{u=0}^{T-1} b(u) \|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_3}. \tag{4.17}$$

It follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 1, \qquad i(T_\lambda, \Omega_{r_3}, K) = 0, \tag{4.18}$$

and hence, $i(T_\lambda, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K) = -1$. Thus, T_λ has a fixed point in $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ for $0 < \lambda < \lambda_0$, which is a positive T-periodic solution of (1.2).

If $f_0 = f_\infty = \infty$, it is easy to see from the above proof that T_λ has a fixed point x_1 in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ and a fixed point x_2 in $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ such that

$$r_2 < \|x_1\| < r_1 < \|x_2\| < r_3. \tag{4.19}$$

Consequently, (1.2) has two positive T-periodic solutions for $0 < \lambda < \lambda_0$ if $f_0 = f_\infty = \infty$.

(c) If $i_0 = 0$, then $f_0 > 0$ and $f_\infty > 0$. It follows that there exist positive numbers η_1, η_2, r_1 and r_2, such that $r_1 < r_2$ and

$$\begin{aligned} f(x) &\geq \eta_1 x \quad \text{for } x \in [0, r_1], \\ f(x) &\geq \eta_2 x \quad \text{for } x \in [r_2, \infty). \end{aligned} \tag{4.20}$$

Let $c_1 = \min\{\eta_1, \eta_2, \min_{r_1 \leq x \leq r_2}\{f(x)/x\}\} > 0$. Thus, we have

$$f(x) \geq c_1 x \quad \text{for } x \in [0, \infty). \tag{4.21}$$

Assume v is a positive T-periodic solution of (1.2). We will show that this leads to a contradiction for $\lambda > \lambda_0$, where $\lambda_0 = (\sigma_L - 1)^2 \sigma_L / c_1(\sigma_l - 1) \sum_{u=0}^{T-1} b(u)$. Since $T_\lambda v(n) = v(n)$ for $n \in [0, T-1]$, it follows from Lemma 2.4 that for $\lambda > \lambda_0$,

$$\|v\| = \|T_\lambda v\| \geq \lambda \frac{(\sigma_l - 1) \sum_{u=0}^{T-1} b(u)}{(\sigma_L - 1)^2 \sigma_L} c_1 \|v\| > \|v\|, \tag{4.22}$$

which is a contradiction.

If $i_\infty = 0$, then $f_0 < \infty$ and $f_\infty < \infty$. It follows that there exist positive numbers ε_1, ε_2, r_1 and r_2 such that $r_1 < r_2$,

$$\begin{aligned} f(x) &\le \varepsilon_1 x \quad \text{for } x \in [0, r_1], \\ f(x) &\le \varepsilon_2 x \quad \text{for } x \in [r_2, \infty). \end{aligned} \tag{4.23}$$

Let $c_2 = \max\{\varepsilon_1, \varepsilon_2, \max_{r_1 \le x \le r_2}\{f(x)/x\}\} > 0$. Thus, we have

$$f(x) \le c_2 x \quad \text{for } x \in [0, \infty). \tag{4.24}$$

Assume v is a positive T-periodic solution of (1.2). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0$, where $\lambda_0 = (\sigma_l - 1)/c_2\sigma_L \sum_{u=0}^{T-1} b(u)$. Since $T_\lambda v(n) = v(n)$ for $n \in [0, T-1]$, it follows from Lemma 2.5 that for $0 < \lambda < \lambda_0$,

$$\|v\| = \|T_\lambda v\| \le \lambda \frac{\sigma_L \sum_{u=0}^{T-1} b(u)}{\sigma_l - 1} c_2 \|v\| < \|v\|, \tag{4.25}$$

which is a contradiction. □

The following result is a direct consequence of Theorem 4.1(c).

Corollary 4.2. *Assume that (H1)-(H2) hold.*

(a) *If there exists a $c_1 > 0$ such that $f(x) \ge c_1 x$ for $x \in [0, \infty)$, then there is a $\lambda_0 = (\sigma_L - 1)^2\sigma_L / c_1(\sigma_l - 1)\sum_{u=0}^{T-1} b(u)$ such that for all $\lambda > \lambda_0$,* (1.2) *has no positive T-periodic solutions.*

(b) *If there exists a $c_2 > 0$ such that $f(x) \le c_2 x$ for $x \in [0, \infty)$, then there is a $\lambda_0 = (\sigma_l - 1)/c_2\sigma_L \sum_{u=0}^{T-1} b(u)$ such that for all $0 < \lambda < \lambda_0$,* (1.2) *has no positive T-periodic solutions.*

