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Research Article

Dynamics of a Higher-Order Nonlinear Difference Equation

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We consider the higher-order nonlinear difference equation $x_{n+1} = (\alpha + x_n)/(A + Bx_n + x_{n-k})$, $n = 0, 1, \ldots$, where parameters are positive real numbers and initial conditions $x_{-k}, \ldots, x_0$ are nonnegative real numbers, $k \geq 2$. We investigate the periodic character, the invariant intervals, and the global asymptotic stability of all positive solutions of the abovementioned equation. We show that the unique equilibrium of the equation is globally asymptotically stable under certain conditions.

1. Introduction and Preliminaries

In this paper, we will investigate the global behavior of solutions of the following nonlinear difference equation:

$$x_{n+1} = \frac{\alpha + x_n}{A + Bx_n + x_{n-k}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where parameters are positive real numbers and initial conditions $x_{-k}, \ldots, x_0$ are nonnegative real numbers, $k \geq 2$.

In 2003, the authors in [1] considered the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.2}$$

with nonnegative parameters α, β, A, B, C and nonnegative initial conditions x_{-1}, x_0. They obtained some global asymptotic stability results for the solutions of (1.2). For (1.2), we can also see [2–4].

For the global behavior of solutions of some related equations, see [5–13]. Other related results can be found in [14–20]. For the sake of convenience, we recall some definitions and theorems which will be useful in the sequel.

Definition 1.1. Let I be some interval of real numbers, and let

$$f : I^{m+1} \longrightarrow I \tag{1.3}$$

be a continuously differential function. Then for every set of initial conditions $y_{-k}, \ldots, y_{-1}, y_0 \in I$, the difference equation

$$y_{n+1} = f(y_n, y_{n-1}, \ldots, y_{n-k}), \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

has a unique solution $\{y_n\}_{n=-k}^{\infty}$.

A point $\overline{y}$ is called an equilibrium point of (1.4) if

$$\overline{y} = f(\overline{y}, \overline{y}, \ldots, \overline{y}). \tag{1.5}$$

That is,

$$y_n = \overline{y}, \quad \text{for } n \geq 0, \tag{1.6}$$

is a solution of (1.4), or equivalently $\overline{y}$ is a fixed point of f.

Definition 1.2. Let $\overline{y}$ be an equilibrium point of (1.4). Then the following are considered.

(i) The equilibrium $\overline{y}$ is called locally stable (or stable) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with $\sum_{i=-k}^{i=0} |y_i - \overline{y}| < \delta$, we have $|y_n - \overline{y}| < \varepsilon$ for all $n \geq k$.

(ii) The equilibrium $\overline{y}$ of (1.4) is called locally asymptotically stable (asymptotic stable) if it is locally stable and if there exists $\gamma > 0$ such that, for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with $\sum_{i=-k}^{i=0} |y_i - \overline{y}| < \gamma$, we have $\lim_{n \to \infty} y_n = \overline{y}$.

(iii) The equilibrium $\overline{y}$ of (1.4) is called a global attractor if, for every $y_{-k}, \ldots, y_{-1}, y_0 \in I$, we have $\lim_{n \to \infty} y_n = \overline{y}$.

(iv) The equilibrium $\overline{y}$ of (1.4) is globally asymptotically stable if it is locally stable and is a global attractor.

(v) The equilibrium $\overline{y}$ of (1.4) is called unstable if it is not stable.

(vi) The equilibrium $\overline{y}$ of (1.4) is called a source, or a repeller, if there exists $r > 0$ such that, for all $y_{-k}, \ldots, y_{-1}, y_0 \in I$ with $\sum_{i=-k}^{i=0} |y_i - \overline{y}| < \gamma$, there exists $N \geq 1$ such that $|y_N - \overline{y}| \geq r$.

An interval $J \subseteq I$ is called an invariant interval for (1.4) if

$$y_{-k}, \dots, y_0 \in J \Longrightarrow y_n \in J \quad \forall n > 0. \tag{1.7}$$

That is, every solution of (1.4) with initial conditions in J remains in J.

The linearized equation associated with (1.4) about the equilibrium $\overline{y}$ is

$$y_{n+1} = \sum_{i=o}^{k} \frac{\partial f}{\partial u_i}(\overline{y}, \dots, \overline{y}) y_{n-i}, \quad n = 0, 1, 2, \dots. \tag{1.8}$$

Its characteristic equation is

$$\lambda^{k+1} = \sum_{i=o}^{k} \frac{\partial f}{\partial u_i}(\overline{y}, \dots, \overline{y}) \lambda^{k-i}. \tag{1.9}$$

Theorem 1.3 (see [10]). *Assume that f is a C^1 function, and let $\overline{y}$ be an equilibrium of* (1.4). *Then the following statements are true.*

(i) *If all the roots of* (1.9) *lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{y}$ of* (1.4) *is asymptotically stable.*

