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Research Article

Optimal Control of Linear Impulsive Antiperiodic Boundary Value Problem on Infinite Dimensional Spaces

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A class of optimal control problems for infinite dimensional impulsive antiperiodic boundary value problem is considered. Using exponential stabilizability and discussing the impulsive evolution operators, without compactness and exponential stability of the semigroup governed by original principle operator, we present the existence of optimal controls. At last, an example is given for demonstration.

1. Introduction

Antiperiodic and periodic motions arise naturally in the mathematical modeling of a variety of physical process. Many authors including us pay great attention to various classes of antiperiodic and periodic systems [1–6]. On the other hand, in order to describe dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, diseases and, some authors have used impulsive differential systems to describe the model since the last century. For the basic theory on impulsive differential equations on finite dimensional spaces, the reader can refer to Lakshmikantham's book (see [7]).

Recently, we have begun to investigate impulsive periodic system on infinite dimensional spaces. The suitable impulsive evolution operator corresponding to homogenous impulsive periodic system was introduced and its properties (boundedness, periodicity, compactness, and exponential stability) were given. Some results including the existence of the periodic PC-mild solutions and alternative theorem, criteria of Massera type, asymptotical stability, and robustness by perturbation for linear impulsive periodic system were established. For semilinear impulsive periodic system and intergrodifferential impulsive periodic system, some fixed point theorems such as Horn fixed point theorem and Leary-Schauder fixed point theorem were applied to obtain the existence of the periodic PC-mild solutions, respectively. In order to do it, we had to construct *Poincaré* operator, discuss its properties, and derive some generalized Gronwall inequalities with impulse for the estimate of the PC-mild solutions [8–11].

However, to our knowledge, optimal control problems arising in systems governed by impulsive antiperiodic system on infinite dimensional spaces have not been extensively investigated. Herein, we study the following optimal control problem (P1):

$$\text{Minimize } L(x,u) : L(x,u) = \int_0^{T_0} \big(g(x(t)) + h(u(t))\big)dt \tag{1.1}$$

subject to impulsive antiperiodic boundary problem

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), \quad t \in [0,T_0] \setminus \widetilde{D}, \\ &\Delta x(\tau_k) = C_k x(\tau_k), \quad k = 1,2,\ldots,\delta, \\ &x(0) = -x(T_0), \quad u \in L^2(0,T_0;U). \end{aligned} \tag{1.2}$$

on real Hilbert spaces H and U, where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$, $\tau_{k+\delta} = \tau_k + T_0$, $\widetilde{D} = \{\tau_1, \tau_2, \ldots, \tau_\delta\} \subset (0,T_0)$, T_0 is a fixed positive number, and $\delta \in \mathbb{N}$ denoted the number of impulsive points between 0 and T_0. The operator A is the infinitesimal generator of a C_0- semigroup $\{T(t), t \geq 0\}$ on H. Operator B belongs to $\pounds_b(U,H)$ and $C_{k+\delta} = C_k \in H$. x denotes the T_0-antiperiodic PC-mild solution of system (1.2) corresponding to the control $u \in L^2([0,T_0];U)$. We have the functions $g : H \to \mathbb{R}$ and $h : U \to \overline{\mathbb{R}} =]-\infty, +\infty]$. In this paper, using exponential stabilizability and discussing the impulsive evolution operators, without compactness and exponential stability of semigroup generated by original principle operator A, we present the existence of antiperiodic optimal controls for problem (P1).

In order to study impulsive antiperiodic system on infinite dimensional spaces, we constructed the impulsive evolution operator $\{S(\cdot,\cdot)\}$ associated with A and $\{C_k; \tau_k\}_{k=1}^{\infty}$ which is very important in sequel. It can be seen from the discussion on linear impulsive antiperiodic system that the invertibility of $[I + S(T_0,0)]$ is the key of the existence of antiperiodic PC-mild solution of system (1.2). For the invertibility of $[I + S(T_0,0)]$, compactness or exponential stability of $\{T(t), t \geq 0\}$ generated by A is needed. By virtue of concept of exponential stabilizibility, which is introduced by Barbu and Pavel in [12] to weaken the assumptions on the existence of antiperiodic PC-mild solutions, we replace the problem (P1) by problem (P2):

$$\text{Minimize } \widetilde{L}(x,v) : \widetilde{L}(x,v) = \int_0^{T_0} \big(g(x(t)) + h(v(t) + Fx(t))\big)dt \tag{1.3}$$

subject to

$$\begin{aligned} \dot{x}(t) &= A_F x(t) + Bv(t), \quad t \in [0, T_0] \setminus \widetilde{D}, \\ \Delta x(\tau_k) &= C_k x(\tau_k), \quad k = 1, 2, \dots, \delta, \\ x(0) &= -x(T_0), \quad v \in L^2(0, T_0; U), \end{aligned} \tag{1.4}$$

where $A_F = A + BF$, $F \in \pounds_b(H, U)$ such that A_F generates an exponentially stable semigroup. Discussing the impulsive evolution operator $\{S_F(\cdot, \cdot)\}$ associated with operator A_F and $\{C_k; \tau_k\}_{k=1}^{\infty}$ and giving some sufficient conditions for invertibility of $[I + S_F(T_0, 0)]$, we prove that every antiperiodic PC-mild solution of (1.2) is an antiperiodic PC-mild solution of (1.4) with $v = u - Fx$ and vice versa. Therefore, the equivalence between problem (P1) and problem (P2) is shown. Utilizing some techniques of semigroup theory and functional analysis, we present the existence of antiperiodic optimal controls for problem (P2), which implies the existence of solutions for problem (P1).

