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Research Article

Large Solutions for Semilinear Parabolic Equations Involving Some Special Classes of Nonlinearities

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We consider a new class of nonlinearities for which a nonlocal parabolic equation with Neumann boundary conditions has finite time blow-up solutions. Our approach is inspired by previous work done by Jazar and Kiwan (2008) and El Soufi et al. (2007) .

1. Introduction

This paper is devoted to the existence of large solutions of the semilinear parabolic problem

$$\begin{gathered} u_t - \Delta u = f(|u|) - \frac{1}{m(\Omega)} \int_\Omega f(|u|)dx, \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega \ , \end{gathered} \tag{1.1}$$

with the initial conditions

$$u(x,0) = u_0(x), \quad \text{on } \Omega, \text{ where } \int_\Omega u_0 dx = 0. \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded regular domain of class C^2, $f : [0,\infty) \mapsto [0,\infty)$ is a locally Lipschitz function, $m(\Omega)$ represents the Lebesgue measure of the domain Ω, and Δ is the Laplace operator.

The above problem was recently studied by El Soufi et al. [1] and Jazar and Kiwan [2], under the assumption that f is a power function of the form $f(u) = u^p$ (with $p > 1$). Under the same restriction on f, some lower bounds estimates for the blow-up time were established in [3]. See also [4, 5].

The aim of our paper is to extend their results to a larger class of nonlinearities whose precise definition is as follows.

Definition 1.1. A real-valued function f defined on an interval $[a, \infty)$ (with $a \geq 0$) satisfies property (C) if it is locally Lipschitz, nonnegative, and its mean value $(1/(t-a)) \int_a^t f(x)dx$ has a superlinear growth in the sense that the ratio

$$\frac{(1/(t-a)) \int_a^t f(x)dx}{(t-a)^{\alpha}} \tag{1.3}$$

is nondecreasing for t large enough and some $\alpha > 1$.

The monotonicity condition on (1.3) means precisely the existence of a constant $C \in (0, 1/2)$ (precisely, $C = (1+\alpha)^{-1}$) such that

$$Cf(t) \geq \frac{1}{t-a} \int_a^t f(x)dx \tag{1.4}$$

for $t > a$ large enough.

For example, if $g \in C^1([0, \infty))$, $g(0) = 0$, and g is nondecreasing, then the function $f(t) = g(t)t^{\alpha}$, with $\alpha > 1$, satisfies property (C). In fact,

$$\begin{aligned} \int_0^t f(x)dx &= \frac{t^{\alpha+1}}{\alpha+1} g(t) - \int_0^t g'(x) \frac{x^{\alpha+1}}{\alpha+1} dx \\ &\leq \frac{t^{\alpha+1}}{\alpha+1} g(t) = Ctf(t), \end{aligned} \tag{1.5}$$

where $C = 1/(\alpha+1) \in (0, 1/2)$.

Assuming that $f(0) = 0$ (which is the case if $a = 0$ and (1.4) works for all $t > 0$), one can infer from (1.4) that

$$\frac{f(t) + f(0)}{2} \geq \frac{1}{t} \int_0^t f(x)dx, \tag{1.6}$$

a fact that reminds of the Hermite-Hadamard inequality in convex functions theory. See [6, page 50]. Thus property (C) can be ascribed to the field of generalized convexity.

The problems of type (1.1) and (1.2) arise naturally in mechanics, biology, and population dynamics. For example, if we consider a couple or a mixture of two equations of the above type, the resulting problem describes the temperatures of two substances, which constitute a combustible mixture, or represents a model for the behavior of densities of two diffusion biological species which interact with each other. This type of problems is connected also with parabolic systems of heat equations with local sources, which arise in population dynamics. See [4, 7–11].

Our paper is organized as follows. In Section 2 we show that every solution u of the problems (1.1) and (1.2) (with u_0 not identically 0 and f satisfying property (C)) is large, provided that its energy at $t = 0$ is nonpositive. See Theorem 2.4. Our approach combines previous work done by El Soufi et al. [1], with a careful analysis of the properties of energy of solutions.

