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Research Article
Topological Entropy of Cournot-Puu Duopoly

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The aim of this paper is to analyze a classical duopoly model introduced by Tönu Puu in 1991. For that, we compute the topological entropy of the model and characterize those parameter values with positive entropy. Although topological entropy is a measure of the dynamical complexity of the model, we will show that such complexity could not be observed.

1. Introduction

The classical Cournot-Puu duopoly [1] is a market which consists in two firms producing equivalent goods, with isoelastic demand function (in inverse form):

$$p = \frac{1}{q_1 + q_2}, \tag{1.1}$$

where q_i, $i = 1,2$, are the outputs of each firm, p is the price, and c_i, $i = 1,2$, are the constant marginal costs. Under these assumptions, both firms maximize their profits provided that

$$q_1 = \sqrt{\frac{q_2}{c_1}} - q_2, \tag{1.2}$$

$$q_2 = \sqrt{\frac{q_1}{c_2}} - q_1, \tag{1.3}$$

obtaining the Cournot equilibrium point $(\overline{q}_1, \overline{q}_2)$ given by

$$\begin{aligned} \overline{q}_1 &= \frac{c_2}{(c_1 + c_2)^2}, \\ \overline{q}_2 &= \frac{c_1}{(c_1 + c_2)^2}. \end{aligned} \tag{1.4}$$

In addition, if $q_i(t)$, $i = 1,2$, is the production of firm i at time t, then the future production of each firm is planned according to (1.2) and (1.3), and hence, taking into account that productions cannot be negative, we obtain the (dynamical) model

$$
\begin{aligned}
q_1(t+1) &= f_1(q_2(t)) = \max\left\{\sqrt{\frac{q_2(t)}{c_1}} - q_2(t), 0\right\}, \\
q_2(t+1) &= f_2(q_1(t)) = \max\left\{\sqrt{\frac{q_1(t)}{c_2}} - q_1(t), 0\right\},
\end{aligned}
\tag{1.5}
$$

where f_1 and f_2 are called reaction functions. In [1], the stability of the Cournot point $(\overline{q}_1, \overline{q}_2)$ was analyzed, proving that it is stable if $a = c_1/c_2 \in (3 - 2\sqrt{2}, 3 + 2\sqrt{2})$. In addition, a bifurcation analysis of the system was also addressed, showing numerically the existence of a period doubling bifurcation scheme.

The aim of this paper is to investigate the parameter value $a = c_1/c_2$ such that the system has a complicated dynamic behavior. To this end, we compute its topological entropy (see [2] or [3]), finding those values of positive entropy. We remark that positive entropy systems are chaotic in the sense of Li and Yorke (see [4, 5] for the definition of Li and Yorke chaos). The existence of Li-Yorke chaotic maps with zero entropy is well known (see, e.g., [6]), but we will show that they cannot exist for Cournot-Puu duopoly and so, the class of positive entropy maps agrees with the class of Li-Yorke chaotic maps.

Finally, recall that positive topological entropy and Li-Yorke chaos are topological chaos notions and hence, it is possible that “topological chaos” may not be observed in practice: for instance, when we consider a suitable unimodal map with positive entropy and an attracting periodic orbit (see e.g., [7]). We show evidences of nonobservable chaos for Cournot-Puu duopoly.

The paper is organized as follows. Section 2 is devoted to introduce the notion of topological entropy and its basic properties and to explain how we compute (approximately) it for our system. In the last section we will compare the results obtained in Section 2 with those obtained from the measure theory (physical) point of view.

2. Computing Topological Entropy

2.1. Topological Entropy

Although topological entropy was introduced first in [2], we give the Bowen definition [3] for compact metric spaces because it is more intuitive, and the connection with dynamical complexity is more clear. Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous map. We say that a set $E \subset X$ is (n, ε, f)-separated if for any $x, y \in E$, $x \neq y$, there exists $k \in \{0, 1, \ldots, n-1\}$ such that $d(f^k(x), f^k(y)) > \varepsilon$. Denote by $s(n, \varepsilon, f)$ the cardinality of any maximal (n, ε, f)-separated set in X. The topological entropy of f is the non negative number

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, f). \tag{2.1}$$

We remark that the definitions of [2, 3] agree for continuous maps on compact metric spaces, which is our case. In addition, the topological entropy fulfills the following properties, which will be useful for us in our computations.