The main result of this paper is the existence of optimal control for problem (P1) (given by Theorem 4.1). However, the novelty of this paper over other related results in literature consists in the fact that the invertibility of $[I + S(T_0, 0)]$ is replaced by weaker condition. In addition some sufficient conditions for invertibility of $[I + S_F(T_0, 0)]$ are presented.

This paper is organized as follows. In Section 2, impulsive evolution operator $\{S_F(\cdot, \cdot)\}$ and its exponential stability are studied and some sufficient conditions guaranteeing $[I + S_F(T_0, 0)]^{-1} \in \pounds_b(H)$ are given. Section 3 is devoted to the equivalence of (P1) and (P2). In Section 4, the existence of optimal antiperiodic arcs for (P2) is presented. Hence, the existence of optimal controls for (P1) is obtained. At last, an example is given to demonstrate the applicability of our results.

2. Invertibility of $[I + S(T_0, 0)]$

Let H be a Hilbert space. $\pounds(H)$ denotes the space of linear operators in H; $\pounds_b(H)$ denotes the space of bounded linear operators in H. $\pounds_b(H)$ is the Hilbert space with the usual supremum norm. Define $\widetilde{D} = \{\tau_1, \dots, \tau_\delta\} \subset [0, T_0]$. We introduce $PC([0, T_0]; H) \equiv \{x : [0, T_0] \to H \mid x$ is continuous at $t \in [0, T_0] \setminus \widetilde{D}$, x is continuous from left and has right hand limits at $t \in \widetilde{D}\}$ and $PC^1([0, T_0]; H) \equiv \{x \in PC([0, T_0]; H) \mid \dot{x} \in PC([0, T_0]; H\}$. Set

$$\|x\|_{PC} = \max\left\{ \sup_{t \in [0, T_0]} \|x(t+0)\|, \sup_{t \in [0, T_0]} \|x(t-0)\| \right\}, \qquad \|x\|_{PC^1} = \|x\|_{PC} + \|\dot{x}\|_{PC}. \tag{2.1}$$

It can be seen that endowed with the norm $\| \cdot \|_{PC} (\| \cdot \|_{PC^1})$, $PC([0, T_0]; H)(PC^1([0, T_0]; H))$ is a Hilbert space.

The basic hypotheses are the following Assumption [H1].

- [H1.1] A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in H with domain $D(A)$.
- [H1.2] There exists δ such that $\tau_{k+\delta} = \tau_k + T_0$.
- [H1.3] For each $k \in \mathbb{Z}_0^+$, $C_k \in \pounds_b(X)$ and $C_{k+\delta} = C_k$.

Under Assumption [H1], we consider the Cauchy problem

$$\begin{gathered} \dot{x}(t) = Ax(t), \quad t \in [0, T_0] \setminus \widetilde{D}, \\ \Delta x(\tau_k) = C_k x(\tau_k), \quad k = 1, 2, \ldots, \delta, \\ x(0) = x_0. \end{gathered} \tag{2.2}$$

For Cauchy problem (2.2), if $x_0 \in D(A)$ and $D(A)$ is an invariant subspace of C_k, using ([13], Theorem 5.2.2, page 144), step by step, one can verify that the Cauchy problem (2.2) has a unique classical solution $x \in PC^1([0, T_0]; H)$ represented by $x(t) = S(t, 0)x_0$ where

$$S(\cdot, \cdot) : \Delta = \{(t, \theta) \in [0, T_0] \times [0, T_0] \mid 0 \le \theta \le t \le T_0\} \longrightarrow \pounds(H) \tag{2.3}$$

given by

$$S(t,\theta) = \begin{cases} T(t-\theta), & \tau_{k-1} \le \theta \le t \le \tau_k, \\ T(t-\tau_k^+)(I+C_k)T(\tau_k-\theta), & \tau_{k-1} \le \theta < \tau_k < t \le \tau_{k+1}, \\ T(t-\tau_k^+)\left[\displaystyle\prod_{\theta<\tau_j<t}(I+C_j)T\left(\tau_j-\tau_{j-1}^+\right)\right](I+C_i)T(\tau_i-\theta), & \\ & \tau_{i-1} \le \theta < \tau_i \le \cdots < \tau_k < t \le \tau_{k+1}. \end{cases} \tag{2.4}$$

Definition 2.1. The operator $\{S(t,\theta), (t,\theta) \in \Delta\}$ given by (2.4) is called the impulsive evolution operator associated with operator A and $\{C_k; \tau_k\}_{k=1}^{\infty}$.