(ii) *If at least one root of* (1.9) *has absolute value greater than one, then the equilibrium $\overline{y}$ of* (1.4) *is unstable.*

Theorem 1.4 (see [10]). *Assume that $P, Q \in R$ and $k \in \{1, 2, \dots\}$. Then*

$$|P| + |Q| < 1 \tag{1.10}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} = Py_n + Qy_{n-k}, \quad n = 0, 1, \dots. \tag{1.11}$$

Lemma 1.5 (see [8]). *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n = 0, 1, \dots, \tag{1.12}$$

where $k \in \{1, 2, \dots\}$. Let $I = [a, b]$ be some interval of real numbers and assume that

$$f : [a, b] \times [a, b] \longrightarrow [a, b] \tag{1.13}$$

is a continuous function satisfying the following properties.

(a) *$f(x, y)$ is nondecreasing in $x \in [a, b]$ for each $y \in [a, b]$, and $f(x, y)$ is nonincreasing in $y \in [a, b]$ for each $x \in [a, b]$.*

(b) *If* $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$f(m, M) = m, \qquad f(M, m) = M, \tag{1.14}$$

then $m = M$.

Then (1.12) *has a unique equilibrium* $\overline{x} \in [a, b]$ *and every solution of* (1.12) *converges to* $\overline{x}$.

Lemma 1.6 (see [8]). *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n = 0, 1, \ldots, \tag{1.15}$$

where $k \in \{1, 2, \ldots\}$. *Let* $I = [a, b]$ *be some interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \longrightarrow [a, b] \tag{1.16}$$

is a continuous function satisfying the following properties.

(a) $f(x, y)$ *is nonincreasing in each of its arguments.*

(b) *If* $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = f(M, M), \qquad M = f(m, m), \tag{1.17}$$

then $m = M$.

Then (1.15) *has a unique equilibrium* $\overline{y} \in [a, b]$ *and every solution of* (1.15) *converges to* $\overline{y}$.

2. Local Stability and Period-Two Solutions

The equilibria of (1.1) are the solutions of the equation

$$\overline{x} = \frac{\alpha + \overline{x}}{A + B\overline{x} + \overline{x}}. \tag{2.1}$$

So (1.1) possesses the unique positive equilibrium

$$\overline{x} = \frac{1 - A + \sqrt{(1 - A)^2 + 4\alpha(B + 1)}}{2(B + 1)}. \tag{2.2}$$

The linearized equation associated with (1.1) about the positive equilibrium is

$$z_{n+1} + \frac{B\overline{x} - 1}{A + B\overline{x} + \overline{x}} z_n + \frac{\overline{x}}{A + B\overline{x} + \overline{x}} z_{n-k} = 0. \tag{2.3}$$

The next result follows from Theorem 1.4.

Theorem 2.1. *Assume that*

$$\text{either } A \geq 1 \text{ or } A < 1, \quad (B-1)(1-A)^2 + 4B^2\alpha > 0. \tag{2.4}$$

Then the positive equilibrium $\overline{x}$ *of* (1.1) *is locally asymptotically stable.*

Theorem 2.2. *Equation* (1.1) *has no nonnegative prime period-two solution.*

Proof. Assume for the sake of contradiction that there exist distinct nonnegative real numbers ϕ and ψ such that

$$\ldots, \phi, \psi, \phi, \psi, \ldots \tag{2.5}$$

is a prime period-two solution of (1.1).

(a) Assume that k is odd. Then $x_{n+1} = x_{n-k}$ and ϕ, ψ satisfy the following system:

$$\phi = \frac{\alpha + \psi}{A + B\psi + \phi}, \qquad \psi = \frac{\alpha + \phi}{A + B\phi + \psi}. \tag{2.6}$$

Subtracting both sides of the above two equations, we obtain

$$(\phi - \psi)\left[\phi + \psi + (A+1)\right] = 0. \tag{2.7}$$

If $\phi \neq \psi$, then $\phi + \psi = -(A+1)$; this contradicts the hypothesis that $\phi, \psi \geq 0$.

(b) Assume that k is even. Then $x_n = x_{n-k}$ and ϕ, ψ satisfy the following system:

$$\phi = \frac{\alpha + \psi}{A + B\psi + \psi}, \qquad \psi = \frac{\alpha + \phi}{A + B\phi + \phi}. \tag{2.8}$$

Subtracting both sides of the above two equations, we obtain

$$(\phi - \psi)(A+1) = 0. \tag{2.9}$$

If $\phi \neq \psi$, then $A = -1$; this contradicts the hypothesis that $A \geq 0$.

The proof is complete. □

3. Boundedness and Invariant Interval

In this section, we will investigate the boundedness and invariant interval of (1.1).

