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Research Article

On the Recursive Sequence $x_{n+1} = \max\{x_n, A\}/x_n^2 x_{n-1}$

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We investigate the periodic nature of the solution of the max-type difference equation $x_{n+1} = \max\{x_n, A\}/x_n^2 x_{n-1}$, $n = 0, 1, 2, \ldots$, where the initial conditions are $x_{-1} = A^{r_1}$ and $x_0 = A^{r_2}$ for $A \in (0, \infty)$, and that r_1 and r_2 are positive rational numbers. The results in this paper solve the Open Problem proposed by Grove and Ladas (2005).

1. Introduction

Max-type difference equations stem from, for example, certain models in automatic control theory (see [1, 2]). Although max-type difference equations are relatively simple in form, it is, unfortunately, extremely difficult to understand thoroughly the behavior of their solutions, see, for example, [1–41] and the relevant references cited therein. Furthermore, difference equation appear naturally as a discrete analogues, and as a numerical solution of differential and delay differential equations having applications various scientific branches, such as in ecology, economy, physics, technics, sociology, and biology. For some papers on periodicity of difference equation see, for example, [8, 9, 11, 12, 15] and the relevant references cited therein.

In [22], the following open problem was posed.

Open Problem 1 (see [22, page 218, Open Problem 6.4]). Assume that $A \in (0, \infty)$, and that r_1 and r_2 are positive rational numbers. Investigate the periodic nature of the solution of the difference equation:

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n^2 x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the initial conditions are $x_{-1} = A^{r_1}$ and $x_0 = A^{r_2}$.

Now, in this paper we give answer to the open problem 6.4.

2. Main Results

2.1. The Case $A > 1$

We consider (1.1) where $A > 1$. It is clear that the change of variables

$$x_n = A^{r_n} \quad \text{for } n \geq -1 \tag{2.1}$$

reduces (1.1) to the difference equation:

$$r_{n+1} = \max\{r_n, 1\} - 2r_n - r_{n-1}, \quad n = 0, 1, 2, \ldots, \tag{2.2}$$

where the initial conditions r_{-1} and r_0 are positive rational numbers.

In this section we consider the behavior of the solutions of (2.2) (or equivalently of (1.1)) in this case $A > 1$. We give the following lemmas which give us explicit solutions for some consecutive terms and show us the behavior of the solutions of (2.2) (or equivalently of (1.1)).

Lemma 2.1. *Assume that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.2)*. If* $2 < r_0 + r_{-1} = r$*, then the following statements are true for some integer* N *such that* $N > 0$*.*

(i) *If* $r_N = -r$*, then* $r_{N-1} > 1$ *and* $r_{N+1} \geq 1$*.*

(ii) *If* $r_N = -r$*, then* $r_N = r_{N+3}$ *or* $r_N = r_{N+5}$*.*

(iii) *If* $r_N = -r$ *and* $r_{N+2} > 1$*, then* $r_{N+3} = -r$ *and* $r_{N+2} = r_{N-1} + r_N - 1$*.*

(iv) *If* $r_N = -r$ *and* $r_{N+2} \leq 1$*, then* $r_{N+5} = -r$ *and* $r_{N+4} = r_{N-1} - r_N - 2$*.*

Proof. (i) Assume that $2 < r_0 + r_{-1} = r$. If $r_0 > 1$, then from (2.2) we get $r_1 = -r$ and $r_2 = 1 + r_0 + 2r_{-1} > 1$.

If $r_0 \leq 1$, then we get $r_1 = 1 - 2r_0 - r_{-1} < 1$, $r_2 = 3r_0 + 2r_{-1} - 1 > 1$, $r_3 = -r$, $r_4 = 2 - r_0 \geq 1$ and $r_5 = 2r_0 + r_{-1} - 2$.

If $r_5 > 1$, then $r_6 = -r$ and $r_7 = r_{-1} + 3 > 1$.

If $r_5 \leq 1$, then $r_6 = 3 - 3r_0 - 2r_{-1} < 1$, $r_7 = 4r_0 + 3r_{-1} - 3 > 1$, $r_8 = -r$ and $r_9 = 4 - 2r_0 - r_{-1} \geq 1$.

Working inductively we have $r_{N-1} > 1$ and $r_{N+1} \geq 1$ for $r_N = -r$. So, the proof of (i) is complete.

(ii) Assume that $r_N = -r$. From (i) and (2.2) we get that

$$\begin{aligned} r_{N+1} &= 1 - 2r_N - r_{N-1} \geq 1, \\ r_{N+2} &= r_N + r_{N-1} - 1. \end{aligned} \tag{2.3}$$

If $r_{N+2} > 1$, then $r_{N+3} = r_N = -r$.