Lemma 2.2. *Impulsive evolution operator $\{S(t,\theta), (t,\theta) \in \Delta\}$ has the following properties.*

(1) *For $0 \le \theta \le t \le T_0$, there exists a constant $M_{T_0} > 0$ such that* $\sup_{0\le\theta\le t\le T_0} \|S(t,\theta)\| \le M_{T_0}$.

(2) *For $0 \le \theta < r < t \le T_0$, $r \ne \tau_k$, $S(t,\theta) = S(t,r)S(r,\theta)$.*

(3) *For $0 \le \theta \le t \le T_0$ and $N \in \mathbb{Z}_0^+$, $S(t+NT_0, \theta+NT_0) = S(t,\theta)$.*

(4) *For $0 \le t \le T_0$ and $N \in \mathbb{Z}_0^+$, $S(NT_0+t, 0) = S(t,0)[S(T_0,0)]^N$.*

(5) *For $0 \le \theta < t$, there exits $M \ge 1$, $\omega \in \mathbb{R}$ such that*

$$\|S(t,\theta)\| \le M \exp\left\{\omega(t-\theta) + \sum_{\theta\le\tau_k<t} \ln(M\|I+C_k\|)\right\}. \tag{2.5}$$

It is well known that if there exist constants $M_0 \ge 0$ and $\omega_0 > 0$ such that the semigroup $\{T(t), t \ge 0\}$ generated by A satisfies $\|T(t)\| \le M_0 e^{-\omega_0 t}, t > 0$, the semigroup $\{T(t), t \ge 0\}$ is said to be exponential stable. In general, a semigroup may not be exponential stable.

Let $B \in \pounds_b(U,H)$. The pair (A,B) is said to be exponentially stabilizable, if there exists $F \in \pounds_b(H,U)$ such that $A_F = A + BF$ generates an exponentially stable C_0-semigroup $\{T_F(t), t \geq 0\}$; that is, there exist $K_F \geq 0$ and $\nu_F > 0$ such that

$$\|T_F(t)\| \leq K_F e^{-\nu_F t}, \quad t > 0. \tag{2.6}$$

Remark 2.3. By [13, Theorem 5.4], the following inequality

$$\int_0^\infty \|T_F(t)\xi\|^p dt < \infty, \quad \text{for every } \xi \in X,\ t > 0,\ 1 \leq p < \infty \tag{2.7}$$

implies that the exponential stability of $\{T_F(t), t \geq 0\}$.

Impulsive evolution operator $S(\cdot,\cdot)$ plays an important role in the sequel. Here, we need to discuss the exponential stability and exponential stabilizability of impulsive evolution operator.

Definition 2.4. $\{S(t,\theta), t \geq \theta \geq 0\}$ is called exponential stability if there exist $K \geq 0$ and $\nu > 0$ such that

$$\|S(t,\theta)\| \leq K e^{-\nu(t-\theta)}, \quad t > \theta \geq 0. \tag{2.8}$$

Consider the Cauchy problem

$$\begin{gathered} \dot{x}(t) = (A+BF)x(t), \quad t \in [0,T_0] \setminus \widetilde{D}, \\ \Delta x(\tau_k) = C_k x(\tau_k), \quad k = 1,2,\ldots,\delta, \\ x(0) = x_0. \end{gathered} \tag{2.9}$$

The impulsive evolution operator $S_F(\cdot,\cdot) : \Delta = \{(t,\theta) \in [0,T_0] \times [0,T_0] \mid 0 \leq \theta \leq t \leq T_0\} \to \pounds(H)$ associated with operator $A_F = A + BF$ and $\{C_k; \tau_k\}_{k=1}^\infty$ can be given by

$$S_F(t,\theta) = \begin{cases} T_F(t-\theta), & \tau_{k-1} \leq \theta \leq t \leq \tau_k, \\ T_F\left(t-\tau_k^+\right)(I+C_k)T_F(\tau_k-\theta), & \tau_{k-1} \leq \theta < \tau_k < t \leq \tau_{k+1}, \\ T_F\left(t-\tau_k^+\right)\left[\prod\limits_{\theta<\tau_j<t}(I+C_j)T_F\left(\tau_j-\tau_{j-1}^+\right)\right](I+C_i)T_F(\tau_i-\theta), & \\ & \tau_{i-1} \leq \theta < \tau_i \leq \cdots < \tau_k < t \leq \tau_{k+1}. \end{cases} \tag{2.10}$$

It is not difficult to verify that $\{S_F(t,\theta), (t,\theta) \in \Delta\}$ also satisfies the similar properties in Lemma 2.2.

Assumption [H2]: The pair (A, B) is exponentially stabilizable.

Under Assumptions [H1] and [H2], by [14, Lemmas 2.4 and 2.5], we can give some sufficient conditions guaranteeing exponential stability of $\{S_F(\cdot,\cdot)\}$ immediately.