In Section 3 we discuss the connection of property (C) with other special classes of nonlinearities, well known in the literature. We prove that every function with generalized regular variation (à la Karamata), as well as every N-function in the sense of Orlicz, satisfies property (C). Meantime property (C) and the classical Keller-Osserman condition have a large overlap (though they are distinct from each other). Thus the class of functions satisfying property (C) provides indeed a natural framework for the existence of large solutions for the problems (1.1) and (1.2).

2. The Existence of Large Solutions

The existence of a solution to the problems (1.1) and (1.2) can be found in [1]. It can be summarized as follows.

Theorem 2.1. *Assume that* $\Omega \subset \mathbb{R}^N$ *is a bounded regular domain of class* C^2 *and* $f : [0,\infty) \mapsto [0,\infty)$ *is a locally Lipschitz function. Then for every* $u_0 \in C(\overline{\Omega})$ *there is an element* $t_{\max} \in [0,\infty]$ *such that the problems* (1.1) *and* (1.2) *has a unique solution*

$$u \in C\left([0,t_{\max}); C\left(\overline{\Omega}\right)\right) \cap C^1\left((0,t_{\max}); C\left(\overline{\Omega}\right)\right), \tag{2.1}$$

which solves the integral equation

$$u(t) = e^{t\delta}u_0 + \int_0^t e^{(t-s)\Delta} f(u(s))ds \tag{2.2}$$

on $[0,t_{\max})$*. Moreover,*

$$\int_\Omega u(t)dx = 0, \quad \forall t \in [0,t_{\max}), \tag{2.3}$$

and if $t_{\max} < \infty$*, then* $\lim_{t\to t_{\max}} \|u(t)\|_{L^\infty(\Omega)} = \infty$.

Each solution u of the problems (1.1) and (1.2) has the property $\int_\Omega u\,dx = 0$ because the integral in the right-hand side of (1.1) is 0 and

$$\frac{d}{dt}\left(\int_\Omega u\,dx\right) = \int_\Omega u_t dx = \int_\Omega \Delta u\,dx = 0. \tag{2.4}$$

Hence, by the initial condition (1.2), we have $\int_\Omega u\,dx = 0$.

Lemma 2.2. *Let $u \in C(\overline{\Omega})$ be a solution of* (1.1) *and* (1.2). *Then the energy of u at the moment t,*

$$E(t) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - \int_0^u f(|t|) dt \right) dx, \tag{2.5}$$

verifies the formula

$$E(t) = E(0) - \int_0^t \int_{\Omega} u_t^2 dx \, dt, \quad \forall t > 0. \tag{2.6}$$

Proof. In fact,

$$\begin{aligned} \frac{dE}{dt}(t) &= \int_{\Omega} (\nabla u_t \nabla u - u_t f(|u|)) dx \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial n} u_t d\sigma - \int_{\Omega} u_t \Delta u \, dx - \int_{\Omega} u_t \; f(|u|) dx \\ &= -\int_{\Omega} u_t (\Delta u + f(|u|)) dx = -\int_{\Omega} u_t^2 dx, \end{aligned} \tag{2.7}$$

and by integrating both sides over $[0, t]$, we obtain formula (2.6). □

According to the previous lemma, if $E(0)$ is nonpositive, then $E(t)$ is nonpositive for all $t > 0$. In the case of functions f satisfying condition (C), this leads to

$$C \int_{\Omega} u f(|u|) dx \geq \int_{\Omega} \int_0^u f(|t|) dt \, dx \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2. \tag{2.8}$$

Lemma 2.3. *Under the assumptions of Lemma 2.2 consider the two auxiliary functions*

$$m(t) := \frac{1}{2} \int_{\Omega} u^2(x, t) dx, \qquad h(t) := \int_0^t m(s) ds. \tag{2.9}$$

Then

$$m'(t) \geq \frac{1}{C} \int_0^t \int_{\Omega} u_t^2 dt, \tag{2.10}$$

$$m'(t) \geq \left(\frac{1}{2C} - 1 \right) \lambda m(t), \quad \text{for some } \lambda > 0, \tag{2.11}$$

$$\frac{1}{2C} (h'(t) - h'(0))^2 \leq h(t) h''(t), \tag{2.12}$$

provided that f satisfies condition (C).