(i) *Conjugacy Invariancy* [8, Chapter 7]. Let $f : X \to X$ and $g : Y \to Y$ be continuous maps on compact metric spaces and let $\pi : X \to Y$ be a continuous surjective map satisfying $\pi \circ f = g \circ \pi$. Then $h(g) \leq h(f)$. If in addition π is a homeomorphism, g and f are said to be conjugated and $h(g) = h(f)$.

(ii) *The Power Formula* [2]. If k is a positive integer, then $h(f^k) = kh(f)$.

(iii) *The Product Formula* [8, Chapter 7]. Let $f_i : X_i \to X_i$, $i = 1, 2$, be continuous maps of metric compact spaces and consider the product map $f_1 \times f_2 : X_1 \times X_2 \to X_1 \times X_2$ defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for all $(x_1, x_2) \in X_1 \times X_2$. Then $h(f_1 \times f_2) = h(f_1) + h(f_2)$.

(iv) *Commutativity Formula* [9]. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. Then $h(f \circ g) = h(g \circ f)$.

Positive topological entropy maps are chaotic in the sense of Li and Yorke (see [4]). Recall that f is Li-Yorke chaotic provided that there is an uncountable set $S \subset X$ such that for any $x, y \in S$, $x \neq y$,

$$0 = \liminf_{n \to \infty} d\left(f^n(x), f^n(y)\right) < \limsup_{n \to \infty} d\left(f^n(x), f^n(y)\right). \tag{2.2}$$

The set S is called scrambled set for f (see [5]). However, zero topological entropy maps can be also Li-Yorke chaotic (see, e.g., [6]). We will discuss these ideas for Cournot-Puu duopoly in Section 3.

2.2. Topological Entropy of the Cournot-Puu Duopoly

The system is given by

$$\left(q_1(t+1), q_2(t+1)\right) = \left(f_1\left(q_2(t)\right), f_2\left(q_1(t)\right)\right). \tag{2.3}$$

If we define $F(q_1, q_2) = (f_1(q_2), f_2(q_1))$, its second iterate is the product map:

$$F^2 = \left(f_1 \circ f_2\right) \times \left(f_2 \circ f_1\right). \tag{2.4}$$

Applying the basic properties of topological entropy, we find that

$$h(F) = \frac{1}{2} h\left(F^2\right) = \frac{1}{2}\left(h\left(f_1 \circ f_2\right) + h\left(f_2 \circ f_1\right)\right), \tag{2.5}$$

and by the commutativity formula, $h(f_1 \circ f_2) = h(f_2 \circ f_1)$. Therefore

$$h(F) = h\left(f_1 \circ f_2\right) = h\left(f_2 \circ f_1\right). \tag{2.6}$$

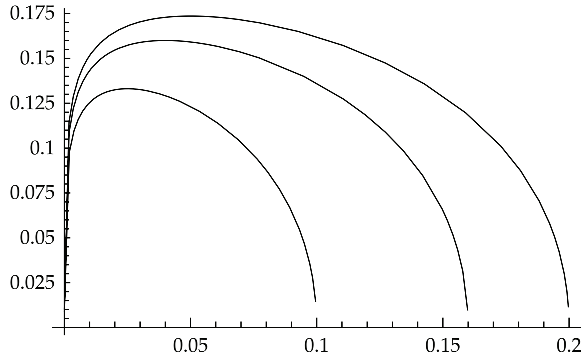


Figure 1: Graphic of g when $c_2 = 1$ and $c_1 = a$ is 5, 6.25, and 10, respectively. We see that g has two monotone pieces.

Now, we recall that

$$f_1(q_2) = \max\left\{\sqrt{\frac{q_2}{c_1}} - q_2, 0\right\},$$
$$f_2(q_1) = \max\left\{\sqrt{\frac{q_1}{c_2}} - q_1, 0\right\}, \tag{2.7}$$

and we assume, for instance, that $c_1 \geq c_2$. If $\varphi_1(q_2) = \sqrt{q_2/c_1} - q_2$, then $\varphi_1(1/c_1) = 0$ and $q_2^M = 1/4c_1$ is the maximum of φ_1 such that $\varphi_1(q_2^M) = q_2^M$. Since $c_1 \geq c_2$, if $\varphi_2(q_1) = \sqrt{q_1/c_2} - q_1$, then $g = \varphi_2 \circ \varphi_1$ (note that g depends on c_1 and c_2 and we should denote it by g_{c_1,c_2}) has a maximum at q_2^M and $q_2^0 = 1/c_1$ is the smallest positive number such that $g(q_2^0) = 0$. Hence, the map $g : [0, q_2^0] \to [0, q_2^0]$ is unimodal map (it has two monotone pieces) and its maximum q_2^M is called the turning point of g (see Figure 1).