Lemma 2.5. *Assumptions [H1] and [H2] hold. There exists* $0 < \lambda < \nu_F$ *such that*

$$\left(\prod_{k=1}^{\delta} K_F \|I + C_k\|\right) e^{-\lambda T_0} < 1. \tag{2.11}$$

Then $\{S_F(t,\theta), t \geq \theta \geq 0\}$ *is exponentially stable.*

Lemma 2.6. *Assumptions [H1] and [H2] hold. Suppose*

$$0 < \mu_1 = \inf_{k=1,2,\ldots,\delta} (\tau_k - \tau_{k-1}) \leq \sup_{k=1,2,\ldots,\delta} (\tau_k - \tau_{k-1}) = \mu_2 < \infty. \tag{2.12}$$

If there exists $\gamma > 0$ *such that*

$$-\nu_F + \frac{1}{\mu} \ln(K_F \|I + C_k\|) \leq -\gamma < 0, \quad k = 1, 2, \ldots, \delta, \tag{2.13}$$

where

$$\mu = \begin{cases} \mu_1, & \gamma - \nu_F < 0, \\ \mu_2, & \gamma - \nu_F \geq 0, \end{cases} \tag{2.14}$$

then $\{S_F(t,\theta), t \geq \theta \geq 0\}$ *is exponentially stable.*

Corollary 2.7. *Let Assumption [H1] and* (2.12) *hold. There exist* $M \geq 1$, $\omega \in \mathbb{R}$ *such that* $\|T_F(t)\| \leq Me^{(\omega + \|BF\|)t}$, $t \geq 0$. *If there exists* $\gamma > 0$ *such that*

$$(\omega + \|BF\|) + \frac{1}{\mu} \ln(M \|I + C_k\|) \leq -\gamma < 0, \quad k = 1, 2, \ldots, \delta, \tag{2.15}$$

where

$$\mu = \begin{cases} \mu_1, & \gamma + \omega + \|BF\| < 0, \\ \mu_2, & \gamma + \omega + \|BF\| \geq 0, \end{cases} \tag{2.16}$$

then $\{S_F(t,\theta), t > \theta \geq 0\}$ *is exponential stable.*

Now some sufficient conditions for the existence of inversion of $[I + S_F(T_0, 0)]$ can be given.

Theorem 2.8. *Under the assumptions of Lemma 2.5 (or Lemma 2.6), the operator $I + S_F(T_0, 0)$ is inverse and $[I + S_F(T_0, 0)]^{-1} \in £_b(H)$.*

Proof. Consider the $Q = \sum_{n=0}^{\infty} [-S_F(T_0, 0)]^n$. Under the assumptions of Lemma 2.5, $\{S_F(\cdot, \cdot)\}$ is exponential stable. It comes from the periodicity of $\{S_F(\cdot, \cdot)\}$ that

$$\left\| [-S_F(T_0, 0)]^n \right\| \leq \|S_F(nT_0, 0)\| \leq Ke^{-\nu n T_0} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.17}$$

Thus, we obtain

$$\|Q\| \leq \sum_{n=0}^{\infty} \left\| [-S_F(T_0, 0)]^n \right\| \leq \sum_{n=0}^{\infty} Ke^{-\nu n T_0}. \tag{2.18}$$

Obviously, the series $\sum_{n=0}^{\infty} Ke^{-\nu n T_0}$ is convergent, thus operator $Q \in £_b(H)$. It comes from

$$[I + S_F(T_0, 0)]Q = Q[I + S_F(T_0, 0)] = I \tag{2.19}$$

that $Q = [I + S_F(T_0, 0)]^{-1} \in £_b(H)$. □

Further, we give a little big stronger condition which will guarantee exponential stability of $\{S_F(\cdot, \cdot)\}$. However, it is more easy to be demonstrated.

Corollary 2.9. *Assumptions [H1] and [H2] hold. If*

$$\nu_F > \frac{\sum_{k=1}^{\delta} \ln\|I + C_k\| + (\delta + 1) \ln K_F}{T_0}, \tag{2.20}$$

then the impulsive evolution operator $S_F(nT_0, 0)$ is strongly convergent to zero at infinity (i.e., $S_F(nT_0, 0) \to 0$ as $n \to \infty$). Further, the operator $I + S_F(T_0, 0)$ is inverse and $[I + S_F(T_0, 0)]^{-1} \in £_b(H)$.

Remark 2.10. If $\|S_F(T_0, 0)\| = L_F < 1$, then $S_F(nT_0, 0) \to 0$ as $n \to \infty$ and the operator $I + S_F(T_0, 0)$ is inverse and $[I + S_F(T_0, 0)]^{-1} \in £_b(H)$.

