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Research Article

Stability and Stabilization of Impulsive Stochastic Delay Difference Equations

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When an impulsive control is adopted for a stochastic delay difference system (SDDS), there are at least two situations that should be contemplated. If the SDDS is stable, then what kind of impulse can the original system tolerate to keep stable? If the SDDS is unstable, then what kind of impulsive strategy should be taken to make the system stable? Using the Lyapunov-Razumikhin technique, we establish criteria for the stability of impulsive stochastic delay difference equations and these criteria answer those questions. As for applications, we consider a kind of impulsive stochastic delay difference equation and present some corollaries to our main results.

1. Introduction

In recent years, stochastic delay difference equations (SDDEs) have been studied by many researchers; a number of results have been reported [1–7]. In these literatures, stability analysis stays on the focus of attention; see [1, 2, 4–6] and the references therein. As we all know, when we adopt an impulsive strategy to an SDDE, the stability of the SDDE may be destroyed or strengthen. Impulsive phenomena exist widely in the real world; therefore, it is important to study the stability problem for SDDEs with impulsive effects [8–10], that is to say, the stability problem for impulsive stochastic delay difference equations (ISDDEs).

For SDDEs, when we take impulsive effects into account, we have at least two problems to deal with. *Problem* 1. When a SDDE is stable, what kind of impulsive effect can the system tolerate so that it remain stable? *Problem* 2. If the SDDE is unstable, then what kind of impulsive effect should be taken to make the system stable? Problems 1 and 2 are called the problem of impulsive stability and the problem of impulsive stabilization, respectively.

As well known, Lyapunov-Razumikhin technique is one of main methods to investigate the stability of delay systems [11, 12]. There are little papers on the stability of ISDDEs [13, 14], and up to our knowledge, there is no paper on the stability of ISDDEs using Lyapunov-Razumikhin technique. In this paper, we study the stability of ISDDEs by Lyapunov-Razumikhin technique. We establish criteria for the r-moment exponential stability; these criteria present the answers to Problems 1 and 2. As for applications, we consider a kind of ISDDE and present some corollaries to our main theorems.

2. Preliminaries

In the sequel, $\mathbb{R}$ denotes the field of real numbers, and $\mathbb{N}$ represents the natural numbers. For some positive integer m and n_0, let $N_{-m} = \{-m, -m+1, \ldots, -1, 0\}$ and $N_{n_0-m} = \{n_0-m, n_0-m+1, \ldots, n_0-1, n_0\}$. Given a matrix A, $\|A\|$ denotes the norm of A induced by the Euclidean vector norm. Let $C([-r,0],\mathbb{R}^n) = \{\psi : [-r,0] \to \mathbb{R}^n, \psi \text{ is continuous}\}$. Given a positive integer m, we define $\|\varphi\|_m = \max_{\theta\in N_{-m}}\{\|\varphi(s)\|\}$ for any $\varphi \in C([-m,0],\mathbb{R}^n)$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $\{\mathcal{F}_n,\ n \in \mathbb{Z}\}$ be a nondecreasing family of sub-σ-algebra of $\mathcal{F}$, that is, $\mathcal{F}_{n_1} \subset \mathcal{F}_{n_2}$ for $n_1 < n_2$.

Consider the impulsive stochastic delay difference equations of the form

$$\begin{gathered} x(n+1) = f(n,x_n) + g(n,x_n)\xi_n, \quad n \neq \eta_k - 1,\ n \geqslant n_0,\ n,k \in \mathbb{N}, \\ x(\eta_k) = H_k(x(\eta_k - 1)), \quad k \in \mathbb{N}, \\ x_{n_0} = \varphi, \end{gathered} \tag{2.1}$$

where $n_0 \in \mathbb{N}$, $f,g \in C(\mathbb{N} \times C((-m,0),\mathbb{R}^n),\mathbb{R}^n)$, and $m \in \mathbb{N}$ represents the delay in system (2.1), $m \geqslant 2$. $x_n \in C([-m,0],\mathbb{R}^n)$ is defined by $x_n(s) = x(n+s)$ for any $s \in [-m,0]$. $\{\xi_n\}$ are $\mathcal{F}_{n+1}$-adapted mutually independent random variables and satisfy $E\xi_n = 0$, $E\xi_n^2 = 1$, where E denotes the mathematical expectation. $H_k \in C(\mathbb{R}^n,\mathbb{R}^n)$. Impulsive moment $\eta_k \in \mathbb{N}$ satisfies: $n_0 < \eta_1 < \eta_2 < \cdots < \eta_n < \cdots$, and $\eta_k \to \infty$ as $k \to \infty$. Let $\eta_0 = n_0$.

Assume that $f(n,0) \equiv 0$, $g(n,0) \equiv 0$, and $H_k(0) = 0$, then system (2.1) admits the trivial solution. We also assume there exists a unique solution of system (2.1), denoted by $x(n) = x(n,n_0,\varphi)$, for any given initial data $x_{n_0} = \varphi$.