We emphasize the fact that the unimodality of g will play an important role in the computations of topological entropy. It is not difficult to see that the other map $\varphi_1 \circ \varphi_2$ is not unimodal, and hence, the algorithm we are going to use to compute the topological entropy has not sense for it.

Note that the dynamics of reaction functions f_1 and f_2 is very simple. It is easy to see that $f_1([0,\infty)) = [0, q_2^M]$ and $f_1|_{[0,q_2^M]}$ is increasing (the same for f_2). So, the dynamics of f_1 is reduced to a repelling fixed point at 0 and an attracting fixed point q_2^M. As we will see later (also in [1]), the dynamics of the joint dynamics given by $f_1 \circ f_2$ and $f_2 \circ f_1$ can be very complicated, in the spirit of Parrondo's paradox (see, e.g., [10, 11] or [12]).

Next, we are going to point out an important property of our unimodal family of maps. The map $g = g_{c_1,c_2}$ depends on c_1 and c_2. For any $x \in [0, q_2^0]$ we have that

$$\begin{aligned} g_{c_1,c_2}(x) &= x - \sqrt{\frac{x}{c_1}} + \sqrt{\frac{\sqrt{x/c_1} - x}{c_2}} \\ &= \frac{1}{c_2}\left(c_2 x - \sqrt{\frac{c_2 x}{c_1/c_2}} + \sqrt{\sqrt{\frac{c_2 x}{c_1/c_2}} - c_2 x}\right) \\ &= \frac{1}{c_2} g_{c_1/c_2,1}(c_2 x). \end{aligned} \tag{2.8}$$

So, if $a = c_1/c_2$, $g_{c_1/c_2,1} = g_a$, and $\pi(x) = c_2 x$, we find that $\pi \circ g_{c_1,c_2} = g_a \circ \pi$. Hence the maps g_{c_1,c_2} and g_a are topologically conjugated and their topological entropies agree. So, our problem is reduced to compute the topological entropy of the family of maps:

$$g_a(x) = x - \sqrt{\frac{x}{a}} + \sqrt{\sqrt{\frac{x}{a}} - x}, \quad a \geq 1. \tag{2.9}$$

A first approach to the computation of topological entropy is given by the following fact: if $a \geq 6.25$, then $g_a(1/4a) = (2\sqrt{a} - 1)/4a \geq 1/a$. Since g_a is unimodal, there are two subintervals $J_1 = [0, x_0]$ and $J_2 = [x_1, 1/a]$ such that $g_a(x_0) = g_a(x_1) = 1/a$, $J_1 \cup J_2 \subset g_a(J_i)$, for $i = 1, 2$, and $[0, x_0) \cap (x_1, 1/a] = \emptyset$. In other words, g_a has a 2-horseshoe and hence $h(g_a) \geq \log 2$ (see, e.g., [13, Chapter 4]. Moreover, since the topological entropy of unimodal is bounded by $\log 2$, we conclude that $h(g_a) = \log 2$.

So, in view of the above results, we must concentrate our efforts in computing the topological entropy for parameter values in $(1, 6.25)$. The next section is devoted to that.

2.3. Practical Computation of Topological Entropy

We have to start this section by recognizing that an exact computation of the topological entropy is not possible. However, we are going to use the algorithm described in [14] to obtain good estimations of topological entropy with prescribed accuracy.

The algorithm is based on several facts. The first one is that the topological entropy of the tent map family

$$t_m(x) = \begin{cases} px & \text{if } 0 \leq x \leq \dfrac{1}{2}, \\ p - px & \text{if } \dfrac{1}{2} \leq x \leq 1 \end{cases} \tag{2.10}$$

is $h(t_p) = \log p$, for $p \in [1, 2]$.