Theorem 3.1. *Every solution of* (1.1) *is bounded from above and from below by positive constants.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of (1.1). Clearly, if the solution is bounded from above by a constant M, then

$$x_{n+1} \geq \frac{\alpha}{A+(B+1)M}, \quad \text{for } n \geq -k, \tag{3.1}$$

and so it is also bounded from below. Now for the sake of contradiction assume that the solution is not bounded from above. Then there exists a subsequence $\{x_{n_m+1}\}_{m=0}^{\infty}$ such that

$$\lim_{m\to\infty} n_m = \infty, \qquad \lim_{m\to\infty} x_{n_m+1} = \infty \tag{3.2}$$

and also

$$x_{n_m+1} = \max\{x_n : n \leq n_m\} \quad \text{for } m \geq 0. \tag{3.3}$$

From (1.1) we see that

$$x_{n+1} < \frac{\alpha}{A} + \frac{1}{A}x_n \quad \text{for } n \geq 0, \tag{3.4}$$

and so

$$\lim_{m\to\infty} x_{n_m+1} = \lim_{m\to\infty} x_{n_m} = \infty. \tag{3.5}$$

Hence, for sufficiently large m,

$$0 \leq x_{n_m+1} - x_{n_m} = \frac{\alpha + x_{n_m}}{A+Bx_{n_m}+x_{n_m-k}} - x_{n_m} = \frac{\alpha + [(1-A) - Bx_{n_m} - x_{n_m-k}]x_{n_m}}{A+Bx_{n_m}+x_{n_m-k}} < 0, \tag{3.6}$$

which is a contradiction.

The proof is complete. □

Let

$$f(x,y) = \frac{\alpha + x}{A+Bx+y}. \tag{3.7}$$

Then the following statements are true.

Lemma 3.2. (a) *Assume that $A \geq B\alpha$. Then $f(x,y)$ is increasing in x for each y and decreasing in y for each x.*

(b) *Assume that $A < B\alpha$. Then $f(x,y)$ is decreasing in y for each x, decreasing in x for $y \in [0, B\alpha - A]$, and increasing in x for $y \in [B\alpha - A, \infty]$.*

Proof. The proofs of (a) and (b) are simple and will be omitted. □

Theorem 3.3. *Equation* (1.1) *possesses the following invariant intervals:*

(a) $[0, 1/B]$, *when* $B\alpha \leq A$;

(b) $[B\alpha - A, 1/B]$, *when* $A < B\alpha < A + 1/B$;

(c) $[0, \alpha/A]$, *when* $B\alpha = A + 1/B$;

(d) $[1/B, B\alpha - A]$, *when* $A + 1/B < B\alpha < A + \alpha/A$;

(e) $[1/B, \alpha/A]$, *when* $B\alpha \geq A + \alpha/A$.

Proof. (a) Set $g(x) = (\alpha + x)/(A + Bx)$, so $g(x)$ is nondecreasing for x and $g(1/B) \leq 1/B$ if $B\alpha \leq A$, when $x_{-k}, \ldots, x_0 \in [0, 1/B]$; then we have

$$x_1 = \frac{\alpha + x_0}{A + Bx_0 + x_{-k}} \leq \frac{\alpha + x_0}{A + Bx_0} \leq g\left(\frac{1}{B}\right) \leq \frac{1}{B}. \tag{3.8}$$

The proof follows by induction.

(b) In view of Lemma 3.2(b), by using the monotonic character of the function $f(x, y)$ and the condition $A < B\alpha < A + (1/B)$, when $x_{-k}, \ldots, x_0 \in [B\alpha - A, 1/B]$, we can get

$$\begin{aligned} x_1 &= \frac{\alpha + x_0}{A + Bx_0 + x_{-k}} = f(x_o, x_{-k}) \geq f\left(B\alpha - A, \frac{1}{B}\right) > B\alpha - A, \\ x_1 &= \frac{\alpha + x_0}{A + Bx_0 + x_{-k}} = f(x_o, x_{-k}) \leq f\left(\frac{1}{B}, B\alpha - \mathrm{A}\right) = \frac{1}{B}. \end{aligned} \tag{3.9}$$

The proof follows by induction.

(c) Set $h(x) = (\alpha + x)/(A + Bx + \alpha/A)$ and $g(x) = (\alpha + x)/(A + Bx)$, so $h(x)$ is increasing and $g(x)$ is decreasing for x if $B\alpha = A + 1/B$. In view of Lemma 3.2(b), by using the monotonic character of the function $f(x, y)$, when $x_{-k}, \ldots, x_0 \in [0, \alpha/A]$, we have

$$\begin{aligned} x_1 &= \frac{\alpha + x_0}{A + Bx_0 + x_{-k}} \geq \frac{\alpha + x_0}{A + Bx_0 + (\alpha/A)} \geq h(0) > 0, \\ x_1 &= \frac{\alpha + x_0}{A + Bx_0 + x_{-k}} \leq \frac{\alpha + x_0}{A + Bx_0} \leq g(0) = \frac{\alpha}{A}. \end{aligned} \tag{3.10}$$

The proof follows by induction.