Definition 2.1. One calls the trivial solution of system (2.1) r-moment exponentially stable if for any initial data $x_{n_0} = \varphi$ there exist two positive constants α and M, such that for all $n \geqslant n_0$, $n \in \mathbb{N}$, the following inequality holds:

$$E\|x(n)\|^r \leqslant M\|\varphi\|_m^r e^{-\alpha n}. \tag{2.2}$$

If the trivial solution of system (2.1) is r-moment exponentially stable, then we also call the system (2.1) r-moment exponentially stable.

3. Main Results

In this section, we will establish two theorems on r-moment exponential stability of system (2.1); these theorems give the answers to Problems 1 and 2.

First, we present the theorem on impulsive stability. The technique adopted in the proof is motivated by [15].

Theorem 3.1. *Assume that there exist a positive function $V(n,x)$ for system* (2.1) *and positive constants r, p, c_1, c_2, and λ, where $p > 1$, $0 < \lambda < 1$, such that.*

(C_1) $c_1\|x\|^r \leqslant V(n,x) \leqslant c_2\|x\|^r$ *for any* $n \in N_{n_0-m} \cup \mathbb{N}$ *and* $x \in \mathbb{R}^n$.

(C_2) *For* $n \neq \eta_k - 1$, *any* $s \in N_{-m}$, $EV(n+1, x(n+1)) \leqslant \lambda EV(n, x(n))$ *whenever* $EV(n+s, x(n+s)) \leqslant pEV(n, x(n))$.

(C_3) *For* $n \neq \eta_k - 1$, *some* $s \in N_{-m} - \{0\}$, $EV(n+1, x(n+1) \leqslant (1/p)\max_{\theta\in N_{-m}}\{EV(n+\theta, x(n+\theta))\}$ *whenever* $EV(n+s, x(n+s)) > e^{\alpha}EV(n, x(n))$, *where* $\alpha = \min\{-\ln\lambda, \ln p/(m+1)\}$.

(C_4) $EV(\eta_k, x(\eta_k)) \leqslant d_k EV(\eta_k - 1, x(\eta_k - 1))$, *where* $d_k > 1$ *and* $d = \max_{k\in\mathbf{N}}\{d_k\} < \infty$.

(C_5) $\eta_{k+1} - \eta_k > m$, $\alpha(1 - 1/m) - \ln d/m = \beta > 0$.

Then for any initial data $x_{n_0} = \varphi$,

$$E\|x(n)\|^r \leqslant \frac{c_2}{c_1}E\|\varphi\|_m^r e^{-\beta n}. \tag{3.1}$$

That is the trivial solution of system (2.1) *that is r-moment exponentially stable.*

Proof. Let $U(n) = \max_{\theta\in N_{-m}}\{e^{\alpha(n+\theta)}EV(n+\theta, x(n+\theta))\}$. For any $n \geqslant n_0$, $n \in [\eta_k, \eta_{k+1} - 1)$, $k \in \mathbb{N}$, define

$$\overline{\theta}_n = \max\left\{\theta \in N_{-m} : e^{\alpha(n+\theta)}EV(n+\theta, x(n+\theta)) = U(n)\right\}, \tag{3.2}$$

then $U(n) = e^{\alpha(n+\overline{\theta}_n)}EV(n+\overline{\theta}_n, x(n+\overline{\theta}_n))$.

Next, we will show that, for any $n \in [\eta_k, \eta_{k+1} - 1)$,

$$U(n+1) \leqslant U(n). \tag{3.3}$$

For a given n, we have two situations to contemplate: $\overline{\theta}_n \leqslant -1$ and $\overline{\theta}_n = 0$.

Case 1 ($\overline{\theta}_n \leqslant -1$). Under this situation, we have $e^{\alpha n}EV(n, x(n)) < e^{\alpha(n+\overline{\theta}_n)}EV(n+\overline{\theta}_n, x(n+\overline{\theta}_n))$, then

$$EV\left(n+\overline{\theta}_n, x\left(n+\overline{\theta}_n\right)\right) > e^{\alpha(-\overline{\theta}_n)}EV(n, x(n)) \geqslant e^{\alpha}EV(n, x(n)). \tag{3.4}$$

Using condition (C_3) and noticing $p \geqslant e^{\alpha(m+1)}$, we obtain

$$\max_{s\in N_{-m}}\{EV(n+s, x(n+s))\} \geqslant e^{\alpha(m+1)}EV(n+1, x(n+1)). \tag{3.5}$$

Multiplying both sides by $e^{\alpha n}$ and rearranging yield

$$e^{\alpha(n-m)}\max_{s\in N_{-m}}\{EV(n+s, x(n+s))\} \geqslant e^{\alpha(n+1)}EV(n+1, x(n+1)). \tag{3.6}$$

Then we get

$$e^{\alpha(n+1)}EV(n+1,x(n+1)) \leqslant \max_{s\in N_{-m}}\left\{e^{\alpha(n+s)}EV(n+s,x(n+s))\right\} = U(n), \tag{3.7}$$

which implies that

$$U(n+1) \leqslant U(n). \tag{3.8}$$

Case 2 ($\overline{\theta}_n = 0$). Making use of the definition of $U(n)$ and $\overline{\theta}_n$, noticing that $p > e^{-\alpha\theta}$ for any $\theta \in N_{-m}$, we have

$$EV(n+\theta,x(n+\theta)) \leqslant e^{\alpha(-\theta)}EV(n,x(n)) < pEV(n,x(n)). \tag{3.9}$$