The second one is the kneading sequence of unimodal map f with maximum (turning point) at c. Let f^n denote the nth iterate of f and $k(f) = (k_1, k_2, k_3, \cdots)$ its kneading sequence given by the rule

$$k_i = \begin{cases} R & \text{if } f^i(c) > c, \\ C & \text{if } f^i(c) = c, \\ L & \text{if } f^i(c) < c. \end{cases} \tag{2.11}$$

We fix that $L < C < R$. For two different unimodal maps f_1 and f_2, we fix their kneading sequences $k(f_1) = (k_n^1)$ and $k(f_2) = (k_n^2)$. We say that $k(f_1) \leq k(f_2)$ provided that there is $m \in \mathbb{N}$ such that $k_i^1 = k_i^2$ for $i < m$ and either an even number of k_i^1's are equal to R and $k_m^1 < k_m^2$ or an odd number of k_i^1's are equal to R and $k_m^2 < k_m^1$. Then one has the following.

(i) If $k(f_1) \leq k(f_2)$, then $h(f_1) \leq h(f_2)$. In addition, if $k_m(f)$ denotes the first m symbols of $k(f)$, then $k_m(f_1) < k_m(f_2)$.

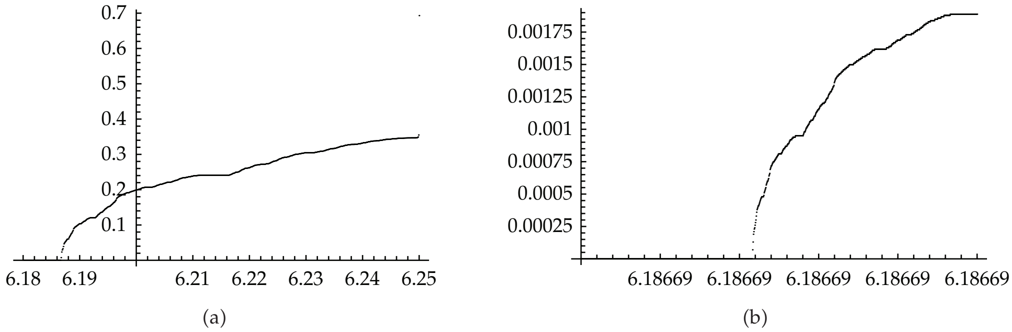


Figure 2: Topological entropy of the map g_a. (a) We can find the entropy for $a \in [6.18, 6.25]$ with accuracy of 10^{-3}. (b) We can see a zoom close to $a = 6.186$, when the topological entropy becomes to be positive. The accuracy for the right picture is 10^{-5}.

So, the algorithm is divided in four steps.

Step 1. Fix $\varepsilon > 0$ (fixed accuracy) and an integer n such that $\delta = 1/n < \varepsilon$.

Step 2. Find the least positive integer m such that $k_m(t_{1+i\delta})$, $0 \leq i \leq n$, are distinct kneading sequences.

Step 3. Compute $k_m(g_a)$ for a fixed $a \in (1, 6.25)$.

Step 4. Find r the largest integer such that $k_m(t_{1+r\delta}) < k_m(g_a)$. Hence $\log(1 + r\delta) \leq h(g_a) \leq \log(1 + (r + 2)\delta)$.

For the practical implementation of the algorithm we use the program Mathematica, which allows us to compute the kneading sequences $k_m(t_{1+i\delta})$ with infinite precision, that is, without round errors. Of course, the computation time increases a lot, but it must be done just one time for each fixed n. Unfortunately, it is not possible to do the computations with infinite precision for our family g_a. Figure 2 shows the topological entropy estimations of g_a.

Let us emphasize that when the accuracy is 10^{-3} (i.e., $n = 1000$), then the length of the kneading sequences of the tent map is $m = 896$. For accuracy of 10^{-6} the length $m = 786432$ distinguishes between the kneading sequence of tent maps with $p = 1+10^{-6}$ and $p = 1+2\cdot 10^{-6}$. So, with accuracy of 10^{-6}, we find that the first parameter a with positive entropy (greater than $\log(1.000001)$) is 6.186688432716827. Finally, note that positive topological entropy is only possible when the constant marginal costs are very different; that is, only "different" firms can produce chaos.