(d) In view of Lemma 3.2(b), by using the monotonic character of the function $f(x, y)$ and the condition $A + 1/B < B\alpha < A + \alpha/A$, when $x_{-k}, \ldots, x_0 \in [1/B, B\alpha - A]$, we obtain

$$\begin{aligned} x_1 &= \frac{\alpha + x_0}{A + Bx_0 + x_{-k}} = f(x_o, x_{-k}) \geq f(B\alpha - A, B\alpha - A) > \frac{1}{B}, \\ x_1 &= \frac{\alpha + x_0}{A + Bx_0 + x_{-k}} = f(x_o, x_{-k}) \leq f\left(\frac{1}{B}, \frac{1}{B}\right) = \frac{B\alpha + 1}{AB + B + 1} < B\alpha - A. \end{aligned} \tag{3.11}$$

The proof follows by induction.

(e) In view of the condition $B\alpha \geq A + \alpha/A$, we can get $B\alpha - A \geq \alpha/A$; by using the monotonic character of the function $f(x, y)$ and the condition $B\alpha \geq A + \alpha/A$, when $x_{-k}, \ldots, x_0 \in [1/B, \alpha/A]$, we have

$$x_1 = \frac{\alpha + x_0}{A + Bx_0 + x_{-k}} = f(x_o, x_{-k}) \geq f\left(\frac{\alpha}{A}, \frac{\alpha}{A}\right) = \frac{A\alpha + \alpha}{A^2 + B\alpha + \alpha} \geq \frac{1}{B},$$
$$x_1 = \frac{\alpha + x_0}{A + Bx_0 + x_{-k}} = f(x_o, x_{-k}) \leq f\left(\frac{1}{B}, \frac{1}{B}\right) = \frac{B\alpha + 1}{AB + B + 1} < \frac{\alpha}{A}. \tag{3.12}$$

The proof follows by induction.

The proof is complete. □

4. Semicycles Analysis

We now give the definitions of positive and negative semicycles of a solution of (1.4) relative to an equilibrium point $\overline{x}$.

A positive semicycle of a solution $\{x_n\}$ of (1.4) consists of a string of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to the equilibrium $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ and such that

$$\text{either } l = -k \text{ or } l > -k, \quad x_{l-1} < \overline{x},$$
$$\text{either } m = \infty \text{ or } m < \infty, \quad x_{m+1} < \overline{x}. \tag{4.1}$$

A negative semicycle of a solution $\{x_n\}$ of (1.4) consists of a string of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than the equilibrium $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ and such that

$$\text{either } l = -k \text{ or } l > -k, \quad x_{l-1} \geq \overline{x},$$
$$\text{either } m = \infty \text{ or } m < \infty, \quad \mathrm{x}_{m+1} \geq \overline{x}. \tag{4.2}$$

Theorem 4.1 (see [12]). *Assume that $f \in C[(0, \infty) \times (0, \infty), (0, \infty)]$ is such that $f(x, y)$ is increasing in x for each fixed y and is decreasing in y for each fixed x. Let $\overline{x}$ be a positive equilibrium of* (1.12). *Then the following are considered.*

(a) *If $k = 1$, then every solution of* (1.12) *has semicycles of length at least two.*

(b) *If $k \geq 2$, then every solution of* (1.12) *has semicycles that are either of length at least $k + 1$ or of length at most $k - 1$.*

Let $\{x_n\}$ be a positive solution of (1.1). Then one has the following identities:

$$x_{n+1}-\frac{1}{B}=\frac{1}{B}\frac{(B\alpha-A)-x_{n-k}}{A+Bx_n+x_{n-k}},\quad n\in\mathbb{N}_0, \tag{4.3}$$

$$x_{n+1}-\frac{\alpha}{A}=-\frac{1}{A}\frac{(B\alpha-A)x_n+\alpha x_{n-k}}{A+Bx_n+x_{n-k}},\quad n\in\mathbb{N}_0, \tag{4.4}$$

$$x_{n+1}-(B\alpha-A)=\frac{B[1/B-(B\alpha-A)]x_n+B\alpha(1/B-x_{n-k})+A[x_{n-k}-(B\alpha-A)]}{A+Bx_n+x_{n-k}},\quad n\in\mathbb{N}_0, \tag{4.5}$$

$$x_{n+1}-\overline{x}=\frac{\overline{x}(\overline{x}-x_{n-k})+B(\overline{x}-1/B)(\overline{x}-x_n)}{A+Bx_n+x_{n-k}},\quad n\in\mathbb{N}_0, \tag{4.6}$$

$$\begin{aligned}
&x_{2(k+1)(n+1)}-x_{2(k+1)n}\\
&\quad=\frac{Bx_{2(k+1)(n+1)-1}\left(1/B-x_{2(k+1)n}\right)\left(A+x_{2(k+1)n}+Bx_{2(k+1)n+k}\right)}{\left(A+Bx_{2(k+1)(n+1)-1}\right)\left(A+Bx_{n+1}+x_{2(k+1)n}\right)+\alpha+x_{2(k+1)n+k}}\\
&\quad+\frac{(1+AB)x_{2(k+1)n+k}\left(\alpha B/(1+AB)-x_{2(k+1)n}\right)+A\left(\alpha-Ax_{2(k+1)n}-x_{2(k+1)n}^2\right)}{\left(A+Bx_{2(k+1)(n+1)-1}\right)\left(A+Bx_{2(k+1)n+k}+x_{2(k+1)n}\right)+\alpha+x_{2(k+1)n+k}},\quad n\in\mathbb{N}_0.
\end{aligned} \tag{4.7}$$