Proof. In fact,

$$
\begin{aligned}
m'(t) &= \int_\Omega u_t u\, dx = \int_\Omega u\big(\Delta u + f(|u|)\big)dx \\
&\geq \int_\Omega \left(-|\nabla u|^2 + \frac{1}{C}\int_0^u f(|t|)dt\right)dx \\
&= -\frac{1}{C}\int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \int_0^u f(|t|)dt\right)dx + \left(\frac{1}{2C}-1\right)\int_\Omega |\nabla u|^2 dx.
\end{aligned}
\tag{2.13}
$$

Hence,

$$
\begin{aligned}
m'(t) &\geq -\frac{1}{C}E(u) + \left(\frac{1}{2C}-1\right)\int_\Omega |\nabla u|^2 dx \\
&\geq -\frac{1}{C}E(u) = -\frac{1}{C}E(u_0) + \frac{1}{C}\int_0^t\int_\Omega u_t^2\, dx\, dt \\
&\geq \frac{1}{C}\int_0^t\int_\Omega u_t^2 dx\, dt.
\end{aligned}
\tag{2.14}
$$

On the other hand, by the Poincaré inequality, we have

$$
m'(t) \geq \left(\frac{1}{2C}-1\right)\int_\Omega |\nabla u|^2 dx \geq \left(\frac{1}{2C}-1\right)\lambda\int_\Omega u^2 dx = \left(\frac{1}{2C}-1\right)\lambda m(t), \tag{2.15}
$$

where λ is a suitable positive constant.

We pass now to the proof of (2.12). Since

$$
\begin{aligned}
h'(t) - h'(0) &= \int_0^t m'(s)ds = \int_0^t\int_\Omega u u_t dx\, dt \\
&\leq \left(\int_0^t\int_\Omega u^2 dx\, dt\right)^{1/2}\left(\int_0^t\int_\Omega u_t^2 dx\, dt\right)^{1/2} \\
&\leq (2h(t))^{1/2}(Cm'(t))^{1/2} = \big(2Ch(t)h''(t)\big)^{1/2},
\end{aligned}
\tag{2.16}
$$

by (2.10) we infer that

$$
h'(t) - h'(0) = \int_0^t m'(s)ds \geq 0, \tag{2.17}
$$

and thus

$$\frac{1}{2C}\left(h'(t)-h'(0)\right)^2 \le h(t)h''(t). \tag{2.18}$$

□

We are now in a position to state the main result of our paper.

Theorem 2.4. *Assume that $f : [0,\infty) \mapsto [0,\infty)$ is a function with property (C), and let u be the solution of the problems* (1.1) *and* (1.2) *corresponding to an initial data $u_0 \in C(\overline{\Omega})$, u_0 not identically zero. If the energy of u at $t = 0$ is nonpositive, then u, as a function of t, cannot be in $L^\infty((0,T);L^2(\Omega))$ for all $T > 0$.*

Proof. Suppose, by *reduction ad absurdum*, that the solution $u(x,\cdot)$ exists in

$$L^\infty\Big((0,T);L^2(\Omega)\Big) \tag{2.19}$$

for all $T > 0$. By (2.11),

$$\lim_{t\to\infty} h'(t) = \lim_{t\to\infty} m(t) = \infty, \tag{2.20}$$

which yields, for each $\beta \in (0,1/C)$, the existence of a number $T_0 > 0$ such that for all $t > T_0$,

$$\beta h'(t)^2 \le \frac{1}{C}\left(h'(t)-h'(0)\right)^2. \tag{2.21}$$

Now, by (2.12) we obtain

$$\beta h'(t)^2 \le 2h(t)h''(t). \tag{2.22}$$

We will show, by considering the function $H(t) = h(t)^{-q}$, for a suitable $q > 0$, that the last inequality leads to a contradiction. In fact,

$$\begin{aligned} H''(t) &= qh(t)^{-q-2}\Big((q+1)\left(h'(t)\right)^2 - h(t)h''(t)\Big) \\ &\le qh(t)^{-q-2}\left(\frac{2(q+1)}{\beta} - 1\right)h(t)h''(t), \end{aligned} \tag{2.23}$$

for all $t \ge T_0$, so that for $\beta \in (0,1/C)$ and $q \in (0,1/(2C)-1)$ with $2(q+1) < \beta < 1/C$, the corresponding function $H(t)$ is concave.