If $r_{N+2} \leq 1$, then $r_{N+3} = 2 - r_{N-1} < 1$, $r_{N+4} = r_{N-1} - r_N - 2 > 1$ and $r_{N+5} = r_N = -r$. So, the proof of (ii) is complete.

As for (iii) and (iv), they are immediately obtained from (ii) and (2.2). □

Lemma 2.2. *Assume that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.2). *If* $2 < r_0 + r_{-1} = k/m$, $GCD(k,m) = 1$, *then the following statements are true.*

(i) *If* $(8k-m) \neq 0(\bmod 3)$ *for* $0 < N < (8k-m+2)$, *then the number of integers* N *for which Lemma 2.1(iii) holds is* $(k-2m)$ *and the number of integers* N *for which Lemma 2.1(iv) holds is* $(k+m)$.

(ii) *If* $(8k-m) = 0(\bmod 3)$ *for* $0 < N < ((8k-m)/3+2)$, *then the number of integers* N *for which Lemma 2.1(iii) holds is* $((k-2m)/3)$ *and the number of integers* N *for which Lemma 2.1(iv) holds is* $((k+m)/3)$.

Proof. (i) Assume that $2 < r_0 + r_{-1} = r = k/m$, $\mathrm{GCD}(k,m) = 1$, $(8k-m) \neq 0(\bmod 3)$ and $0 < N < (8k-m+2)$.

Assume that number of integers N satisfying Lemma 2.1(iii) is $(k-2m+1)$.This assumption made Lemma 2.1(iii) be applied consecutively for $(k-2m+1)$ times such that;

$$\begin{gathered}
r_N = r_{N+3} = r_{N+6} = \cdots = r_{N+3(k-2m)} = r_{N+3(k-2m+1)} = -r,\\
r_{N+2} = r_{N-1} + r_N - 1 > 1,\\
r_{N+5} = r_{N-1} + 2(r_N - 1) > 1,\\
r_{N+8} = r_{N-1} + 3(r_N - 1) > 1,\\
\vdots\\
r_{N+3(k-2m)+2} = r_{N-1} + (k-2m+1)(r_N - 1) > 1.
\end{gathered} \tag{2.4}$$

Thus, $r_{N-1} > 1 + (k-2m+1)(-r_N+1) > -2r_N$. But, from Lemma 2.1(i) we have $r_{N+1} = 1 - 2r_N - r_{N-1} \geq 1$ and $(-2r_N) \geq r_{N-1}$. This means that Lemma 2.1(iii) cannot be applied consecutively for $(k-2m+1)$ times. So, the number of integers N satisfying Lemma 2.1(iii) is not more than $(k-2m)$.

Similarly, assume that the number of integers N satisfying Lemma 2.1(iv) is $(k+m+1)$. So, we can apply Lemma 2.1(iv) consecutively for $(k+m+1)$ times such that

$$\begin{gathered}
r_N = r_{N+5} = r_{N+10} = \cdots = r_{N+5(k+m)} = r_{N+5(k+m+1)} = -r,\\
r_{N+4} = r_{N-1} - r_N - 2 > 1,\\
r_{N+9} = r_{N-1} + 2(-r_N - 2) > 1,\\
r_{N+14} = r_{N-1} + 3(-r_N - 2) > 1,\\
\vdots\\
r_{N+5(k+m-1)+4} = r_{N-1} + (k+m)(-r_N - 2) > 1.
\end{gathered} \tag{2.5}$$

Thus, we have $r_{N+5(k+m)+2} = r_{N-1} + (k+m)(-r_N - 2) + r_N - 1 \leq 1$ and $r_{N-1} \leq 1 + (k+m)(r_N + 2) - r_N + 1 < 1$. But, it contradicts Lemma 2.1(i). So, the number of integers N satisfying Lemma 2.1(iv) is not more than $(k+m)$.

Now, assume that the number of integers N satisfying Lemma 2.1(iii) is $(k - 2m - 1)$. We have just had the number of integers N satisfying Lemma 2.1(iii) is less than $(k - 2m + 1)$. From the proof of Lemma 2.1(i), if j is the smallest integer N satisfying Lemma 2.1(i), then we have $r_j = -r$ for $j \in \{1, 3\}$. We apply Lemma 2.1(iii) for $(k - 2m - 1)$ times such that

$$\begin{aligned} r_j &= r_{j+3} = r_{j+6} = \cdots = r_{j+3(k-2m-2)} \\ &= r_{j+3(k-2m-1)} = -r \end{aligned} \tag{2.6}$$

and from Lemma 2.1(ii)

$$\begin{aligned} r_{j+3(k-2m-1)} &= r_{j+3(k-2m-1)+5} = \cdots = r_{j+3(k-2m-1)+5(k+m)} \\ &= r_{j+3(k-2m-1)+5(k+m+1)} = -r. \end{aligned} \tag{2.7}$$

Thus, the number of integers N satisfying Lemma 2.1(iv) is $(k + m + 1)$. But it is not possible. So, the number of integers N satisfying Lemma 2.1(iii) is $(k - 2m)$.