Under condition (C_2), the above inequality implies that

$$EV(n+1,x(n+1)) \leqslant \lambda EV(n,x(n)). \tag{3.10}$$

Multiplying both sides by $e^{\alpha(n+1)}$, we have

$$\begin{aligned} e^{\alpha(n+1)}EV(n+1,x(n+1)) &\leqslant e^{\alpha(n+1)}\lambda EV(n,x(n)) \\ &= e^{\alpha n}EV(n,x(n))e^{\alpha}\lambda \\ &\leqslant e^{\alpha n}EV(n,x(n)) = U(n). \end{aligned} \tag{3.11}$$

Thus

$$U(n+1) \leqslant U(n), \tag{3.12}$$

which is the desired assertion.

When $n = \eta_{k+1}$, under condition (C_4) and using the definition of $U(n)$, we get

$$\begin{aligned} U(\eta_{k+1}) &= \max_{\theta\in N_{-m}}\left\{e^{\alpha(\eta_{k+1}+\theta)}EV\left(\eta_{k+1}+\theta,x\left(\eta_{k+1}+\theta\right)\right)\right\} \\ &= \max\Bigg\{e^{\alpha\eta_{k+1}}EV\left(\eta_{k+1},x\left(\eta_{k+1}\right)\right), \\ &\qquad \max_{\theta\in N_{-m}-\{0\}}\left\{e^{\alpha(\eta_{k+1}+\theta)}EV\left(\eta_{k+1}+\theta,x\left(\eta_{k+1}+\theta\right)\right)\right\}\Bigg\} \end{aligned}$$

$$\begin{aligned}
&\leqslant \max\Big\{ d_{k+1}e^{\alpha}e^{\alpha(\eta_{k+1}-1)}EV(\eta_{k+1}-1,x(\eta_{k+1}-1)),\\
&\qquad \max_{\theta\in N_{-m}-\{0\}}\Big\{e^{\alpha(\eta_{k+1}+\theta)}EV(\eta_{k+1}+\theta,x(\eta_{k+1}+\theta))\Big\}\Big\}\\
&\leqslant d_{k+1}e^{\alpha}\max_{\theta\in N_{-m}-\{0\}}\Big\{e^{\alpha(\eta_{k+1}+\theta)}EV(\eta_{k+1}+\theta,x(\eta_{k+1}+\theta))\Big\}\\
&\leqslant d_{k+1}e^{\alpha}U(\eta_{k+1}-1)\leqslant d_{k+1}e^{\alpha}U(\eta_k).
\end{aligned} \tag{3.13}$$

By induction and taking (3.3) into account, when $n\in[\eta_l,\eta_{l+1})$, for all $l\in\mathbb{N}$, we have

$$U(n)\leqslant U(\eta_l)\leqslant d_l e^{\alpha}U(\eta_{l-1})\leqslant \prod_{i=1}^{l}(d_i e^{\alpha})U(n_0), \tag{3.14}$$

which yields

$$e^{\alpha n}EV(n,x(n))\leqslant \prod_{i=1}^{l}(d_i e^{\alpha})c_2E\|\varphi\|_m^r. \tag{3.15}$$

By virtue of condition (C_5),

$$EV(n,x(n))\leqslant e^{-n(\alpha(1-l/n)(-1/n)\sum_{i=1}^{l}\ln d_i)}c_2E\|\varphi\|_m^r\leqslant c_2E\|\varphi\|_m^r e^{-\beta n}. \tag{3.16}$$

The desired result follows when we take condition (C_1) into account. □

Now, we are in position to state the theorem on impulsive stabilization. The method used in the proof is motivated by [16].

Theorem 3.2. *Assume that there exist a function $V(n,x)$ for system* (2.1) *and constants $r>0$, $c_1>0$, $c_2>0$, $\lambda>0$, and natural number $\alpha>1$, such that the following conditions hold.*

- (C_1) $c_1\|x\|^r\leqslant V(n,x)\leqslant c_2\|x\|^r$ *for any $n\in N_{n_0-m}\cup\mathbb{N}$ and $x\in\mathbb{R}^n$.*
- (C_2) *For $n\neq\eta_k-1$, any $s\in N_{-m}$, $EV(n+1,x(n+1))\leqslant(1+\lambda)EV(n,x(n))$ whenever $qEV(n+1,x(n+1))\geqslant EV(n+s,x(n+s))$, where $q>e^{2\lambda\alpha}$.*
- (C_3) $EV(\eta_k,x(\eta_k))\leqslant d_kEV(\eta_k-1,x(\eta_k-1))$, *where $d_k>0$.*
- (C_4) $m\leqslant\eta_{k+1}-\eta_k\leqslant\alpha$, $\ln d_k+\alpha\lambda<-\lambda(\eta_{k+1}-\eta_k)$.