If $1/B=B\alpha-A$, then $\overline{x}=1/B$ and (4.3), (4.7) change into

$$x_{n+1}-\frac{1}{B}=\frac{1}{B}\frac{1/B-x_{n-k}}{A+Bx_n+x_{n-k}},\quad n\in\mathbb{N}_0, \tag{4.8}$$

$$\begin{aligned}
&x_{2(k+1)(n+1)}-x_{2(k+1)n}\\
&\quad=\frac{\left(1/B-x_{2(k+1)n}\right)\left(ABx_{2(k+1)(n+1)-1}+Bx_{2(k+1)n}x_{2(k+1)(n+1)-1}\right)}{\left(A+Bx_{2(k+1)(n+1)-1}\right)\left(A+Bx_{2(k+1)n+k}+x_{2(k+1)n}\right)+\alpha+x_{2(k+1)n+k}}\\
&\quad+\frac{\left(1/B-x_{2(k+1)n}\right)\left(B^2x_{2(k+1)n+k}x_{2(k+1)(n+1)-1}+B^2\alpha x_{2(k+1)n+k}+Ax_{2(k+1)n}+AB\alpha\right)}{\left(A+Bx_{2(k+1)(n+1)-1}\right)\left(A+Bx_{2(k+1)n+k}+x_{2(k+1)n}\right)+\alpha+x_{2(k+1)n+k}},\\
&\qquad n\in\mathbb{N}_0.
\end{aligned} \tag{4.9}$$

The following lemmas are straightforward consequences of identities (4.3)–(4.9).

Lemma 4.2. *Assume that*

$$B\alpha\le A, \tag{4.10}$$

and let $\{x_n\}$ *be a solution of* (1.1). *Then the following statements are true.*

(i) $x_n \leq 1/B$ *for all* $n \geq 1$.

(ii) *If for some* $N \geq 0$, $x_{N-k} \leq \overline{x}$ *and* $x_N \geq \overline{x}$, *then* $x_{N+1} \geq \overline{x}$.

(iii) *If for some* $N \geq 0$, $x_{N-k} > \overline{x}$ *and* $x_N < \overline{x}$, *then* $x_{N+1} < \overline{x}$.

(iv) $0 < \overline{x} < 1/B$.

Lemma 4.3. *Assume that*

$$A < B\alpha < A + \frac{1}{B}, \tag{4.11}$$

and let $\{x_n\}$ *be a solution of* (1.1). *Then the following statements are true.*

(i) *If for some* $N \geq 0$, $x_N < B\alpha - A$, *then* $x_{N+k+1} > 1/B$.

(ii) *If for some* $N \geq 0$, $x_N = B\alpha - A$, *then* $x_{N+k+1} = 1/B$.

(iii) *If for some* $N \geq 0$, $x_N > B\alpha - A$, *then* $x_{N+k+1} < 1/B$.

(iv) *If for some* $N \geq 0$, $B\alpha - A < x_N < 1/B$, *then* $B\alpha - A < x_{N+k+1} < 1/B$.

(v) *If for some* $N \geq 0$, $x_{N-k} \leq \overline{x}$ *and* $x_N \geq \overline{x}$, *then* $x_{N+1} \geq \overline{x}$.

(vi) *If for some* $N \geq 0$, $x_{N-k} > \overline{x}$ *and* $x_N < \overline{x}$, *then* $x_{N+1} < \overline{x}$.

(vii) *If for some* $N \geq 0$, $x_{2(k+1)N} < B\alpha - A$, *then* $x_{2(k+1)(N+1)} > x_{2(k+1)N}$.

(viii) *If for some* $N \geq 0$, $x_{2(k+1)N} > 1/B$, *then* $x_{2(k+1)(N+1)} < x_{2(k+1)N}$.

(ix) $B\alpha - A < \overline{x} < 1/B$.

Lemma 4.4. *Assume that*

$$B\alpha = A + \frac{1}{B}, \tag{4.12}$$

and let $\{x_n\}$ *be a solution of* (1.1). *Then the following statements are true.*

(i) *If for some* $N \geq 0$, $x_N > 1/B$, *then* $x_{N+k+1} < 1/B$.

(ii) *If for some* $N \geq 0$, $x_N = 1/B$, *then* $x_{N+k+1} = 1/B$.