Similarly, assume that number of integers N satisfying Lemma 2.1(iv) is $(k + m - 1)$. From Lemma 2.1(ii)–(iv), we have

$$\begin{aligned} r_j &= r_{j+5} = r_{j+10} = \cdots = r_{j+5(k+m-2)} \\ &= r_{j+5(k+m-1)} = -r, \\ r_{j+5(k+m-1)} &= r_{j+5(k+m-1)+3} = \cdots = r_{j+5(k+m-1)+3(k-2m+1)} \\ &= r_{j+5(k+m-1)+3(k-2m+2)} = -r. \end{aligned} \tag{2.8}$$

Thus, the number of integers N satisfying Lemma 2.1(iii) is $(k - 2m + 2)$. But it is not possible. So, the number of integers N satisfying Lemma 2.1(iv) is $(k + m)$. So, the proof of (i) is completed.

(ii) Proof of (ii) is similar to the proof of (i). So, it is omitted. □

We omit the proofs of Lemmas 2.3 and 2.4 since they can easily be obtained in a way similar to the proofs of Lemmas 2.1 and 2.2.

Lemma 2.3. *Assume that $\{r_n\}_{n=-1}^{\infty}$ is a solution of (2.2). If $1 < r_0 + r_{-1} = r < 2$, then the following statements are true for some integer N such that $N > 0$.*

(i) *If $r_N = -r$, then $r_{N-1} > 1$ and $r_{N+1} \geq 1$.*

(ii) *If $r_N = -r$, then $r_N = r_{N+5}$ or $r_N = r_{N+7}$.*

(iii) *If $r_N = -r$ and $r_{N+4} > 1$, then $r_{N+5} = -r$ and $r_{N+4} = r_{N-1} - r_N - 2$.*

(iv) *If $r_N = -r$ and $r_{N+4} \leq 1$, then $r_{N+7} = -r$ and $r_{N+6} = r_{N-1} - 3r_N - 3$.*

Lemma 2.4. *Assume that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.2). *If* $1 < r_0 + r_{-1} = k/m < 2$, $GCD(k,m) = 1$, *then the following statements are true.*

(i) *If* $(8k - m) \neq 0 (\bmod 3)$ *for* $0 < N < (8k - m - 1)$, *then number of integers* N *for which Lemma 2.3(iii) holds is* $3(k - m)$ *and the number of integers* N *for which Lemma 2.3(iv) holds is* $(2m - k)$.

(ii) *If* $(8k - m) = 0 (\bmod 3)$ *for* $0 < N < ((8k - m)/3 - 1)$, *then the number of integers* N *for which Lemma 2.3(iii) holds is* $(k - m)$ *and the number of integers* N *for which Lemma 2.3(iv) holds is* $((2m - k)/3)$.

Lemma 2.5. *Assume that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.2). *If* $r_0 + r_{-1} < 1$ *and* $r_0 + r_{-1} \neq 1/2$, *then the following statements are true for some integer* N *such that* $N \geq -1$.

(i) *Assume that* $r_N > 1$, *and* $r_{N+2} \geq 1$. *If* $r_0 + r_{-1} < 1/2$, *then* $r_{N+1} = r_0 + r_{-1} - 1$ *or if* $r_0 + r_{-1} > 1/2$, *then* $r_{N+1} = -r_0 - r_{-1}$.

(ii) *If* $r_N > 1$ *and* $r_{N+2} \geq 1$, *then* $r_{N+3} < 1$, $r_{N+4} < 1$, $r_{N+5} < 1$, $r_{N+6} = 3 + 2r_{N+1} - r_N < 1$ *and* $r_{N+7} = r_N - 3(r_{N+1} + 1)$,

(iii) *If* $r_N \notin (r_{N_i}, r_{N_i+1}, r_{N_i+2}, r_{N_i+3}, r_{N_i+4}, r_{N_i+5}, r_{N_i+6})$ *such that* $r_{N_i} > 1$ *and* $r_{N_i+2} \geq 1$, *then* $r_N \leq 1$.

Proof. (i) Assume that $r_0 + r_{-1} < 1/2$. From (2.2), we get that $r_1 = 1 - 2r_0 - r_{-1} < 1$, $r_2 = 3r_0 + 2r_{-1} - 1 < 1$ and $r_3 = 2 - 4r_0 - 3r_{-1}$.

If $r_3 > 1$, then $r_4 = r_0 + r_{-1} - 1$, $r_5 = 1 + 2r_0 + r_{-1} > 1$.