3. On Chaos and Further Discussions

When g_a has positive topological entropy, it is chaotic in the sense of Li and Yorke. The Cournot-Puu map F is Li-Yorke chaotic if and only if both maps $f_1 \circ f_2$ and $f_2 \circ f_1$ are also Li-Yorke chaotic (see [15]). Although the existence of Li-Yorke chaotic maps with zero topological entropy is known (see, e.g. [6]), we will show that such maps cannot exist in the Cournot-Puu model. The argument is as follows.

It was proved in [16] that zero topological entropy interval maps are Li-Yorke chaotic if they have wandering intervals (recall that a subinterval $J \subset [a, b]$ is wandering for a continuous map $f : [a, b] \subset \mathbb{R} \to [a, b]$ if for any $n, m \in \mathbb{N}$, $f^n(J) \cap f^m(J) = \emptyset$). It was

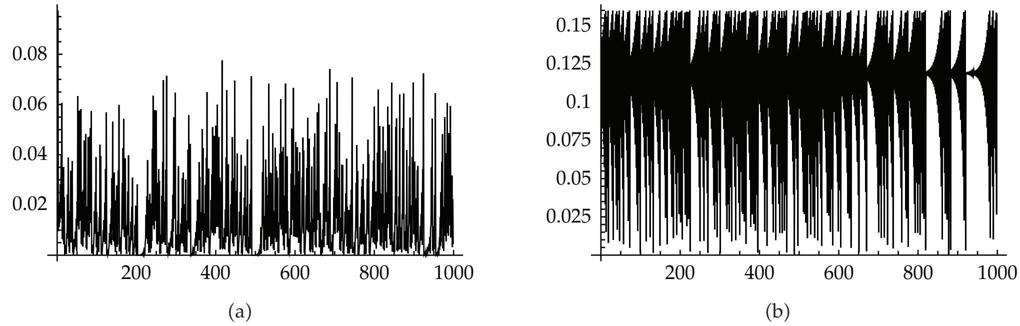


Figure 3: The picture shows the distance between the first 1000 iterates of two points for the parameter values $a = 6.24$ (a) and $a = 6.25$ (b).

proved in [17] that C^∞ maps with nonflat turning points do not have wandering intervals (A turning point c is nonflat if some derivative of f at c does not vanish.). Of course, our map g_a, $a \in (1, 6.25)$, has no finite derivative at 0, but the dynamics of g_a is concentrated in the invariant (by g_a) interval $[g_a^2(1/4a), g_a(1/4a)]$, in which the map is C^∞. The second derivative at the turning point is given by

$$g_a''\left(\frac{1}{4a}\right) = 2a(1 - \sqrt{a}), \tag{3.1}$$

which is different from zero for all $a \in (1, 6.25)$. As a consequence, the turning point of g_a is nonflat and therefore g_a has not wandering intervals. Hence, in the case of the Cournot-Puu model, the class of positive topological entropy maps agrees with the class of Li-Yorke chaotic maps.

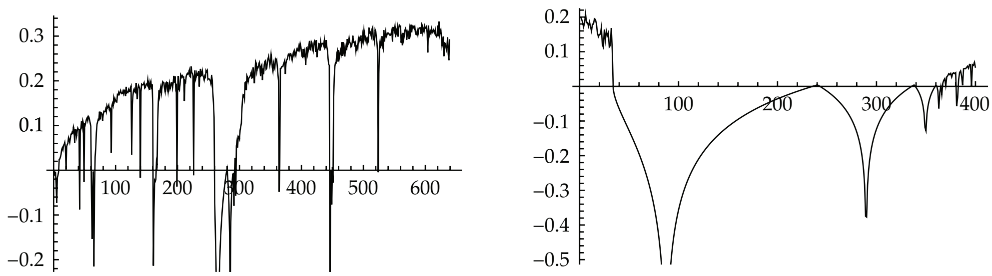
Figure 3 shows the distance $\|F^n(x_1, y_1) - F^n(x_2, y_2)\|$ for different parameter values. Recall that the existence of two points holding the Li–Yorke chaos conditions (called a Li-Yorke pair) is enough to guarantee the chaoticity of the map (see [15]).

Topological entropy and Li-Yorke chaos are purely topological notions. We must point out that such topological chaos could not be observed, say on a computer simulation. More precisely, we are going to show that there are parameter values for which the Cournot-Puu duopoly displays positive topological entropy while almost any point (in the sense of the Lebesgue measure) is attracted by a periodic orbit. In other words, the probability of a single orbit to be attracted by a periodic point is one. This leads us to the philosophical question on whether chaos exists when it cannot be observed.