Then for any initial data $x_{n_0}=\varphi$ there exists positive constant C; for any $n\in\mathbb{N}$, the following inequality holds:

$$E\|x(n)\|^r\leqslant CE\|\varphi\|_m^r e^{-\lambda n}, \tag{3.17}$$

that is, the trivial solution of system (2.1) *is r-moment exponentially stable.*

Proof. Choose $M > 1$ such that

$$(1+\lambda)c_2E\|\varphi\|_m^r \leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_1}e^{-\alpha\lambda} < ME\|\varphi\|_m^r e^{-\lambda\eta_1} \leqslant qc_2E\|\varphi\|_m^r. \tag{3.18}$$

We will show that, for any $n \in [\eta_k, \eta_{k+1})$, $k = 1, 2, \ldots$,

$$EV(n, x(n)) \leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_{k+1}}. \tag{3.19}$$

Write $EV(n, x(n)) = EV(n)$ for the sake of brevity.

First we will show that, for any $n \in [0, \eta_1)$,

$$EV(n) \leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_1}. \tag{3.20}$$

Obviously, when $n \in [-m, 0]$, $EV(n) \leqslant ME\|\xi\|_m^r e^{-\lambda\eta_1}$. If (3.20) is not true, then there exists $\overline{n} \in [0, \eta_1 - 1)$ such that

$$EV(\overline{n}+1) > ME\|\varphi\|_m^r e^{-\lambda\eta_1}. \tag{3.21}$$

And when $n \leqslant \overline{n}$, $EV(n) \leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_1}$. At the same time there exists $n^* \geqslant 0$ such that $EV(n^*) \leqslant c_2E\|\varphi\|_m^r$, and when $n^* < n \leqslant \overline{n}$,

$$c_2E\|\varphi\|_m^r < EV(n) \leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_1}. \tag{3.22}$$

Note that there may not exist the natural number n that satisfies $n^* < n \leqslant \overline{n}$ such that (3.22) holds. However, we claim that there must be a natural number n satisfing $n^* < n \leqslant \overline{n}$ such that (3.22) holds. If not, we have $n^* = \overline{n}$; then we get

$$EV(n) \leqslant c_2E\|\varphi\|_m^r, \quad n \leqslant \overline{n}. \tag{3.23}$$

Obviously,

$$qEV(\overline{n}+1) \geqslant EV(\overline{n}+s), \quad \forall s \in N_{-m}. \tag{3.24}$$

Under condition (C_2) we get

$$EV(\overline{n}+1) \leqslant (1+\lambda)EV(\overline{n}). \tag{3.25}$$

That is

$$\begin{aligned} EV(\overline{n}) &\geqslant \frac{1}{1+\lambda}EV(\overline{n}+1) > \frac{1}{1+\lambda}ME\|\varphi\|_m^r e^{-\lambda\eta_1} \\ &= \frac{e^{\alpha\lambda}}{1+\lambda}ME\|\varphi\|_m^r e^{-\lambda\eta_1}e^{-\alpha\lambda} \\ &> ME\|\varphi\|_m^r e^{-\lambda\eta_1}e^{-\alpha\lambda} \geqslant c_2E\|\varphi\|_m^r, \end{aligned} \tag{3.26}$$

which contradicts with (3.23). Then there must be an n satisfing $n^* < n \leqslant \overline{n}$ such that (3.22) holds.

For any $n \in [n^* + 1, \overline{n}]$,

$$EV(n+s) \leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_1} < qc_2 E\|\varphi\|_m^r < qEV(n). \tag{3.27}$$

By virtue of (C_2), for any $n \in [n^* + 1, \overline{n}]$,

$$EV(n) \leqslant (1+\lambda)EV(n-1), \tag{3.28}$$

and for $s \in N_{-m}$, we have

$$\begin{aligned} qEV(\overline{n}+1) &\geqslant EV(\overline{n}+s), \\ qEV(n^*+1) &\geqslant EV(n^*+s). \end{aligned} \tag{3.29}$$

Making use of (3.28), we get

$$\begin{aligned} EV(\overline{n}+1) &\leqslant (1+\lambda)EV(\overline{n}) \leqslant (1+\lambda)^{\overline{n}-n^*} EV(n^*+1) \\ &\leqslant (1+\lambda)^{\alpha} EV(n^*) < e^{\alpha\lambda} c_2 E\|\varphi\|_m^r. \end{aligned} \tag{3.30}$$

Taking (3.3) into account, the above inequality yields

$$EV(\overline{n}+1) > ME\|\varphi\|_m^r e^{-\lambda\eta_1}, \tag{3.31}$$

which implies that

$$ME\|\varphi\|_m^r e^{-\lambda\eta_1} < e^{\alpha\lambda} c_2 E\|\varphi\|_m^r. \tag{3.32}$$

It contradicts with (3.18); then (3.20) holds, that is, (3.19) holds for $k = 1$.