If $r_3 \leq 1$, then $r_4 = 5r_0 + 4r_{-1} - 2 < 1$, $r_5 = 3 - 6r_0 - 5r_{-1}$ and $r_5 > 1$ or $r_5 \leq 1$.

If $r_5 > 1$, then $r_6 = r_0 + r_{-1} - 1$ and $r_7 = 4r_0 + 3r_{-1} \geq 1$.

If $r_5 \leq 1$, then $r_6 = 7r_0 + 6r_{-1} - 3 < 1$, $r_7 = 4 - 8r_0 - 7r_{-1}$ and $r_7 > 1$ or $r_7 \leq 1$.

Working inductively, we have $r_{N+1} = r_0 + r_{-1} - 1$ for $r_0 + r_{-1} < 1/2$, $r_N > 1$ and $r_{N+2} \geq 1$.

Assume that $r_0 + r_{-1} > 1/2$. From (2.2), we get that $r_1 = 1 - 2r_0 - r_{-1} < 1$ and $r_2 = 3r_0 + 2r_{-1} - 1$.

If $r_2 > 1$, then $r_3 = -r_0 - r_{-1}$ and $r_4 = 2 - r_0 > 1$.

If $r_2 \leq 1$, then $r_3 = 2 - 4r_0 - 3r_{-1} < 1$ and $r_4 = 5r_0 + 4r_{-1} - 2$.

If $r_4 > 1$, then $r_5 = -r_0 - r_{-1}$ and $r_6 = 3 - 3r_0 - 2r_{-1} > 1$.

Working inductively, we have $r_{N+1} = -r_0 - r_{-1}$ for $1/2 < r_0 + r_{-1} < 1$, $r_N > 1$ and $r_{N+2} \geq 1$. So, the proof of (i) is completed.

(ii) Assume that $r_0 + r_{-1} < 1$ and that $r_N > 1$, $r_{N+2} \geq 1$. From (2.2) and (i), we get that

$$r_{N+2} = \max\{r_{N+1}, 1\} - 2r_{N+1} - r_N = 1 - 2r_{N+1} - r_N, \quad r_N \leq -2r_{N+1}. \tag{2.9}$$

Then,

$$\begin{aligned}
r_{N+3} &= \max\{r_{N+2}, 1\} - 2r_{N+2} - r_{N+1} = r_N + r_{N+1} - 1 < 1, \\
r_{N+4} &= \max\{r_{N+3}, 1\} - 2r_{N+3} - r_{N+2} = 2 - r_N < 1, \\
r_{N+5} &= \max\{r_{N+4}, 1\} - 2r_{N+4} - r_{N+3} = r_N - r_{N+1} - 2 < 1, \\
r_{N+6} &= \max\{r_{N+5}, 1\} - 2r_{N+5} - r_{N+4} = -r_N + 2r_{N+1} + 3 < 1, \\
r_{N+7} &= \max\{r_{N+6}, 1\} - 2r_{N+6} - r_{N+5} = r_N - 3(r_{N+1} + 1).
\end{aligned} \tag{2.10}$$

So, the proof of (ii) is completed.

(iii) Assume that the smallest integer of integers N satisfying (i) is N_1. From the proof of (i), we have $r_j < 1$ for $j < N_1$. Also, from this assumption we have the subsequence $(r_{N_1}, r_{N_1+1}, r_{N_1+2}, r_{N_1+3}, r_{N_1+4}, r_{N_1+5}, r_{N_1+6})$ such that $r_{N_1} > 1$ and $r_{N_1+2} \geq 1$. Then, from (ii) we get that $r_{N_1+7} = r_{N_1} - 3(r_{N_1+1} + 1)$ and $r_{N_1+7} \leq 1$ or $r_{N_1+7} > 1$. If $r_{N_1+7} \leq 1$, we get that $r_{N_1+8} = -r_{N_1} + 4(r_{N_1+1} + 1) < 1$. It means that r_{N_1+7} and r_{N_1+8} are not the element previous and later subsequences satisfying (ii). If $r_{N_1+7} > 1$, then we get that $r_{N_1+9} \geq 1$ and that r_{N_1+7} is a element of the subsequence $(r_{N_1+7}, r_{N_1+8}, r_{N_1+9}, r_{N_1+10}, r_{N_1+11}, r_{N_1+12}, r_{N_1+13})$ such that $r_{N_1+7} > 1$ and $r_{N_1+9} \geq 1$. If this proceeds, we have $r_N \leq 1$, for $r_N \notin (r_{N_i}, r_{N_i+1}, r_{N_i+2}, r_{N_i+3}, r_{N_i+4}, r_{N_i+5}, r_{N_i+6})$ such that $r_{N_i} > 1$ and $r_{N_i+2} \geq 1$. □

Lemma 2.6. *Assume that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.2). *If* $r_0 + r_{-1} = k/m < 1/2$, $GCD(k, m) = 1$, *then the following statements are true.*

(i) *If* $(7m - 8k) \neq 0(\bmod 3)$ *for* $N < (7m - 8k - 1)$, *then number of integers N for which Lemma 2.5(ii) holds is* $(m - 2k)$ *and the number of integers N for which Lemma 2.5(iii) holds is* $6k$.