3. Optimal Control Problem of Impulsive Antiperiodic System

We study the following optimal control problem (P1):

$$\text{(P1)}: \quad \text{Minimize } L(x, u) : L(x, u) = \int_0^{T_0} (g(x(t)) + h(u(t)))dt \tag{3.1}$$

subject to

$$\begin{gathered}\dot{x}(t)=Ax(t)+Bu(t),\quad t\in[0,T_0]\setminus\widetilde{D},\ x\in PC([0,T_0];H),\\ \Delta x(\tau_k)=C_kx(\tau_k),\quad k=1,2,\ldots,\delta,\\ x(0)=-x(T_0),\quad u\in L^2(0,T_0;U).\end{gathered}\tag{3.2}$$

Definition 3.1. A function $x\in PC([0,T_0];H)$ is said to be a T_0-antiperiodic PC-mild solution of the controlled system (3.2) if x satisfies

$$x(t)=S(t,0)x(0)+\int_0^t S(t,\theta)Bu(\theta)d\theta,\quad \text{for } t\in[0,T_0];\quad x(0)=-x(T_0).\tag{3.3}$$

If system (3.2) has a T_0-antiperiodic PC-mild solution corresponding to u, $(x,u)\in PC([0,T_0];H)\times L^2(0,T_0;U)$ is said to be an admissible pair. Set

$$U_{\mathrm{ad}}=\{(x,u)\mid (x,u)\text{ is admissible}\}\tag{3.4}$$

which is called admissible set. Problem (P1) can be rewritten as follows.

Find $(x^*,u^*)\in U_{\mathrm{ad}}$ such that

$$L(x^*,u^*)\le L(x,u)\quad \forall(x,u)\in U_{\mathrm{ad}}.\tag{3.5}$$

In fact, if the condition

$$[I+S(T_0,0)]^{-1}\in \pounds_b(H)\tag{3.6}$$

is satisfied, then for every $u\in L^2(0,T_0;U)$ the T_0-antiperiodic PC-mild solution of system (3.2) can be given by

$$x(t)=S(t,0)x_0+\int_0^t S(t,\theta)Bu(\theta)d\theta,\quad \forall t\in[0,T_0],\tag{3.7}$$

where

$$x_0=-[I+S(T_0,0)]^{-1}\int_0^{T_0} S(T_0,\theta)Bu(\theta)d\theta.\tag{3.8}$$

If the condition (3.6) fails, then system (3.2) has no solutions for every $u\in L^2(0,T_0;U)$.

Under Assumptions [H1] and [H2], we can write system (3.2) formally in the form

$$\begin{aligned} \dot{x}(t) &= A_F x(t) + B(u(t) - Fx(t)), \quad t \in [0, T_0] \setminus \widetilde{D},\ x \in PC([0, T_0]; H), \\ \Delta x(\tau_k) &= C_k x(\tau_k), \quad k = 1, 2, \ldots, \delta, \\ x(0) &= -x(T_0), \quad u \in L^2(0, T_0; U) \end{aligned} \tag{3.9}$$

and substitute $u - Fx = v$ so $u = v + Fx$. Therefore, we led to the problem (P2):

$$\text{Minimize } \widetilde{L}(x, v) : \widetilde{L}(x, v) = \int_0^{T_0} \big(g(x(t)) + h(v(t) + Fx(t))\big) dt \tag{3.10}$$

subject to

$$\begin{aligned} \dot{x}(t) &= A_F x(t) + Bv(t), \quad t \in [0, T_0] \setminus \widetilde{D}, x \in PC([0, T_0]; H), \\ \Delta x(\tau_k) &= C_k x(\tau_k), \quad k = 1, 2, \ldots, \delta, \\ x(0) &= -x(T_0), \quad v \in L^2(0, T_0; U). \end{aligned} \tag{3.11}$$

It can be seen from the proof of Theorem 2.8 that if $\{S_F(\cdot, \cdot)\}$ is exponentially stable, then $[I + S_F(T_0, 0)]^{-1}$ exists and $[I + S_F(T_0, 0)]^{-1} \in \pounds_b(H)$. Set

$$x(0) = -[I + S_F(T_0, 0)]^{-1} \int_0^{T_0} S_F(T_0, \theta) Bv(\theta) d\theta; \tag{3.12}$$

then $x \in PC([0, T_0]; H)$ given by

$$x(t) = S_F(t, 0)x(0) + \int_0^t S_F(t, \theta) Bv(\theta) d\theta \tag{3.13}$$

is the antiperiodic PC-mild solution of (3.11).

By Theorem 2.8, we have the following existence result immediately.

Theorem 3.2. *For every $v \in L^2(0, T_0; U)$, system* (3.11) *has a unique T_0-antiperiodic PC-mild solution provided that assumptions of Lemma 2.2 (or Lemma 2.5) are satisfied.*

In order to show the equivalence of problem (P1) and problem (P2), we have to prove that every PC-mild solution of (3.2) is a PC-mild solution of (3.11) with $v = u - Fx$ and vice versa. It is not obvious for PC-mild solution. Here is the equivalence.