(iii) *If for some* $N \geq 0$, $x_N < 1/B$, *then* $x_{N+k+1} > 1/B$.

(iv) *If for some* $N \geq 0$, $x_{2(k+1)N} > 1/B$, *then* $x_{2(k+1)(N+1)} < x_{2(k+1)N}$.

(v) *If for some* $N \geq 0$, $x_{2(k+1)N} < 1/B$, *then* $x_{2(k+1)(N+1)} > x_{2(k+1)N}$.

(vi) $\overline{x} = 1/B$.

Lemma 4.5. *Assume that*

$$A + \frac{1}{B} < B\alpha < A + \frac{\alpha}{A}, \tag{4.13}$$

and let $\{x_n\}$ *be a solution of* (1.1). *Then the following statements are true.*

(i) *If for some* $N \geq 0$, $x_N < B\alpha - A$, *then* $x_{N+k+1} > 1/B$.

(ii) *If for some* $N \geq 0$, $x_N = B\alpha - A$, *then* $x_{N+k+1} = 1/B$.

(iii) *If for some* $N \geq 0$, $x_N > B\alpha - A$, *then* $x_{N+k+1} < 1/B$.

(iv) *If for some* $N \geq 0$, $1/B < x_N < B\alpha - A$, *then* $1/B < x_{N+k+1} < B\alpha - A$.

(v) *If for some* $N \geq 0$, $x_{N-k} \leq \overline{x}$ *and* $x_N \leq \overline{x}$, *then* $x_{N+1} \geq \overline{x}$.

(vi) *If for some* $N \geq 0$, $x_{N-k} > \overline{x}$ *and* $x_N > \overline{x}$, *then* $x_{N+1} < \overline{x}$.

(vii) *If for some* $N \geq 0$, $x_{2(k+1)N} < 1/B$, *then* $x_{2(k+1)(N+1)} > x_{2(k+1)N}$.

(viii) *If for some* $N \geq 0$, $x_{2(k+1)N} > B\alpha - A$, *then* $x_{2(k+1)(N+1)} < x_{2(k+1)N}$.

(ix) $1/B < \overline{x} < B\alpha - A$.

Lemma 4.6. *Assume that*

$$B\alpha \geq A + \frac{\alpha}{A}, \tag{4.14}$$

and let $\{x_n\}$ *be a solution of* (1.1). *Then the following statements are true.*

(i) $x_n < \alpha/A$ *for all* $n \geq 1$.

(ii) *If for some* $N \geq 0$, $x_N < \alpha/A$, *then* $x_{N+k+1} > 1/B$.

(iii) *If for some* $N \geq 0$, $1/B < x_N < \alpha/A$, *then* $1/B < x_{N+k+1} < \alpha/A$.

(iv) *If for some* $N \geq 0$, $x_{N-k} \leq \overline{x}$ *and* $x_N \leq \overline{x}$, *then* $x_{N+1} \geq \overline{x}$.

(v) *If for some* $N \geq 0$, $x_{N-k} > \overline{x}$ *and* $x_N > \overline{x}$, *then* $x_{N+1} < \overline{x}$.

(vi) *If for some* $N \geq 0$, $x_{2(k+1)N} < 1/B$, *then* $x_{2(k+1)(N+1)} > x_{2(k+1)N}$.

(vii) *If for some* $N \geq 0$, $x_{2(k+1)N} > \alpha/A$, *then* $x_{2(k+1)(N+1)} < x_{2(k+1)N}$.

(viii) $1/B < \overline{x} < \alpha/A$.

The following result is a consequence of Theorem 4.1 and Lemmas 4.2–4.6.

Theorem 4.7. *Let* $\{x_n\}_{n=-k}^{\infty}$ *be a nontrivial solution of* (1.1). *Then the following statements are true.*

(a) *Assume that* $B\alpha \leq A$. *Then, except possibly for the first semicycle, every oscillatory solution of* (1.1) *has semicycles that are either of length at least* $k+1$, *or of length at most* $k-1$.

(b) *Assume that* $A < B\alpha < A + 1/B$. *Then, except possibly for the first semicycle, every oscillatory solution of* (1.1) *which lies in the invariant interval* $[B\alpha - A, 1/B]$ *has semicycles that are either of length at least* $k+1$, *or of length at most* $k-1$.

(c) *Assume that* $B\alpha = A + 1/B$. *Then, except possibly for the first semicycle,* $\{x_n\}_{n=-k}^{\infty}$ *is oscillatory and the sum of the lengths of two consecutive semicycles is equal to* $2(k+1)$.

(d) *Assume that* $A + 1/B < B\alpha < A + \alpha/A$. *Then, except possibly for the first semicycle, every oscillatory solution of* (1.1) *which lies in the invariant interval* $[1/B, B\alpha - A]$ *has semicycles at most* $k+1$.

(e) *Assume that* $B\alpha \geq A + \alpha/A$. *Then, except possibly for the first semicycle, every oscillatory solution of* (1.1) *which lies in the invariant interval* $[1/B, \alpha/A]$ *has semicycles at most* $k+1$.