(ii) *If* $(7m-8k) = 0(\bmod 3)$ *for* $N < ((7m-8k)/3-1)$, *then number of integers N for which Lemma 2.5(ii) holds is* $((m-2k)/3)$ *and the number of integers N for which Lemma 2.5(iii) holds is* $2k$.

Lemma 2.7. *Assume that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.2). *If* $1/2 < r_0 + r_{-1} = k/m < 1$, $GCD(k, m) = 1$, *then the following statements are true.*

(i) *If* $(8k - m) \neq 0(\bmod 3)$ *for* $N < (8k - m - 1)$, *then number of integers N for which Lemma 2.5(ii) holds is* $(2k - m)$ *and the number of integers N for which Lemma 2.5(iii) holds is* $6(m - k)$.

(ii) *If* $(8k - m) = 0(\bmod 3)$ *for* $N < ((8k - m)/3 - 1)$, *then number of integers N for which Lemma 2.5(ii) holds is* $((2k-m)/3)$ *and the number of integers N for which Lemma 2.5(iii) holds is* $2(m - k)$.

Theorem 2.8. *Consider* (2.2). *If* $r_{-1} + r_0 = k/m \leq 1/2$ *and* $GCD(k, m) = 1$, *then the following statements are true.*

(i) *If* $(7m - 8k) \neq 0(\bmod 3)$, *then* $\{r_n\}_{n=-1}^{\infty}$ *is periodic with prime period* $(7m - 8k)$,

(ii) *If* $(7m - 8k) = 0(\bmod 3)$, *then* $\{r_n\}_{n=-1}^{\infty}$ *is periodic with prime period* $((7m - 8k)/3)$.

Proof. (i) Assume that $r_{-1}+r_0=k/m<1/2$, $\mathrm{GCD}(k,m)=1$ and $(7m-8k)\neq 0(\bmod 3)$. It suffices to prove $r_n=r_{n+7m-8k}$ for $n\geq -1$. We must show that $r_{-1}=r_{7m-8k-1}$ and $r_0=r_{7m-8k}$. From Lemmas 2.5 and 2.6, for getting the last two terms we need, we can assume that

$$r_n\leq 1 \quad \text{for } n=-1,0,\dots,(6k-2) \tag{2.11}$$

and that

$$r_{6k-1+7l}>1, \quad r_{6k+1+7l}\geq 1 \quad \text{for } l=0,1,\dots,(m-2k-1). \tag{2.12}$$

From this assumption and (2.2), we get that for $n=1,2,\dots,(3k-1)$

$$\begin{aligned} r_{2n-1}&=r_{-1}+\sum_{i=1}^{n}[1-2(r_{-1}+r_0)]\leq 1\\ r_{2n}&=r_0+\sum_{i=1}^{n}[2(r_{-1}+r_0)-1]\leq 1 \end{aligned} \tag{2.13}$$

and that

$$\begin{aligned} r_{6k-1}&=r_{-1}+\sum_{i=1}^{3k}[1-2(r_{-1}+r_0)]>1,\\ r_{6k}&=(r_{-1}+r_0)-1,\\ r_{6k+1}&=2-r_{-1}+\sum_{i=1}^{3k-1}[2(r_{-1}+r_0)-1]\geq 1,\\ &\vdots\\ r_{6k+5}&=3r_{-1}+4r_0+\sum_{i=1}^{3k-1}[2(r_{-1}+r_0)-1]\leq 1,\\ r_{6k+6}&=1-(4r_{-1}+5r_0)-\sum_{i=1}^{3k-1}[2(r_{-1}+r_0)-1]>1,\\ r_{6k+7}&=(r_{-1}+r_0)-1,\\ &\vdots\\ r_{6k+12}&=6r_{-1}+7r_0+\sum_{i=1}^{3k-1}[2(r_{-1}+r_0)-1]\leq 1,\\ r_{6k+13}&=1-(7r_{-1}+8r_0)-\sum_{i=1}^{3k-1}[2(r_{-1}+r_0)-1]>1,\\ &\vdots \end{aligned}$$

$$
\begin{aligned}
r_{6k-1+7(m-2k-1)} &= 1 - r_{-1} - 2r_0 - 3(m-2k-1)(r_{-1}+r_0) - \sum_{i=1}^{3k-1}[2(r_{-1}+r_0)-1] > 1,\\
&\ \vdots\\
r_{6k-1+7(m-2k-1)+6} &= r_0 + 3(m-2k)(r_{-1}+r_0) + \sum_{i=1}^{3k-1}[2(r_{-1}+r_0)-1] \le 1,\\
r_{6k-1+7(m-2k)} &= 1 - [-r_{-1} + (3m-6k+2)(r_{-1}+r_0)] - \sum_{i=1}^{3k-1}[2(r_{-1}+r_0)-1].
\end{aligned}
\tag{2.14}
$$