Theorem 3.3. *Under Assumptions [H1] and [H2], if $\{S_F(\cdot, \cdot)\}$ is exponentially stable, then every PC-mild solution of* (3.2) *is a PC-mild solution of* (3.11) *with $v = u - Fx$ and vice versa. Therefore, the problem (P1) is equivalent to problem (P2).*

Proof. It is obvious that every strong solution of system (3.2) is a strong solution of system (3.11). We prove only that (3.3) implies

$$x(t) = S_F(t,0)x(0) + \int_0^t S_F(t,\theta)Bv(\theta)d\theta, \tag{3.14}$$

$$x(0) = -[I + S_F(T_0,0)]^{-1}\int_0^{T_0} S_F(T_0,\theta)Bv(\theta)d\theta, \tag{3.15}$$

as the inverse statement will have the same proof. Therefore, let x satisfy (3.3) and denote the Yosida approximation of A by A_λ. Let x_λ be the strong solution of

$$\begin{gathered}
\dot{x}_\lambda(t) = A_\lambda x_\lambda(t) + Bu(t), \quad t \in [0,T_0] \setminus \widetilde{D},\ x_\lambda \in PC([0,T_0];H), \\
\Delta x_\lambda(\tau_k) = C_k x_\lambda(\tau_k), \quad k = 1,2,\ldots,\delta, \\
x_\lambda(0) = x(0), \quad u \in L^2(0,T_0;U).
\end{gathered} \tag{3.16}$$

Taking into account that

$$T_\lambda(t)x(0) \longrightarrow T(t)x(0) \quad \text{as } \lambda \longrightarrow 0, \quad \text{uniformly in } t \in [0,T_0], \tag{3.17}$$

it follows that for each $t \in [0,T_0]$ but fixed,

$$S_\lambda(t,\theta)x(0) \longrightarrow S(t,\theta)x(0) \quad \text{as } \lambda \longrightarrow 0, \quad \text{uniformly in } \theta \in [0,t], \tag{3.18}$$

where the operator $\{S_\lambda(t,\theta), (t,\theta) \in \Delta\}$ is the impulsive evolution operator associated with A_λ and $\{C_k;\tau_k\}_{k=1}^{\infty}$.

In fact, for $\tau_{k-1} \le \theta \le t \le \tau_k$,

$$\begin{aligned}
S_\lambda(t,\theta)x(0) &= T_\lambda(t-\theta)x(0) \longrightarrow T(t-\theta)x(0) \\
&= S(t,\theta)x(0) \quad \text{as } \lambda \longrightarrow 0, \quad \text{uniformly in } \theta \in [0,t].
\end{aligned} \tag{3.19}$$

For $\tau_{k-1} \le \theta < \tau_k < t \le \tau_{k+1}$, $S_\lambda(t,\theta)x(0) = T_\lambda(t-\tau_k^+)(I+C_k)T_\lambda(\tau_k-\theta)x(0)$.

Since $T_\lambda(\tau_k-\theta)x(0) \to T(\tau_k-\theta)x(0)$ as $\lambda \to 0$, uniformly in $\theta \in [0,\tau_k]$,

$$(I+C_k)T_\lambda(\tau_k-\theta)x(0) \longrightarrow (I+C_k)T(\tau_k-\theta)x(0) \quad \text{as } \lambda \longrightarrow 0, \quad \text{uniformly in } \theta \in [0,\tau_k]. \tag{3.20}$$

Further,

$$S_\lambda(t,\theta)x(0) \longrightarrow S(t,\theta)x(0) \quad \text{as } \lambda \longrightarrow 0, \quad \text{uniformly in } \theta \in [0,t], \tag{3.21}$$

For $\tau_{i-1} \leq \theta < \tau_i \leq \cdots < \tau_k < t \leq \tau_{k+1}$, step by step,

$$\begin{aligned} &\left[\prod_{\theta<\tau_j<t}(I+C_j)T_\lambda\left(\tau_j-\tau_{j-1}^{+}\right)\right](I+C_i)T_\lambda(\tau_i-\theta)x(0) \\ &\longrightarrow \left[\prod_{\theta<\tau_j<t}(I+C_j)T\left(\tau_j-\tau_{j-1}^{+}\right)\right](I+C_i)T(\tau_i-\theta)x(0) \end{aligned} \tag{3.22}$$

as $\lambda \to 0$, uniformly in $\theta \in [0,\tau_k]$. Of course, we have

$$S_\lambda(t,\theta)x(0) \longrightarrow S(t,\theta)x(0) \quad \text{as } \lambda \longrightarrow 0, \quad \text{uniformly in } \theta \in [0,t]. \tag{3.23}$$

On the other hand, define

$$q^\lambda(\theta) = S_\lambda(t,\theta)Bu(\theta) - S(t,\theta)Bu(\theta), \tag{3.24}$$

then

$$\left\|q^\lambda(\theta)\right\| = \|(S_\lambda(t,\theta) - S(t,\theta))Bu(\theta)\| \leq 2M_{T_0}\|B\|\|u\|_{L^2(U;H)} \in L^1(0,T_0;H). \tag{3.25}$$

Since $q^\lambda(\theta) \to 0$ a.e. $\theta \in [0,t]$ as $\lambda \to 0$, by virtue of Majorized Convergence theorem, we obtain

$$\int_0^t q^\lambda(\theta)d\theta \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0. \tag{3.26}$$

This implies that $x_\lambda \to x$ in $PC([0,T_0];H)$ as $\lambda \to 0$.