Fix $x \in [0, 1/a]$ and denote the set of limit points of the sequence $(g_a^n(x))$ by $\omega(x, g_a)$, which will be called the ω-limit set of x under g_a. Recall that a metric attractor is a subset $A \subset [0, 1/a]$ such that $g_a(A) \subseteq A$, $\mathcal{B}(A) = \{x : \omega(x, g_a) \subset A\}$ has positive Lebesgue measure, and there is no proper subset $A' \subsetneq A$ with the same properties. By [18], the regularity properties of g_a imply that there are three possibilities for its metric attractors.

(A1) The first is a periodic orbit (recall that x is periodic if $g_a^n(x) = x$ for some $n \in \mathbb{N}$).

(A2) The second is a solenoidal attractor, which is basically a Cantor set in which the dynamics is quasiperiodic. More precisely, the dynamics on the attractor is conjugated to a minimal translation, in which each orbit is dense on the attractor. The dynamics of g_a restricted to the attractor is simple; neither positive topological entropy nor Li-Yorke chaos can be obtained.



(a) (b)

Figure 4: Estimation of the Lyapunov exponents for $a = 6.186 + k/1000$, $0 \leq k < 640$ (a) and $6.2115 + k/100000$, $0 \leq k \leq 400$ (b).

(A3) The third is a union of periodic intervals $J_1, \ldots, J_k$, such that $g_a^k(J_i) = J_i$ and $g_a^k(J_i) = J_j$, $1 \leq i < j \leq k$, and such that g_a^k is topologically mixing. Topologically mixing property implies the existence of dense orbits on each periodic interval (under the iteration of g_a^k).

Moreover, the map g_a can have at most one attractor of type A2 and A3 for any $a \in (1, 6.25)$.

In order to find positive topological entropy maps for which almost all points in $[0, 1/a]$ are attracted by a periodic orbit, we have to exclude the possibility of existence of attractors of types A2 and A3. Since an attracting periodic orbit $\{x_0, g_a(x_0), \ldots, g_a^{k-1}(x_0)\}$ holds that

$$d = \left| g_a'(x_0) g_a'(g_a(x_0)) \cdots g_a'\left(g_a^{k-1}(x_0)\right) \right| < 1, \tag{3.2}$$

we estimate the Lyapunov exponent

$$\lambda(x, a) = \limsup_{n \to \infty} \frac{1}{n} \log \left| g_a'(g_a^n(x)) \right| \tag{3.3}$$

for $x = g_a^{10^6}(1/4a)$. Since the metric attractor either contains the turning point or is an attracting periodic orbit (see [18]), we have that those values of a for which $\lambda(x, a) < 0$ are good candidates to have only periodic attractors. In Figure 4 we show the estimations of the Lyapunov exponent with $n = 1000$.

So, in view of Figure 4, for instance, we consider $a_0 = 6.21236$. We find that $x_0 \cong 0.160363$ is a period 6 such that

$$d = \left| \Pi_{k=0}^{5} g_{a_0}'\left(g_{a_0}^k(0.160363)\right) \right| \cong 0.003, \tag{3.4}$$

and hence we have a periodic attractor, while the topological entropy of g_a is positive. So, if almost all orbits are attracted by a periodic point, Li-Yorke pairs cannot be observed because

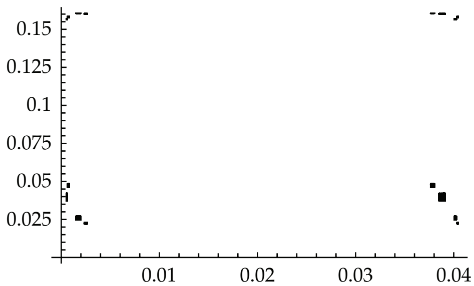


Figure 5: We show, for $a = 6.1872$, an attractor of type A3. It consists of 16 periodic rectangles, each of them being transitive under the iteration of the two dimensional map F_a^{16}.