Then we get $r_{7m-8k-1} = r_{-1}$ and that

$$
\begin{aligned}
r_{7m-8k} &= 1 - 2r_{7m-8k-1} - r_{7m-8k-2}\\
&= 1 - 2r_{-1} - r_{7m-8k-2} - \left[r_0 + 3(m-2k)(r_{-1}+r_0) + \sum_{i=1}^{3k-1}[2(r_{-1}+r_0)-1]\right] = r_0.
\end{aligned}
\tag{2.15}
$$

So, we have $r_n = r_{n+7m-8k}$ for $n \ge -1$.

(ii) Assume that $r_{-1} + r_0 = 1/2$. From (2.2), we get immediately $r_1 = r_{-1}$ and $r_2 = r_0$. So, we have $r_n = r_{n+(7m-8k)/3}$ for $n \ge -1$. If $r_{-1} + r_0 < 1/2$, $\mathrm{GCD}(k, m) = 1$ and $(7m - 8k) = 0 (\mathrm{mod}\, 3)$, then the proof of (ii) is similar to the proof of (i). So, it is omitted.

□

Theorem 2.9. *Consider* (2.2). *If* $r = r_{-1} + r_0 = k/m > 1/2$ *and* $GCD(k, m) = 1$, *then the following statements are true.*

(i) *If* $(8k - m) \ne 0(\mathrm{mod}\, 3)$, *then* $\{r_n\}_{n=-1}^{\infty}$ *is periodic with prime period* $(8k - m)$.

(ii) *If* $(8k - m) = 0(\mathrm{mod}\, 3)$, *then* $\{r_n\}_{n=-1}^{\infty}$ *is periodic with prime period* $((8k - m)/3)$.

Proof. (i) If $r_{-1}+r_0 = 1$, then from (2.2) we get that $r_1 = -r_0, r_2 = 1+r_0,\ r_3 = -1,\ r_4 = 2-r_0,\ r_5 = r_0 - 1,\ r_6 = r_{-1}$ and $r_7 = r_0$. So, we have $r_n = r_{n+8k-m}$ for $n \ge -1$.

In the case $1/2 < r_{-1} + r_0 < 1$, the proof is similar to the proof of Theorem 2.8(i) such that; from Lemmas 2.5 and 2.7, we can assume that

$$
r_n \le 1 \quad \text{for } n = -1, 0, \dots, [6(m-k) - 3] \tag{2.16}
$$

and that

$$
r_{6(m-k)-2+7l} > 1, \qquad r_{6(m-k)+7l} \ge 1, \qquad r_{6(m-k)-2+7(2k-m)} \le 1 \tag{2.17}
$$

for $l = 0, 1, \ldots, (2k - m - 1)$. Thus, we get that

$$\begin{aligned} r_{2n} &= r_0 + \sum_{i=1}^{n} [2(r_{-1} + r_0) - 1] \leq 1, \\ r_{2n+1} &= r_{-1} + \sum_{i=1}^{n+1} [1 - 2(r_{-1} + r_0)] \leq 1 \end{aligned} \tag{2.18}$$

for $n = 1, 2, \ldots, [3(m - k) - 2]$ and that

$$\begin{aligned} r_{6(m-k)-2} &= r_0 + \sum_{i=1}^{3(m-k)-1} [2(r_{-1} + r_0) - 1] > 1, \\ r_{6(m-k)-1} &= -(r_{-1} + r_0), \\ r_{6(m-k)} &= 1 + 2r_{-1} + r_0 + \sum_{i=1}^{3(m-k)-1} [1 - 2(r_{-1} + r_0)] \geq 1, \\ &\vdots \\ r_{6(m-k)-2+7(2k-m)-1} &= r_{-1} + 3(2k - m)(1 - r_{-1} - r_0) - \sum_{i=1}^{3(m-k)-1} [2(r_{-1} + r_0) - 1] \leq 1, \\ r_{6(m-k)-2+7(2k-m)} &= r_0 + 3(2k - m)(r_{-1} + r_0 - 1) + \sum_{i=1}^{3(m-k)-1} [2(r_{-1} + r_0) - 1] \leq 1, \\ r_{6(m-k)+7(2k-m)-1} &= 1 - 2r_{6(m-k)+7(2k-m)-2} - r_{6(m-k)+7(2k-m)-3} = r_{-1}, \\ r_{6(m-k)+7(2k-m)} &= 1 - 2r_{6(m-k)+7(2k-m)-1} - r_{6(m-k)+7(2k-m)-2} = r_0 \end{aligned} \tag{2.19}$$

So, we have $r_n = r_{n+8k-m}$ for $n \geq -1$.