However, (3.16) can be written as

$$\begin{gathered} \dot{x}_\lambda(t) = (A_\lambda + BF)x_\lambda(t) + Bv_\lambda(t), \quad t \in [0,T_0] \setminus \widetilde{D},\ x_\lambda \in PC([0,T_0];H), \\ \Delta x_\lambda(\tau_k) = C_k x_\lambda(\tau_k), \quad k = 1,2,\ldots,\delta, \\ x_\lambda(0) = x(0), \quad u \in L^2(0,T_0;U) \end{gathered} \tag{3.27}$$

with $v_\lambda = u - Fx_\lambda$.

Similarly, one can obtain that x_λ in (3.27) is also convergent to the solution of (3.14) with $v = u - Fx$.

At the same time, it is easy to see that $U_{\mathrm{ad}} \neq \emptyset$ and problem (P1) is equivalent to problem (P2). □

4. Existence of Optimal Controls

In this section, we present the existence of optimal controls for problem (P1) which is the main result of this paper.

We make the following assumptions.

[H3] The function $h : U \to \overline{\mathbb{R}}$ is convex and lower semicontinuous; $\operatorname{Int} D(h) \neq \emptyset$, where $D(h) = \{u \in U; h(u) < +\infty\}$. Moreover, $h : U \to [0, +\infty)$ has the the following growth properties:

$$\lim_{\|u\|_U \to \infty} \frac{h(u)}{\|u\|_U} = +\infty. \tag{4.1}$$

[H4] The function $g : H \to \mathbb{R}$ is convex and lower semicontinuous; for arbitrary $x \in H$,

$$\varpi\|x\| + \mathcal{C} \leq g(x) < +\infty, \tag{4.2}$$

for some $\varpi > 0$ and $\mathcal{C} \geq 0$.

Theorem 4.1. *In addition to assumptions of Theorem 3.3, Assumptions [H3] and [H4] hold. Then problem (P1) has at least one optimal control pair (x^*, u^*).*

Proof. By virtue of Theorem 3.3, it is sufficient to show the existence of optimal controls for problem (P2). Set

$$\inf\left\{\widetilde{L}(x, v) \mid \widetilde{L}(x, v), \text{overall } (x, v) \text{ as in } (3.14)\right\} = d. \tag{4.3}$$

If $d = +\infty$, there is nothing to prove. By Assumptions [H3] and [H4], we know $d \geq 0$.

Let (x_n, v_n) with $x_n \in PC([0, T_0]; H)$ and $v_n \in L^2(0, T_0; U)$ be a minimizing sequence for problem (P2). This means

$$d \leq \int_0^{T_0} \left(g(x_n(t)) + h(v_n(t) + F x_n(t))\right) dt \leq d + \frac{1}{n}, \quad n = 1, 2, \ldots. \tag{4.4}$$

Set

$$u_n(t) = v_n(t) + F x_n(t). \tag{4.5}$$

It is obvious that (4.4) implies that

$$\int_0^{T_0} h(u_n(t)) dt \leq d + 1. \tag{4.6}$$

Let E be any measurable subset of $[0,T_0]$ and $\sigma > 0$. Clearly, $E = E_1 \cup E_2$ with $E_1 = E \cap \{t; \|u_n(t)\|_U < \sigma\}$ and $E_2 = E \cap \{t; \|u_n(t)\|_U \geq \sigma\}$.

It can be seen from Assumption [H3] that there exists $\phi(\cdot)$ such that

$$h(u) \geq \phi(\sigma)\|u\|_U, \quad \forall \|u\|_U \geq \sigma, \tag{4.7}$$

where

$$\lim_{\sigma\to\infty} \phi(\sigma) = +\infty. \tag{4.8}$$

By standard argument, we have

$$\begin{aligned}
&\int_E \|u_n(t)\|_U dt \\
&\quad = \int_{E_1} \|u_n(t)\|_U dt + \int_{E_2} \|u_n(t)\|_U dt \leq \sigma m(E_1) + \frac{1}{\phi(\sigma)}\int_0^{T_0} h(u_n(t))dt \leq \sigma m(E) + \frac{d+1}{\phi(\sigma)}.
\end{aligned} \tag{4.9}$$

This implies that the set $\{u_n\}$ is uniformly integrable on $[0,T_0]$. In view of the Dunford-Petties theorem, (4.9) implies that $\{u_n\}$ is sequentially weakly compact in $L^1(0,T_0;U)$. Say $u_n \to u^*$ weakly in $L^1(0,T_0;U)$.