for any $x, y \in (0, 1/a)$ we have that x and either y are proximal, that is,

$$\lim_{n \to \infty} \left| g_a^n(x) - g_a^n(y) \right| = 0, \tag{3.5}$$

or distal

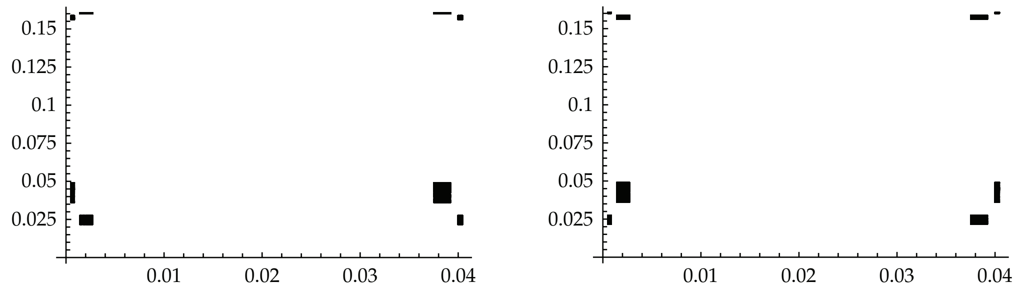
$$\liminf_{n \to \infty} \left| g_a^n(x) - g_a^n(y) \right| > 0. \tag{3.6}$$

Note that this periodic attractor of g_a generates six different periodic attractors, with period 12, of the two-dimensional Cournot-Puu map. The spatial distribution of these periodic attractors was characterized in [19]. In [20], the existence of different attractor for two dimensional maps was also pointed out.

It is well known that the existence of a periodic point of period 6 of g_a implies that $h(g_a) > 0$ (see, e.g., [13, Chapter 4]). So, for the parameter value $a = 6.21236$ the reader may suspect the existence of chaos for g_a. The next example is even more paradoxical. For the $a = 6.2085$ there is an attracting periodic point of period 64 and, since periodic points whose periods are power of 2 do not guarantee that g_a had positive topological entropy, the existence of the above periodic orbit of period 64 does not offer any implication of topological chaos. Since almost all orbits are attracted to this periodic orbit (at least numerical experiments show it), the topological chaos could not be detected by evaluating the orbits.

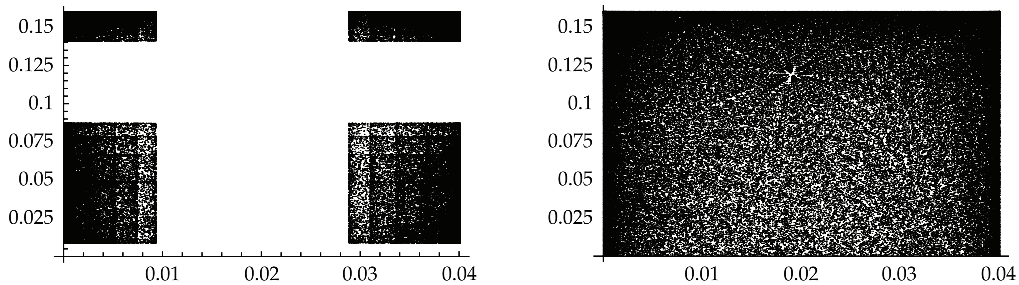
In the case of positive Lyapunov exponent, we can address the existence of periodic rectangles as metric attractors, which follow the spatial distributions characterized in [21]. We call them attractors of type A3, because they are generated by attractors of type A3 of g_a. For instance, for $a = 6.189$, we find two different metric attractors given by four periodic rectangles. For $a = 6.24$ we have a unique attractor given by four periodic topologically mixing rectangles. In the limit case $a = 6.25$ we find a topologically mixing map on a rectangle. Figures 5, 6, and 7 show these attractors. It is unclear for us whether solenoidal type attractors exist in this model.

We must point out that different ω-limit sets of the system can be obtained by computer simulations, even when they are not attractors of the system. For instance, for parameters values $a = 6.24$ and $a = 6.25$, we can obtain in Figure 8 the following limit sets of the point $(0.001, f_2(0.001))$, which is in the so-called MPE set of the system given by the



(a) (b)

Figure 6: Two different attractors of type A3 for the parameter value $a = 6.189$.