Assume that $1 < r_{-1} + r_0 < 2$. From Lemmas 2.3 and 2.4, for getting the last two terms, we can assume that

$$\begin{aligned} r_{N+5l_1} &= r_{N+5(l_1+1)} = -r, \\ r_{N+5[3(k-m)]+7l_2} &= r_{N+5[3(k-m)]+7(l_2+1)} = -r \end{aligned} \tag{2.20}$$

for $l_1 = 0, 1, \ldots, [3(k-m)-1]$ and $l_2 = 0, 1, \ldots, (2m-k-1)$. From this assumption and Lemma 2.4(iii)-(iv), we get that

$$\begin{aligned} r_{N+4} &= r_{N-1} - r_N - 2, \\ r_{N+9} &= r_{N-1} + 2(-r_N - 2), \\ &\vdots \\ r_{N+5[3(k-m)-1]+4} &= r_{N-1} + 3(k-m)(-r_N - 2), \end{aligned} \tag{2.21}$$

and that

$$\begin{aligned} r_{N+5[3(k-m)]+6} &= r_{N+5[3(k-m)-1]+4} - 3r_N - 3, \\ r_{N+5[3(k-m)]+13} &= r_{N+5[3(k-m)-1]+4} + 2(-3r_N - 3), \\ &\vdots \\ r_{N+5[3(k-m)]+7(2m-k-1)+6} &= r_{N+5[3(k-m)-1]+4} + (2m-k)(-3r_N - 3) > 1 \end{aligned} \tag{2.22}$$

Thus, we get $r_{N+8k-m-1} = r_{N-1}$ and $r_{N+8k-m} = r_N$. From $r_{n-1} = \max\{r_n, 1\} - 2r_n - r_{n+1}$, we have $r_n = r_{n+8k-m}$ for $n \geq -1$. Also, it is easy to see that $r_{N-1} \neq r_{N+5l_1-1}$ and $r_{N-1} \neq r_{N+5[3(k-m)]+7l_2-1}$ for $l_1 = 1, 2, \ldots, [3(k-m)]$ and $l_2 = 1, 2, \ldots, (2m-k)$ which imply that $8k-m$ is the smallest period.

In the case $2 < r_{-1} + r_0$, the proof is similar. So, it is omitted.

(ii) Assume that $r_{-1} + r_0 = 2$. If $r_0 > 1$, then from (2.2) we get that $r_1 = -2$, $r_2 = 5 - r_0$, $r_3 = r_0 - 3$, $r_4 = r_{-1}$ and $r_5 = r_0$. If $r_0 \leq 1$, then $r_1 = -1 - r_0$, $r_2 = 3 + r_0$, $r_3 = -2$, $r_4 = r_{-1}$ and $r_5 = r_0$. So, we have $r_n = r_{n+(8k-m)/3}$ for $n \geq -1$. The rest of proof is similar to the proof of (i) and is omitted. □

2.2. *The Case* $A < 1$

We consider (1.1) where $A < 1$. It is clear that the change of variables

$$x_n = A^{r_n} \quad \text{for } n \geq -1 \tag{2.23}$$

reduces (1.1) to the difference equation:

$$r_{n+1} = \min\{r_n, 1\} - 2r_n - r_{n-1}, \quad n = 0, 1, 2, \ldots, \tag{2.24}$$

where the initial conditions r_{-1} and r_0 are positive rational numbers.

In this section we consider the behavior of the solutions of (2.24) (or equivalently of (1.1)) in this case $A < 1$. We omit the proofs of the following results since they can easily be obtained in a way similar to the proofs of the lemmas and theorems in the previous section.