5. Global Stability Proof

In this section, we will investigate the global stability of all positive solutions of (1.1).

Theorem 5.1. *Let* $\{x_n\}_{n=-k}^{\infty}$ *be a positive solution of* (1.1). *Then the following statements are true.*

(a) *Assume that* $B\alpha \leq A$. *Then every solution of* (1.1) *eventually enters the interval* $[0, 1/B]$.

(b) *Assume that* $A < B\alpha < A + 1/B$. *Then every solution of* (1.1) *eventually enters the interval* $[B\alpha - A, 1/B]$.

(c) *Assume that* $A + 1/B < B\alpha < A + \alpha/A$. *Then every solution of* (1.1) *eventually enters the interval* $[1/B, B\alpha - A]$.

(d) *Assume that* $B\alpha \geq A + \alpha/A$. *Then every solution of* (1.1) *eventually enters the interval* $[1/B, \alpha/A]$.

Proof. (a) In view of Lemma 4.2, we know that $x_n \leq 1/B$ for all $n \geq 1$ and $\overline{x} \in [0, 1/B]$; that is, all solutions of (1.1) eventually enter the interval $[0, 1/B]$.

(b) If $x_{-k}, \dots, x_0 \in [B\alpha - A, 1/B]$, by Theorem 3.3(b), then we have $x_n \in [B\alpha - A, 1/B]$, for all $n \geq 0$. If the initial conditions are not in the interval $[B\alpha - A, 1/B]$, then we consider the $2(k+1)$th subsequences $\{x_{2(k+1)n+j}\}_{j=0}^{2k+1}$ of the solution $\{x_n\}$. We will give the proof for the subsequence $\{x_{2(k+1)n}\}$. The proof for all the other subsequences is similar and will be omitted. Without loss of generality, we assume that there exists N sufficiently large such that $x_{2(k+1)N} < B\alpha - A$ if ($x_{2(k+1)N} > 1/B$, then the proof is similar and will be omitted); then in view of Lemmas 4.3(ii) and (iv), we know that $x_{(k+1)(2N+1)} > 1/B > B\alpha - A$ and $x_{2(k+1)(N+1)} < 1/B$. If $x_{2(k+1)(N+1)} \geq B\alpha - A$, then, by induction, we know that the former assertion implies that the result is true. If $x_{2(k+1)(N+1)} < B\alpha - A$, by Lemma 4.3(viii), then we can get $x_{2(k+1)(N+1)} > x_{2(k+1)N}$. It follows by induction that the subsequence $\{x_{2(k+1)(N+m)}\}_{m=0}^{\infty}$ is increasing, and because $x_{2(k+1)(N+m)} < B\alpha - A$, so $\lim_{m\to\infty} x_{2(k+1)(N+m)}$ exists and $\lim_{m\to\infty} x_{2(k+1)(N+m)} \leq B\alpha - A$. However, taking limits by (4.7), we get a contradiction.

(c) The proof is similar to (b), so will be omitted.

(d) In view of Lemma 4.6, we know that $x_n < \alpha/A$ for all $n \geq 1$; that is, all solutions of (1.1) eventually enter the interval $[0, \alpha/A]$. Furthermore, by Theorem 3.3, $[1/B, \alpha/A]$ is an invariant interval of (1.1). Now, assume for the sake of contradiction that all solutions never enter the interval $[1/B, \alpha/A]$, then the subsequence $\{x_{2(k+1)(N+m)}\}_{m=0}^{\infty}$ enters the interval $[0, 1/B]$. Because $x_{2(k+1)N} \leq 1/B$ and $B\alpha \geq A + \alpha/A$, then, by Lemma 4.6, we know that $x_{2(k+1)(N+1)} > x_{2(k+1)N}$; it follows by induction that the subsequence $\{x_{2(k+1)(N+m)}\}_{m=0}^{\infty}$ is increasing in the interval $[0, 1/B]$. So $\lim_{m\to\infty} x_{2(k+1)(N+m)}$ exists and $\lim_{m\to\infty} x_{2(k+1)(N+m)} \leq 1/B$, which is a contradiction because (1.1) has no equilibrium point in the interval $[0, 1/B]$.

The proof is complete. □

Theorem 5.2. *Assume that* (2.4) *holds. Then the positive equilibrium* $\overline{x}$ *of* (1.1) *is a global attractor of all positive solutions of* (1.1).

We consider the following five cases.

Case 1. Assume that $B\alpha \leq A$. By Theorems 3.3(a) and 5.1(a), we know that (1.1) possesses an invariant interval $[0, 1/B]$ and every solution of (1.1) eventually enters the interval $[0, 1/B]$. Further, it is easy to see that $f(x, y)$ increases in x and decreases in y in $[0, 1/B]$.