(a) (b)

Figure 7: Attractors of type A3 for $a = 6.24$ (a). For the limit case $a = 6.25$ we receive a dense orbit on the rectangle, which is the attractor of the system (b).

points in the set $\{(x, f_2(x)) : x \geq 0\} \cup \{(f_1(x), x) : x \geq 0\}$. In general, the ω-limit sets for points in the sets

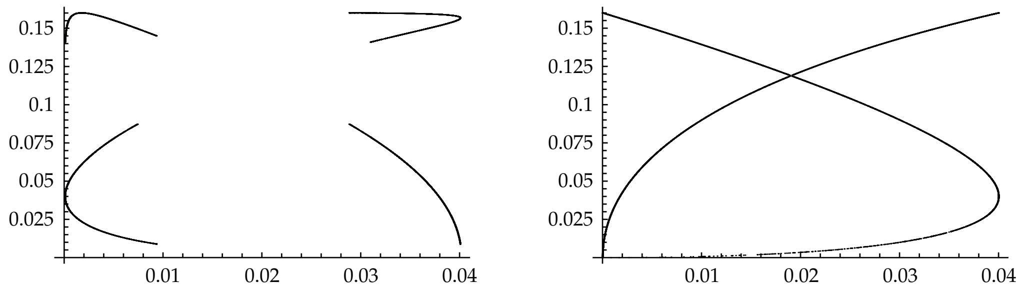
$$\mathcal{M}_k = \left\{ \left(x, f_2\left((f_1 \circ f_2)^k(x)\right)\right) : x \geq 0 \right\} \cup \left\{ \left(f_1\left((f_2 \circ f_1)^k(x)\right), x\right) : x \geq 0 \right\} \tag{3.7}$$

for $k = 1, 2, \ldots$ are also one-dimensional. Figure 9 shows some examples. These ω-limit sets are obtained when the associated one dimensional orbits of the model satisfy some synchronization properties.

Finally, if the parameter value a is greater than 6.25, although the entropy is positive, we can see that the orbit of almost any point is eventually the fixed point 0. The reason for that is the existence of an interval $J \subset [0, 1/a]$ such that $g_a(x) > 1/a$ for any $x \in J$, and therefore $g_a^2(x) = 0$. We say that this map has a "hole" and all the interval $[0, 1/a]$ except for a Cantor set goes eventually to 0, and chaos cannot be detected. We refer the reader to [22, 23] for more information about the dynamics of unimodal maps with holes.

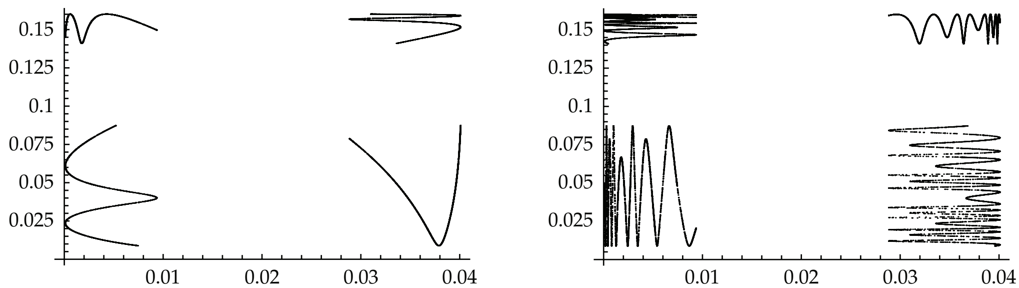
4. Conclusion

We analyze in detail the Cournot-Puu model introduced in [1]. We use very recent and powerful mathematical results to characterize the attractors of the model and to compute its topological entropy. Although the existence of topological chaos is proved when the



(a) (b)

Figure 8: Limit sets for the parameter values 6.24 (a) and 6.25 (b). They can be observed although they are not properly an attractor of the system.



(a) (b)

Figure 9: ω-limit sets for parameter value $a = 6.24$ and $k = 6$ (a) and $k = 16$ (b).

difference between the marginal costs of both firms is big enough, we check that it may not be physically observed. The following questions remain open for this model. On one hand, it would be useful to get another algorithm for computing the topological entropy in order to improve the accuracy of the computations. On the other hand, it is unclear how many attractors in the system. Note that the map g_a might have more than one attractor because its Schwartzian derivative need not be negative (recall that unimodal maps with negative Schwartzian derivative have only one attractor [7]). It is also an open question whether the existence of solenoidal attractors is possible, and how they are distributed in the square in the case of the two-dimensional model.

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