Lemma 2.10. *Assume that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.24)*. If* $r_{-1}, r_0 > 1$ *and* $r = r_0 + r_{-1}$*, then the following statements are true for some integer* $N > 0$*.*

(i) *If* $r_N = r - 1$*, then* $r_{N-1} \leq 1$ *and* $r_{N+1} \leq 1$*.*

(ii) *If* $r_N = r - 1$*, then* $r_N = r_{N+3}$ *or* $r_N = r_{N+5}$*.*

(iii) *If* $r_N = r - 1$ *and* $r_{N+2} < 1$*, then* $r_{N+3} = r - 1$ *and* $r_{N+2} = r_{N-1} + r_N - 1$*.*

(iv) *If* $r_N = r - 1$ *and* $r_{N+2} \geq 1$*, then* $r_{N+5} = r - 1$ *and* $r_{N+4} = r_{N-1} - r_N - 2$*.*

Lemma 2.11. *Assume that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.24) *for which Lemma 2.10 holds. If* $r = r_0 + r_{-1} = k/m$*, GCD*$(k, m) = 1$*, then the following statements are true.*

(i) *If* $(8k - 7m) \neq 0(\bmod 3)$ *for* $0 < N < (8k - 7m - 1)$*, then number of integers* N *for which Lemma 2.10(iii) holds is* $(k + m)$ *and the number of integers* N *for which Lemma 2.10(iv) holds is* $(k - 2m)$*.*

(ii) *If* $(8k - 7m) = 0(\bmod 3)$ *for* $0 < N < ((8k - 7m)/3 - 1)$*, then number of integers* N *for which Lemma 2.10(iii) holds is* $((k + m)/3)$ *and the number of integers* N *for which Lemma 2.10(iv) holds is* $((k - 2m)/3)$*.*

Lemma 2.12. *Assume that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.24)*. If* $\max\{r_{-1}, r_0\} = r > 1$ *and one of the initial contitions* $r_{-1},\ r_0$ *is less than or equal to one, then the following statements are true for some integer* $N > 0$*.*

(i) *If* $r_N = r$*, then* $r_{N-1} \leq 1$ *and* $r_{N+1} \leq 1$*.*

(ii) *If* $r_N = r$*, then* $r_N = r_{N+3}$ *or* $r_N = r_{N+5}$*.*

(iii) *If* $r_N = r$ *and* $r_{N+2} < 1$*, then* $r_{N+3} = r$ *and* $r_{N+2} = r_{N-1} + r_N - 1$*.*

(iv) *If* $r_N = r$ *and* $r_{N+2} \geq 1$*, then* $r_{N+5} = r$ *and* $r_{N+4} = r_{N-1} - r_N - 2$*.*

Lemma 2.13. *Assume that* $\{r_n\}_{n=-1}^{\infty}$ *is a solution of* (2.24) *for which Lemma 2.12 holds. If* $r = \max\{r_{-1}, r_0\} = k/m$*, GCD*$(k, m) = 1$*, then the following statements are true.*

(i) *If* $(8k + m) \neq 0(\bmod 3)$ *for* $0 < N < (8k + m - 1)$*, then number of integers* N *for which Lemma 2.12(iii) holds is* $(2m + k)$ *and the number of integers* N *for which Lemma 2.12(iv) holds is* $(k - m)$*.*

(ii) *If* $(8k + m) = 0(\bmod 3)$ *for* $0 < N < ((8k + m)/3 - 1)$*, then number of integers* N *for which Lemma 2.12(iii) holds is* $((2m + k)/3)$ *and the number of integers* N *for which Lemma 2.12(iv) holds is* $((k - m)/3)$*.*

Theorem 2.14. *Consider* (2.24)*. If* $r_{-1}, r_0 > 1,\ r = (r_0 + r_{-1}) = k/m$ *and GCD*$(k, m) = 1$*, then the following statements are true.*

(i) *If* $(8k - 7m) \neq 0(\bmod 3)$*, then* $\{r_n\}_{n=-1}^{\infty}$ *is periodic with prime period* $(8k - 7m)$*.*

(ii) *If* $(8k - 7m) = 0(\bmod 3)$*, then* $\{r_n\}_{n=-1}^{\infty}$ *is periodic with prime period* $((8k - 7m)/3)$*.*

Theorem 2.15. *Consider* (2.24)*. If at least one of the initial conditions of* (2.24) *is less than or equal to one,* $\max\{r_{-1}, r_0, 1\} = k/m$ *and GCD*$(k, m) = 1$*, then the following statements are true.*

(i) *If* $(8k + m) \neq 0(\bmod 3)$*, then* $\{r_n\}_{n=-1}^{\infty}$ *is periodic with prime period* $(8k + m)$*.*

(ii) *If* $(8k + m) = 0(\bmod 3)$*, then* $\{r_n\}_{n=-1}^{\infty}$ *is periodic with prime period* $((8k + m)/3)$*.*

3. Conclusion

Difference equation appears naturally as a discrete analogue and as a numerical solutions of differential and delay differential equations having applications various scientific branches, such as in ecology, economy, physics, technics, sociology, and biology. Specially, max-type difference equations stem from, for example, certain models in automatic control theory. In this paper, periodic nature of the solution of (1.1) which was open problem proposed by Grove and Ladas [22] was investigated. We describe a new method in investigating periodic character of max-type difference equations. It is expected that after some modifications our method will be applicable to Open Problem 6.3 in [22].

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