Assume that (3.19) holds for $k = 1, 2, \ldots, h$, that is, when $n \in [\eta_{k-1}, \eta_k)$, $k = 1, 2, \ldots, h$,

$$EV(n) \leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_k}. \tag{3.33}$$

Under conditions (C_3) and (C_4), we have

$$\begin{aligned} EV(\eta_h) &\leqslant d_h EV(\eta_h - 1) \leqslant d_h ME\|\varphi\|_m^r e^{-\lambda\eta_h} \\ &\leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}} e^{-\alpha\lambda} \leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}}. \end{aligned} \tag{3.34}$$

Now we will show that, when $n \in [\eta_h, \eta_{h+1})$,

$$EV(n) \leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}}. \tag{3.35}$$

If (3.35) is not true, then there must be an $\overline{n} \in (\eta_h, \eta_{h+1} - 1)$, such that

$$EV(\overline{n}+1) > ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}}, \tag{3.36}$$

and for $n \in [\eta_h, \overline{n}]$

$$EV(n) \leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}}. \tag{3.37}$$

At the same time, there exists an $n^* \in [\eta_h, \overline{n}]$ such that

$$EV(n^*) \leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}} e^{-\alpha\lambda}, \tag{3.38}$$

And, when $n^* < n \leqslant \overline{n}$,

$$EV(n) > ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}} e^{-\alpha\lambda}. \tag{3.39}$$

If there does not exist an n satisfing $n^* < n \leqslant \overline{n}$ such that (3.39) holds, then $n^* = \overline{n}$. Obviously, for any $s \in N_{-m}$, $qEV(\overline{n}+1) \geqslant EV(\overline{n}+s)$. Using condition (C_2) yields $EV(\overline{n}+1) \leqslant (1+\lambda)EV(\overline{n})$, that is,

$$EV(\overline{n}) \geqslant \frac{1}{1+\lambda} EV(\overline{n}+1) \geqslant \frac{e^{\lambda\alpha}}{1+\lambda} ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}} e^{-\alpha\lambda} > ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}} e^{-\alpha\lambda}, \tag{3.40}$$

which contradicts with the definition of $\overline{n}$; then there exists at least one number n satisfing $n^* < n \leqslant \overline{n}$ such that (3.39) holds.

For $n \in [n^*+1, \overline{n}]$ and $s \in N_{-m}$, we have

$$\begin{aligned} EV(n+s) &\leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_h} \\ &= e^{\lambda(\eta_{h+1}-\eta_h)} ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}} \\ &\leqslant e^{2\lambda\alpha} ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}} e^{-\alpha\lambda} \\ &< qEV(n), \end{aligned} \tag{3.41}$$

which implies that, under condition (C_2),

$$EV(n) \leqslant (1+\lambda)EV(n-1). \tag{3.42}$$

Obviously, $qEV(\overline{n}+1) \geqslant EV(\overline{n})$. Using condition (C_2) again, we get

$$EV(\overline{n}+1) \leqslant (1+\lambda)EV(\overline{n}). \tag{3.43}$$

since $qEV(n^*+1) > EV(n^*+s)$, $s \in N_{-m}$, we have, under condition (C_2)

$$EV(n^*+1) \leqslant (1+\lambda)EV(n^*). \tag{3.44}$$

Then

$$\begin{aligned} EV(\overline{n}+1) &\leqslant (1+\lambda)EV(\overline{n}) \leqslant (1+\lambda)^{\overline{n}-n^*}EV(n^*+1) \\ &\leqslant (1+\lambda)^{\overline{n}-n^*+1}EV(n^*) \leqslant (1+\lambda)^{\alpha}EV(n^*) \\ &< e^{\alpha\lambda}ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}}e^{-\alpha\lambda} \\ &= ME\|\varphi\|_m^r e^{-\lambda\eta_{h+1}} < EV(\overline{n}), \end{aligned} \tag{3.45}$$

which conflicts with the definition of $\overline{n}$. Then (3.19) holds for $k = h+1$.

By induction, we know that (3.19) holds for $n \in [\eta_k, \eta_{k+1})$, $k \in \mathbb{N}$.

Using condition (C_1), for any $n \in [\eta_k, \eta_{k+1})$, $k \in \mathbb{N}$, we have

$$c_1E\|x(n)\|^r \leqslant EV(n) \leqslant ME\|\varphi\|_m^r e^{-\lambda\eta_{k+1}} \leqslant ME\|\varphi\|_m^r e^{-\lambda n}. \tag{3.46}$$

That is the desired result. □

4. Applications

In this section, we consider a kind of impulsive stochastic delay difference equation as follows:

$$\begin{gathered} x(n+1) = f(n, x(n), x(n-m)) + g(n, x(n), x(n-m))\xi_n, \quad n \neq \eta_k - 1, \\ x(\eta_k) = H_k(x(\eta_k - 1)), \\ x(n_0 + s) = \varphi(s), \quad s \in N_{-m}. \end{gathered} \tag{4.1}$$

Using the obtained results, we present three corollaries for system (4.1).