Theorem 4.3. *Assume that (H1)-(H2) hold and $i_0 = i_\infty = 0$. If*

$$\frac{(\sigma_L - 1)^2 \sigma_L}{(\sigma_l - 1)\sum_{u=0}^{T-1} b(u)} \cdot \frac{1}{\max\{f_0, f_\infty\}} < \lambda < \frac{\sigma_l - 1}{\sigma_L \sum_{u=0}^{T-1} b(u)} \cdot \frac{1}{\min\{f_0, f_\infty\}}, \tag{4.26}$$

then (1.2) *has a positive T-periodic solutions.*

Proof. If $f_\infty > f_0$, then $(\sigma_L - 1)^2\sigma_L/(\sigma_l - 1)\sum_{u=0}^{T-1} b(u) f_\infty < \lambda < (\sigma_l - 1)/\sigma_L \sum_{u=0}^{T-1} b(u) f_0$. It is easy to see that there exists $\varepsilon : 0 < \varepsilon < f_\infty$ such that

$$\frac{(\sigma_L - 1)^2 \sigma_L}{(\sigma_l - 1)\sum_{u=0}^{T-1} b(u)(f_\infty - \varepsilon)} < \lambda < \frac{\sigma_l - 1}{\sigma_L \sum_{u=0}^{T-1} b(u)(f_0 + \varepsilon)}. \tag{4.27}$$

Now, turning to f_0 and f_∞, there is an $r_1 > 0$, such that $f(x) \le (f_0 + \varepsilon)x$ for $0 \le x \le r_1$. Thus, $f(x(n)) \le (f_0 + \varepsilon)x(n)$ for $x \in \partial\Omega_{r_1}$ and $n \in [0, T-1]$. We have by Lemma 2.5 that

$$\|T_\lambda u\| \le \lambda\left(f_0 + \varepsilon\right) \frac{\sigma_L \sum_{u=0}^{T-1} b(u)}{\sigma_l - 1} \|x\| < \|x\| \quad \text{for } x \in \partial\Omega_{r_1}. \tag{4.28}$$

On the other hand, there is an $\widehat{H} > r_1$ such that $f(x) \ge (f_\infty - \varepsilon)x$ for $x \ge \widehat{H}$. Let $r_2 = \max\{2r_1, \widehat{H}/((\sigma_l - 1)/\sigma_L(\sigma_L - 1))\}$, and it follows that $x(n) \ge ((\sigma_l - 1)/\sigma_L(\sigma_L - 1))\|x\| \ge \widehat{H}$ for $x \in \partial\Omega_{r_2}$ and $n \in [0, T-1]$. In view of Lemma 2.4, we have

$$\|T_\lambda u\| \ge \lambda\left(f_\infty - \varepsilon\right) \frac{(\sigma_l - 1) \sum_{u=0}^{T-1} b(u)}{(\sigma_L - 1)^2 \sigma_L} \|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_2}. \tag{4.29}$$

It follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 1, \qquad i(T_\lambda, \Omega_{r_2}, K) = 0. \tag{4.30}$$

Thus, $i(T_\lambda, \Omega_{r_2} \setminus \overline{\Omega}_{r_1}, K) = -1$. Hence, T_λ has a fixed point in $\Omega_{r_2} \setminus \overline{\Omega}_{r_1}$. Consequently, (1.2) has a positive T-periodic solution.

If $f_\infty < f_0$, then $(\sigma_L - 1)^2\sigma_L/(\sigma_l - 1)\sum_{u=0}^{T-1} b(u) f_0 < \lambda < (\sigma_l - 1)/\sigma_L \sum_{u=0}^{T-1} b(u) f_\infty$. It is easy to see that there exists an $\varepsilon : 0 < \varepsilon < f_0$ such that

$$\frac{(\sigma_L - 1)^2 \sigma_L}{(\sigma_l - 1) \sum_{u=0}^{T-1} b(u)\left(f_0 - \varepsilon\right)} < \lambda < \frac{\sigma_l - 1}{\sigma_L \sum_{u=0}^{T-1} b(u)\left(f_\infty + \varepsilon\right)}. \tag{4.31}$$

Now, turning to f_0 and f_∞, there is an $r_1 > 0$ such that $f(x) \ge (f_0 - \varepsilon)x$ for $0 \le x \le r_1$. Thus, $f(x(n)) \ge (f_0 - \varepsilon)x(n)$ for $x \in \partial\Omega_{r_1}$ and $n \in [0, T-1]$. We have by Lemma 2.4 that

$$\|T_\lambda u\| \ge \lambda\left(f_0 - \varepsilon\right) \frac{(\sigma_l - 1) \sum_{u=0}^{T-1} b(u)}{(\sigma_L - 1)^2 \sigma_L} \|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_1}. \tag{4.32}$$

On the other hand, there is an $\widehat{H} > r_1$ such that $f(x) \le (f_\infty + \varepsilon)x$ for $x \ge \widehat{H}$. Let $r_2 = \max\{2r_1, \widehat{H}/((\sigma_l - 1)/\sigma_L(\sigma_L - 1))\}$, and it follows that $x(n) \ge ((\sigma_l - 1)/\sigma_L(\sigma_L - 1))\|x\| \ge \widehat{H}$ for $x \in \partial\Omega_{r_2}$ and $n \in [0, T-1]$. Thus, $f(x(n)) \le (f_\infty + \varepsilon)x(n)$ for $x \in \partial\Omega_{r_2}$ and $n \in [0, T-1]$. In view of Lemma 2.5, we have

$$\|T_\lambda u\| \le \lambda\left(f_\infty + \varepsilon\right) \frac{\sigma_L \sum_{u=0}^{T-1} b(u)}{\sigma_l - 1} \|x\| < \|x\| \quad \text{for } x \in \partial\Omega_{r_2}. \tag{4.33}$$

It follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 0, \qquad i(T_\lambda, \Omega_{r_2}, K) = 1. \tag{4.34}$$

Thus, $i(T_\lambda, \Omega_{r_2} \setminus \overline{\Omega}_{r_1}, K) = 1$. Hence, T_λ has a fixed point in $\Omega_{r_2} \setminus \overline{\Omega}_{r_1}$. Consequently, (1.2) has a positive T-periodic solution. □

Next, we pay our attention to (1.3), that is

$$x(n+1) = \left[1 - a(n)g(x(n))\right]x(n) + \lambda b(n) f(x(n-\tau(n))), \quad n \in \mathbb{Z}, \tag{4.35}$$

where λ, a, b, $f(x)$, and $g(x)$ satisfy the same assumptions stated for (1.2) except that $0 \leq la(n) \leq La(n) < 1$ for all $n \in [0, T-1]$. In view of (1.3), we have that

$$x(n) = \lambda \sum_{u=n}^{n+T-1} K_x(n,u) b(u) f(x(u-\tau(u))), \tag{4.36}$$

where

$$K_x(n,u) = \frac{\prod_{s=u+1}^{n+T-1}(1 - a(s)g(x(s)))}{1 - \prod_{s=0}^{T-1}(1 - a(s)g(x(s)))}, \quad u \in [n, n+T-1]. \tag{4.37}$$

Note that since $0 \leq la(n) \leq La(n) < 1$ for all $n \in [0, T-1]$, we have

$$\frac{\rho_L}{1-\rho_L} \leq K_x(n,u) \leq \frac{1}{1-\rho_l}, \quad n \leq u \leq n+T-1, \tag{4.38}$$

here

$$\rho_l = \prod_{s=0}^{T-1}(1 - a(s)l), \qquad \rho_L = \prod_{s=0}^{T-1}(1 - a(s)L), \tag{4.39}$$

and $0 < \rho_L(1-\rho_l)/(1-\rho_L) < 1$.

Similarly, we can get the following theorems.

Theorem 4.4. *Assume that (H1)-(H2) hold and $0 \leq la(n) \leq La(n) < 1$ for $n \in [0, T-1]$.*

(a) *If $i_0 = 1$ or 2, then (1.3) has i_0 positive T-periodic solutions for $\lambda > (1-\rho_L)/\rho_L m(1) \sum_{u=0}^{T-1} b(u) > 0$.*

(b) *If $i_\infty = 1$ or 2, then (1.3) has i_∞ positive T-periodic solutions for $0 < \lambda < (1-\rho_l)/M(1) \sum_{u=0}^{T-1} b(u)$.*

(c) *If $i_0 = 0$ or $i_\infty = 0$, then (1.3) has no positive T-periodic solution for sufficiently large or small $\lambda > 0$, respectively.*

The following result is a direct consequence of Theorem 4.4(c).

Corollary 4.5. *Assume that (H1)-(H2) hold and $0 \leq la(n) \leq La(n) < 1$ for $n \in [0, T-1]$.*

(a) *If there exists a $c_1 > 0$ such that $f(x) \geq c_1 x$ for $x \in [0, \infty)$, then there is a $\lambda_0 = ((1-\rho_L)/\rho_L)^2 \cdot (1/c_1(1-\rho_l) \sum_{u=0}^{T-1} b(u))$ such that for all $\lambda > \lambda_0$ (1.3) has no positive T-periodic solutions.*

(b) *If there exists a* $c_2 > 0$ *such that* $f(x) \le c_2 x$ *for* $x \in [0, \infty)$, *then there is a* $\lambda_0 = (1-\rho_l)/c_2 \sum_{u=0}^{T-1} b(u)$ *such that for all* $0 < \lambda < \lambda_0$ (1.3) *has no positive* T*-periodic solutions.*

Theorem 4.6. *Assume that (H1)-(H2) hold and* $i_0 = i_\infty = 0$. *Let* $0 \le la(n) \le La(n) < 1$ *for* $n \in [0, T-1]$. *If*

$$\left(\frac{1-\rho_L}{\rho_L}\right)^2 \cdot \frac{1}{(1-\rho_l)\sum_{u=0}^{T-1} b(u)} \cdot \frac{1}{\max\{f_0, f_\infty\}} < \lambda < \frac{1-\rho_l}{\sum_{u=0}^{T-1} b(u)} \cdot \frac{1}{\min\{f_0, f_\infty\}}, \tag{4.40}$$

then (1.3) *has a positive* T*-periodic solution.*

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