By (2.20), $\lim_{t\to\infty} h(t) = \infty$, whence $\lim_{t\to\infty} H(t) = 0$. Thus H provides an example of a concave and strictly positive function which tends to 0 at infinity, a fact which is not possible. The proof is done. □

3. Classes of Functions with Property (C)

The aim of this section is to comment on how large is the class of functions which plays property (C). In this respect we will discuss here several particular classes of functions with this property.

We start with the class of regularly varying functions, introduced by Karamata in [12].

Definition 3.1. A positive measurable function f defined on interval $[a, \infty)$ (with $a \geq 0$) is said to be regularly varying at infinity, of index $\sigma \in \mathbb{R}$ (abbreviated, $f \in \mathrm{RV}_\infty(\sigma)$), provided that

$$\lim_{x \to \infty} \frac{f(tx)}{f(x)} = t^\sigma, \quad \forall t > 0. \tag{3.1}$$

All functions of index σ are of the form

$$f(x) = x^\sigma \exp\left(a(x) + \int_0^x \frac{\varepsilon(s)}{s} ds \right), \tag{3.2}$$

where $a(x)$ and $\varepsilon(x)$ are bounded and measurable, $a(x) \to \alpha \in \mathbb{R}$, and $\varepsilon(x) \to 0$ as $x \to \infty$. In particular, so are

$$x^\sigma \log x, \qquad x^\sigma \mathrm{loglog} x, \qquad x^\sigma \exp\left(\frac{\log x}{\mathrm{loglog} x} \right), \qquad x^\sigma \exp\left((\log x)^{1/3} \left(\cos\,(\log x)^{1/3} \right) \right). \tag{3.3}$$

See [13] for details.

Semilinear problems with nonlinearities in the class of regularly varying functions have been studied by Cîrstea and Rădulescu [14].

Proposition 3.2. *If $f \in RV_\infty(\sigma)$ with $\sigma > 1$, then*

$$\lim_{x \to \infty} \frac{F(x)}{x f(x)} = \frac{1}{\sigma + 1} < \frac{1}{2}, \tag{3.4}$$

where

$$F(x) := \int_0^x f(s) ds. \tag{3.5}$$

Under these assumptions, f satisfies condition (C) (and thus Theorem 2.4 applies to it).

Proof. To prove this, consider the change of variable $s = tx$, which yields

$$F(x) = \int_0^x f(s) ds = \int_0^1 x f(tx) dt. \tag{3.6}$$

The continuity of f and the fact that $f \in \mathrm{RV}_\infty(\sigma)$ assure the existence of a $\delta > 0$ such that for every $x > \delta$ we have

$$\frac{f(tx)}{f(x)} \leq t^\sigma + 1, \tag{3.7}$$

whence the integrability of the function $t \to \dfrac{f(tx)}{f(x)}$ on $[0,1]$. Then

$$\begin{aligned}\lim_{x\to\infty} \frac{F(x)}{xf(x)} &= \lim_{x\to\infty} \int_0^1 \frac{f(tx)}{f(x)} dt \\ &= \int_0^1 \lim_{x\to\infty} \frac{f(tx)}{f(x)} dt = \int_0^1 t^\sigma dt = \frac{1}{\sigma+1},\end{aligned} \tag{3.8}$$

where the commutation of the limit with the integral is motivated by the Lebesgue dominated convergence theorem. □

An important class of nonlinearities which appeared in connection with the study of boundary blow-up problems for elliptic equations is that of functions satisfying the Keller-Osserman condition. See the papers by Rădulescu [15] and Dumont et al. [16].

Definition 3.3. A nonnegative and nondecreasing function $f \in C^1([0,\infty))$ with $f(0) = 0$ satisfies the generalized Keller-Osserman condition of order $p > 1$ if

$$\int_1^\infty \frac{1}{(F(t))^{1/p}} dt < \infty, \tag{3.9}$$

where F is the primitive of f given by formula (3.5).