Corollary 4.1. *Assume that conditions* (C_1), (C_4), *and* (C_5) *of Theorem 3.1 hold, but conditions* (C_2) *and* (C_3) *are replaced with the following conditions:*

(C_2^*) *There exist constants* λ_1 *and* λ_2, $0 < \lambda_1, \lambda_2 < 1$, *such that*

$$EV(n+1, x(n+1)) \leqslant \lambda_1 EV(n, x(n)) + \lambda_2 EV(n-m, x(n-m)). \tag{4.2}$$

If $\lambda_1 + \lambda_2 < 1$, *then the trivial solution of system* (4.1) *is* r*-moment exponentially stable.*

Proof. Let $x(n)$ be a solution of system (4.1). Take $(p = \sqrt{\lambda_1^2 + 4\lambda_2} + \lambda_2 - \lambda_1)/3\lambda_2$. It is easy to see that, under the conditions $0 < \lambda_1, \lambda_2 < 1$, and $0 < \lambda_1 + \lambda_2 < 1$,

$$1 < p < \frac{\sqrt{\lambda_1^2 + 4\lambda_2} - \lambda_1}{2\lambda_2} < \frac{1}{\lambda_1 + \lambda_2} < \frac{1 - \lambda_1}{\lambda_2}. \tag{4.3}$$

If $EV(n+\theta, x(n+\theta)) \leqslant pEV(n, x(n))$ for any $\theta \in N_{-m}$, it follows from the condition (C_2^*) that

$$EV(n+1, x(n+1)) \leqslant (\lambda_1 + p\lambda_2) EV(n, x(n)). \tag{4.4}$$

Then condition (C_2) of Theorem 3.1 follows under (4.3).

Let $\lambda = \lambda_1 + p\lambda_2$; using inequality (4.3), we get α in Theorem 3.1: $\alpha = \min\{-\ln\lambda, \ln p/(m+1)\} = \ln p/(m+1)$.

Now we assume that $V(n+\theta, x(n+\theta)) > e^{\alpha} V(n, x(n))$ for some $\theta \in N_{-m}$; by virtue of (C_2^*) and inequality (4.3),

$$\begin{aligned} EV(n+1, x(n+1)) &\leqslant \lambda_1 EV(n, x(n)) + \lambda_2 EV(n-m, x(n-m)) \\ &< \lambda_1 e^{-\alpha} EV(n+\theta, x(n+\theta)) + \lambda_2 EV(n-m, x(n-m)) \\ &< (\lambda_1 + \lambda_2) \max_{s\in N_{-m}} \{EV(n+s, x(n+s))\} \\ &< \frac{1}{p} \max_{s\in N_{-m}} \{EV(n+s, x(n+s))\}. \end{aligned} \tag{4.5}$$

Then condition (C_3) of Theorem 3.1 follows, which completes the proof. □

From the above proof, we know that constant β in Theorem 3.1 equals to

$$\frac{m-1}{m(m+1)} \ln \frac{\sqrt{\lambda_1^2 + 4\lambda_2} + \lambda_2 - \lambda_1}{3\lambda_2} - \frac{\ln d}{m}. \tag{4.6}$$

Corollary 4.2. *Assume that conditions (C_1), (C_4), and (C_5) of Theorem 3.1 hold, but conditions (C_2) and (C_3) are replaced with the following condition.*

(C_2^{**}) *There exists a constant* $0 < \lambda < 1$ *such that*

$$EV(n+1, x(n+1)) \leqslant \lambda \max_{s\in N_{-m}} \{EV(n+s, x(n+s))\}. \tag{4.7}$$

Then the trivial solution of system (4.1) *is r-moment exponentially stable.*

Proof. Let $x(n)$ be a solution of system (4.1). Take

$$p = \left(\frac{1}{\lambda}\right)^{(m+1)/(m+2)}. \tag{4.8}$$

Since $0 < \lambda < 1$, we have $1 < p < 1/\lambda$ and

$$\frac{\ln p}{(m+1)} = \frac{\ln(1/\lambda)}{(m+2)} < \ln\left(\frac{1}{\lambda}\right). \tag{4.9}$$

For any $s \in N_{-m}$, assume that $V(n+s, x(n+s)) \leqslant pV(n, x(n))$; by virtue of condition (C_2^{**}), we get

$$EV(n+1, x(n+1)) \leqslant p\lambda EV(n), \tag{4.10}$$

that is, condition (C_2) of Theorem 3.1.

Since $1 < p < 1/\lambda$ we have $1/p > \lambda$. Under condition (C_2^{**}), for any $n \in \mathbb{N}$, we get

$$\begin{aligned} EV(n+1, x(n+1)) &\leqslant \lambda \max_{s \in N_{-m}} \{EV(n+s, x(n+s))\} \\ &< \left(\frac{1}{p}\right) \max_{s \in N_{-m}} \{EV(n+s, x(n+s))\}, \end{aligned} \tag{4.11}$$

that is condition (C_3) of Theorem 3.1. □

From the above proof, we know that constant α in Theorem 3.1 equals to

$$\min\left\{-\ln(\lambda p), \frac{\ln p}{(m+1)}\right\} = -\ln(\lambda p) = \frac{\ln p}{(m+1)} = -\frac{\ln \lambda}{(m+2)}. \tag{4.12}$$

Then constant β in Theorem 3.1 equals

$$-\frac{\ln \lambda}{(m+2)}\left(1 - \frac{1}{m}\right) - \frac{\ln d}{m}. \tag{4.13}$$

Now, we present a corollary of Theorem 3.2 which establishes a criterion of mean square exponential stability for system (4.1).