Let $m, M \in [0, 1/B]$ be a solution of the system

$$\frac{\alpha + m}{A + Bm + M} = m, \qquad \frac{\alpha + M}{A + BM + m} = M, \tag{5.1}$$

which is equivalent to

$$\alpha + m = Am + Bm^2 + Mm, \qquad \alpha + M = AM + BM^2 + Mm. \tag{5.2}$$

Hence

$$(m - M)[1 - A - B(M + m)] = 0. \tag{5.3}$$

Now if $m + M \neq (1 - A)/B$, then $m = M$. For instance, this is the case if $A \geq 1$ is satisfied.

If $m + M = (1 - A)/B$, then m and M satisfy the system

$$m + M = \frac{1 - A}{B}, \qquad mM = \frac{\alpha}{1 - B} \tag{5.4}$$

and the equation

$$B(B - 1)m^2 + (B - 1)(A - 1)m - B\alpha = 0, \tag{5.5}$$

whose discriminant is

$$\Delta = (B - 1)\left[(B - 1)(A - 1)^2 + 4B^2\alpha\right]. \tag{5.6}$$

Clearly, in this case, $B < 1$, and in view of condition (2.4), we have $\Delta < 0$, from which it follows that $m = M$. In view of Lemma 1.5, (1.1) has a unique equilibrium $\overline{x} \in [0, 1/B]$ and every solution of (1.1) converges to $\overline{x}$.

Case 2. Assume that $A < B\alpha < A + 1/B$. By Theorems 3.3(b) and 5.1(b), we know that (1.1) possesses an invariant interval $[B\alpha - A, 1/B]$ and every solution of (1.1) eventually enters the interval $[B\alpha - A, 1/B]$. Further, it is easy to see that $f(x, y)$ increases in x and decreases in y in $[B\alpha - A, 1/B]$. Then using the same argument in Case 1, (1.1) has a unique equilibrium $\overline{x} \in [B\alpha - A, 1/B]$ and every solution of (1.1) converges to $\overline{x}$.

Case 3. Assume that $B\alpha = A + 1/B$. Considering the $2(k+1)$th subsequences $\{x_{2(k+1)n+j}\}_{n=0}^{\infty}$ $(j \in \{0, 1, \ldots, 2k + 1\})$, $n \geq 0$, then by Lemma 4.4, we know that each one of the $2(k + 1)$th subsequences is above $1/B$, below $1/B$, or identically equal to $1/B$. Furthermore, by the identity (4.9), we can get that all $2(k + 1)$th subsequences converge monotonically to limits, and for all $n \in \mathbb{N}$,

$$x_{2(k+1)(n+1)} = x_{2(k+1)n} \quad \text{iff } x_{2(k+1)n} = \frac{1}{B}. \tag{5.7}$$

So all the $2(k+1)$th subsequences $\{x_{2(k+1)(n+1)+j}\}_{n=0}^{\infty}$ $(j \in \{0,1,\ldots,2k+1\})$ converge to $1/B$. That is, $\overline{x} = 1/B$ is a global attractor of (1.1).

Case 4. Assume that $A + 1/B < B\alpha < A + \alpha/A$. By Theorems 3.3(d) and 5.1(c), we know that (1.1) possesses an invariant interval $[1/B, B\alpha - A]$ and every solution of (1.1) eventually enters the interval $[1/B, B\alpha - A]$. Furthermore, it is easy to see that the function $f(x,y)$ decreases in each of its arguments in the interval $[1/B, B\alpha - A]$. Let $m, M \in [1/B, B\alpha - A]$ be a solution of the system

$$\frac{\alpha + m}{A + Bm + m} = M, \qquad \frac{\alpha + M}{A + BM + M} = m, \tag{5.8}$$

that is, the solution of the system

$$\alpha + m = AM + (B+1)mM, \qquad \alpha + M = Am + (B+1)mM. \tag{5.9}$$

Then $(m - M)(A+1) = 0$, which implies that $m = M$. Employing Lemma 1.6, we see that (1.1) has a unique equilibrium $\overline{x} \in [1/B, B\alpha - A]$ and every solution of (1.1) converges to $\overline{x}$.

Case 5. Assume that $B\alpha \geq A + \alpha/A$. By Theorems 3.3(e) and 5.1(d), we know that (1.1) possesses an invariant interval $[1/B, \alpha/A]$ and every solution of (1.1) eventually enters the interval $[1/B, \alpha/A]$. Further, it is clear to see that the function $f(x,y)$ decreases in each of its arguments in the interval $[1/B, \alpha/A]$. Then, using the same argument as in Case 4, (1.1) has a unique equilibrium $\overline{x} \in [1/B, \alpha/A]$ and every solution of (1.1) converges to $\overline{x}$.

The proof is complete.

In view of Theorems 2.1 and 5.2, we have the following result.

Theorem 5.3. *Assume that* (2.4) *holds. Then the unique positive equilibrium* $\overline{x}$ *of* (1.1) *is globally asymptotically stable.*

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