If $f \in \mathrm{RV}_\infty(\sigma+1)$ with $\sigma + 2 > p > 1$ a nondecreasing and continuous function, then $F \in \mathrm{RV}_\infty(\sigma+2)$ and $F^{-1/p} \in \mathrm{RV}_\infty((-\sigma-2)/p)$. Since $(-\sigma-2)/p < -1$, we infer that $F^{-1/p} \in L^1([1,\infty))$ and thus f satisfies the generalized Keller-Osserman condition.

It is worth to notice that the function $\exp(t)$ is not regularly varying at infinity though satisfies the generalized Keller-Osserman condition and also the hypothesis of Proposition 3.4.

Necessarily, if a function f satisfies the generalized Keller-Osserman condition of order $p > 1$, then

$$\lim_{t\to\infty} \frac{F(t)}{t^p} = \infty, \tag{3.10}$$

while $F(t)/t^p$ may be (or may be not) a monotonic function.

As noticed in the Introduction, property (C) is intimately related to the monotonicity of $F(t)/t^p$ in the following way.

Proposition 3.4. *If $F(t)/t^p$ is nondecreasing for some $p > 2$, then the function f satisfies condition (C) with $C = 1/p$ (and thus Theorem 2.4 applies to it).*

According to Proposition 3.4, the function $f(t) = pt^{p-1}\log(t+1)+(t^p/(t+1))$ satisfies for $p > 2$ condition (C) but not the generalized Keller-Osserman condition of order p. Indeed, f admits the primitive $F(t) = t^p \log(t+1)$.

We end our paper by discussing the connection of property (C) with a class of functions due to Orlicz.

Definition 3.5. An N-function is any function $M : [0,\infty) \to \mathbb{R}$ of the form

$$M(x) = \int_0^x p(t)dt, \tag{3.11}$$

where p is nondecreasing and right continuous, $p(0) = 0$, $p(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} p(t) = \infty$.

An N-function M satisfies the Δ_2-*condition* if there exist constants $k > 0$ and $x_0 \geq 0$ such that

$$M(2x) \leq kM(x), \quad \forall x \geq x_0. \tag{3.12}$$

Any N-function M is convex and plays the following properties:

(N1) $M(0) = 0$ and $M(x) > 0$ for $x > 0$;

(N2) $M(x)/x \to 0$ as $x \to 0$ and $M(x)/x \to \infty$ as $x \to \infty$.

Two examples of N-functions which satisfy the Δ_2-condition are x^p/p (for $p \geq 1$) and $t(\log t)^+$.

The N-functions which satisfy the Δ_2-condition are instrumental in the theory of Orlicz spaces (which extend the $L^p(\mu)$ spaces). Their theory is available in many books, such as [17, 18], and has important applications to interpolation theory [19] and Fourier analysis [20].

According to [18, page 23], the constant k which appears in the formulation of Δ_2-condition is always greater than or equal to 2.

Proposition 3.6. *Every N-function $M : [0,\infty) \to \mathbb{R}$ which satisfies the Δ_2-condition has property (C) (and thus Theorem 2.4 applies to it).*

Proof. Since M is nondecreasing,

$$M(tx) = M\left(2^{\log_2 t}x\right) \leq M\left(2^{[\log_2 t]+1}x\right), \tag{3.13}$$

and taking into account the Δ_2-condition we infer that

$$M(tx) \leq M(x)k^{[\log_2 t]+1} \leq M(x)k^{\log_2 t+1} \leq M(x)t^{2\log_2 k}, \tag{3.14}$$

for all x big enough and $t \geq 2$. Hence,

$$\begin{aligned}\int_0^t M(x)dx &= \int_0^1 tM(ts)ds \\ &\leq \int_0^1 tM(t)s^{2\log_2 k}ds = \frac{1}{2\log_2 k+1}tM(t) \\ &\leq \frac{1}{3}tM(t),\end{aligned} \tag{3.15}$$

and the proof is done. □

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