Corollary 4.3. *Assume that there exist positive constants λ, α, and q where α is a natural number and $\alpha > 1$, $q \geqslant e^{2\lambda\alpha}$such that system* (4.1) *satisfies the following.*

(1)

$$\begin{aligned} &E\|f(n, x(n), x(n-m))\|^2 + E\|g(n, x(n), x(n-m))\|^2 \\ &\quad \leqslant \frac{1}{2}\left(aE\|x(n)\|^2 + bE\|x(n-m)\|^2\right), \end{aligned} \tag{4.14}$$

where a, b are positive constants, $b < 1/q$, and

$$0 < \frac{a + bq - 1}{1 - bq} \leqslant \lambda. \tag{4.15}$$

(2) $\|H_k(x)\| \leqslant \beta_k\|x\|$, *for any* $x \in \mathbb{R}^n$, $\beta_k > 0$, *and* $2\ln\beta_k + \lambda(\eta_{k+1} - \eta_k) \leqslant -\lambda\alpha$. *The impulsive moments satisfy* $m \leqslant \eta_{k+1} - \eta_k \leqslant \alpha$.

Then, for any initial data $x_{n_0} = \varphi$, *the solution* $x(n)$ *of system* (4.1) *satisfies*

$$E\|x(n)\|^2 \leq E\|\varphi\|_m^2 e^{-(\lambda/2)n}. \tag{4.16}$$

That is to say, the trivial solution of system (4.1) *is mean square exponentially stable.*

Proof. Let $V(n,x) = \|x\|^2$, then,

$$\begin{aligned}
&EV(n+1, x(n+1)) \\
&\quad = E\|x(n+1)\|^2 \\
&\quad = E\big\|f(n,x(n),x(n-m)) + g(n,x(n),x(n-m))\xi_n\big\|^2 \\
&\quad \leqslant 2\Big(E\big\|f(n,x(n),x(n-m))\big\|^2 + E\Big(\big\|g(n,x(n),x(n-m))\big\|^2\xi_n^2\Big)\Big) \\
&\quad \leqslant aE\|x(n)\|^2 + bE\|x(n-m)\|^2 \\
&\quad = aEV(n) + bEV(n-m).
\end{aligned} \tag{4.17}$$

Assume that $qEV(n+1,x(n+1)) \geqslant EV(n+s,x(n+s))$ holds for any $s \in N_{-m}$, then

$$EV(n+1) = E\|x(n+1)\|^2 \leqslant \frac{a}{1-bq}EV(n) \leqslant (1+\lambda)EV(n). \tag{4.18}$$

The other conditions of Theorem 3.2 are easy to be verified and the conclusion of this corollary now follows. □

5. Examples

Now we study some examples to illustrate our results.

We consider a linear impulsive stochastic delay difference equation as following:

$$\begin{gathered}
x(n+1) = ax(n) + bx(n-m) + cx(n-m)\xi_n, \quad n \neq \eta_k - 1,\ n \geqslant 0, \\
x(\eta_k) = \beta_k x(\eta_k - 1), \quad k \in \mathbb{N}, \\
x(s) = \varphi(s), \quad s \in N_{-m}.
\end{gathered} \tag{5.1}$$

First we take $a = 0.5$, $b = 0.25$, $c = 0.25$, $m = 9$, $\eta_k = 10k$, $\beta_k = 1.1$, $k = 1,2,\ldots$, and $\varphi(s) = 1/(100+s^2)$. By virtue of Corollary 4.1, taking $V(x,n) = x^2(n)$, we can get the mean square exponential stability of (5.1). The stability is shown in Figure 1.

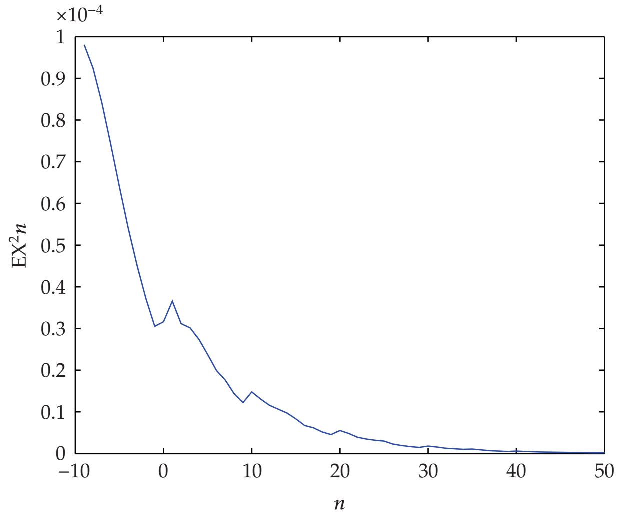

Figure 1: Mean square exponential stability of (5.1): $a = 0.5$, $b = 0.25$, $c = 0.25$, $m = 9$, $\eta_k = 10k$, and $\beta_k = 1.1$.

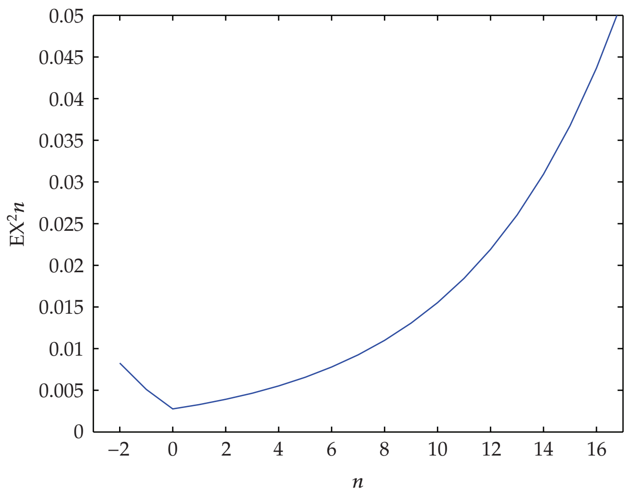

Figure 2: Instability without impulsive effects of (5.1): $a = 1.09$, $b = 0$, $c = e^{-19}$, and $m = 3$.

Now we take $a = 1.09$, $b = 0$, $c = e^{-19}$, $m = 3$, and $\varphi(s) = 1/(10 + s^2)$ in (5.1) without impulsive effects. It is easy to see that equation is unstable. This property is shown in Figure 2. Then we take an impulsive strategy: $\eta_k = 4k$, $\beta_k = e^{-19}$. In light of Corollary 4.3, we can see that equation is mean square exponentially stable. The stability is shown in Figure 3.

It should be pointed that the conditions of Corollary 4.3 are sufficient but not necessary. If we take $a = 1.09$, $b = 0$, $c = e^{-19}$, $m = 3$, $\eta_k = 4k$, and $\beta_k = 0.7$ and $\varphi(s) = 1/(10 + s^2)$, then it is not difficult to show that the conditions of Corollary 4.3 are not satisfied again, but under this situation, the equation is still stable. The stability is shown in Figure 4.

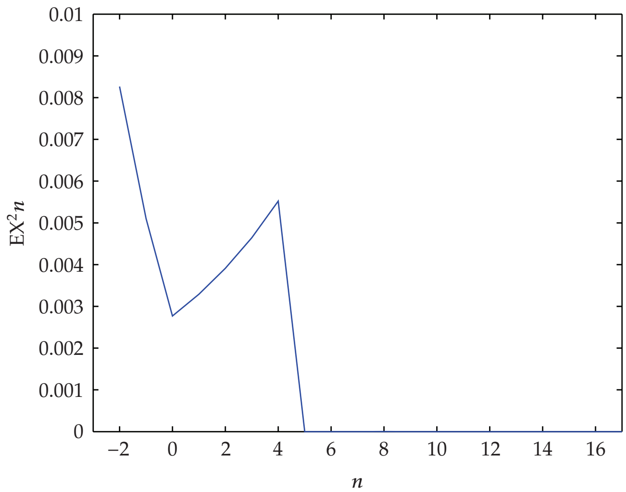

Figure 3: Mean square exponential stability of (5.1): $a = 1.09$, $b = 0$, $c = e^{-19}$, $m = 3$, $\eta_k = 4k$, and $\beta_k = e^{-19}$.

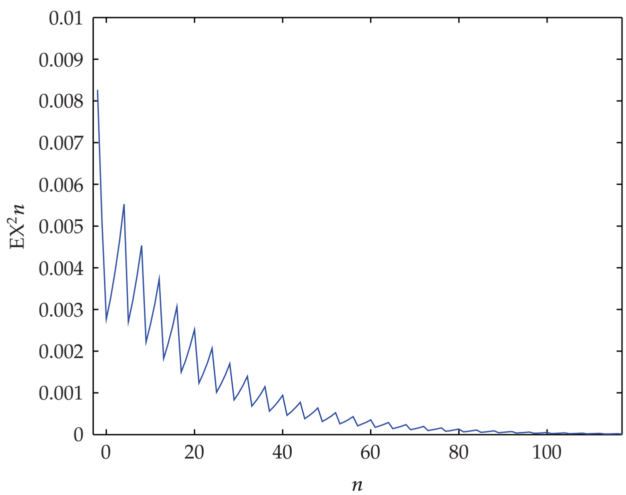

Figure 4: Stability of (5.1): $a = 1.09$, $b = 0$, $c = e^{-19}$, $m = 3$, $\eta_k = 4k$, and $\beta_k = 0.7$.

6. Conclusions

In this paper, we considered the r-moment exponential stability for impulsive stochastic delay difference equations. Using the Lyapunov-Razumikhin method, we established criteria of r-moment exponential stability and these criteria presented the answers for the problem of impulsive stability and the problem of impulsive stabilization. As for applications, we considered a kind of impulsive stochastic delay difference equation and obtained three corollaries for our main theorems. The results we got may work in the study of stability of numerical method for the impulsive delay differential equations.

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