Moreover, (4.2) and (4.4) imply

$$\int_0^{T_0} \|x_n(t)\| dt \leq \frac{1}{\varpi}\int_0^{T_0} \left(g(x_n(t)) + h(u_n(t))\right)dt \leq \frac{d+1}{\varpi}. \tag{4.10}$$

Taking into account that the pair (x_n, v_n) satisfies

$$\begin{aligned}
x_n(t) &= S_F(t,0)x_n(0) + \int_0^t S_F(t,\theta)Bv_n(\theta)d\theta, \\
x_n(0) &= -[I + S_F(T_0,0)]^{-1}\int_0^{T_0} S_F(T_0,\theta)Bv_n(\theta)d\theta.
\end{aligned} \tag{4.11}$$

It comes from (4.11) and (4.10) that

$$\begin{aligned}
\|x_n(t)\| &\leq \|S_F(t,0)x_n(0)\| + \int_0^t \|S_F(t,\theta)Bv_n(\theta)\|d\theta \\
&\leq M_{T_0}\|x_n(0)\| + M_{T_0}\int_0^t \|Bv_n(\theta)\|d\theta \\
&\leq M_{T_0}\left\|[I + S_F(T_0,0)]^{-1}\right\| M_{T_0}\int_0^{T_0} \|Bv_n(\theta)d\theta\| + M_{T_0}\int_0^t \|Bv_n(\theta)\|d\theta \\
&\leq M_{T_0}\left(M_{T_0}\left\|[I + S_F(T_0,0)]^{-1}\right\| + 1\right)\|B\|_{\pounds_b(U,H)}\left(\int_0^{T_0} \|v_n(\theta)\|^2 d\theta\right)^{1/2} \\
&\leq M_{T_0}(M_{T_0}\|Q\| + 1)\|B\|_{\pounds_b(U,H)}\|v_n\|_{L^2(0,T_0;U)},
\end{aligned} \tag{4.12}$$

which deduce that there exists $M > 0$ such that

$$\|x_n(t)\| \leq M, \quad \text{for } t \in [0, T_0], \tag{4.13}$$

that is, $\{x_n\}$ is bounded in Banach space $(L^1(0, T_0; H))^* = L^\infty(0, T_0; H)$. By Alaoglu theorem, we have $x_n \to x^*$ weakly star convergent in $L^\infty(0, T_0; H)$.

Set $v_n = u_n - Fx_n$ and $F \in \pounds_b(H, U)$, then

$$v_n \longrightarrow u^* - Fx^* = v^* \quad \text{weakly in } L^1(0, T_0; U). \tag{4.14}$$

There exists a function $\tilde{x}(\cdot) : [0, T_0] \to H$ such that

$$\tilde{x}(t) = S_F(t, 0)\tilde{x}(0) + \int_0^t S_F(t, \theta) B v^*(\theta) d\theta \tag{4.15}$$

with

$$\tilde{x}(0) = -[I + S_F(T_0, 0)]^{-1} \int_0^{T_0} S_F(T_0, \theta) B v^*(\theta) d\theta. \tag{4.16}$$

Clearly,

$$x_n(t) \longrightarrow \tilde{x}(t) \quad \text{weakly convergent in } H, \text{for each } t \in [0, T_0]. \tag{4.17}$$

One can verify $x_n \to \tilde{x}$ weakly convergent in $L^1(0, T_0; H)$. This implies that $\tilde{x} = x^*$. Hence x^* is the T_0-antiperiod PC-mild solution of system (3.11) corresponding to the control $v \in L^2(0, T; U)$ given by

$$x^*(t) = S_F(t, 0)x^*(0) + \int_0^t S_F(t, \theta) B v^*(\theta) d\theta \tag{4.18}$$

with

$$x^*(0) = -[I + S_F(T_0, 0)]^{-1} \int_0^{T_0} S_F(T_0, \theta) B v^*(\theta) d\theta. \tag{4.19}$$

Letting $n \to \infty$ in (4.4), using Assumptions [H3] and [H4] again, by [15, Theorem 2.1], we can obtain

$$d = \lim_{n \to \infty} \int_0^{T_0} \left(g(x_n(t)) + h(v_n(t) + Fx_n(t))\right) dt \geq \int_0^{T_0} \left(g(x^*(t)) + h(v^*(t) + Fx^*(t))\right) dt \geq d. \tag{4.20}$$

Thus, we can conclude that $d = \tilde{L}(x^*, v^*)$. In fact, let $u^* = v^* + Fx^*$; $(x^*, u^*) \in U_{\text{ad}}$ is the optimal pair for problem (P1). □

5. An Example

Let $H = L^2(0,1)$ and let $\phi_n(x)$, $n = 1,2,\ldots$, be an orthogonal basis for $L^2(0,1)$.

Minimize

$$\int_0^{T_0}\int_0^1 g_0(y,x)\,dy\,dt + \int_0^{T_0} h(u(t))\,dt \tag{5.1}$$

subject to

$$u \in L^2((0,1)\times(0,T_0)), \qquad x \in PC([0,T_0];H), \tag{5.2}$$

related by the following antiperiodic boundary value problem with impulse:

$$\begin{gathered}
\frac{\partial}{\partial t}x(t,y) = Ax(t,y) + 2Iu(t,y), \quad y \in (0,1), t > 0, t \in [0,2\pi] \setminus \widetilde{D} = \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}, \\
x(t,0) = x(t,1) = 0, \quad t > 0, \\
\Delta x(t,y) = \begin{cases} 0.05Ix(t,y), & k = 1, \\ -0.05Ix(t,y), & k = 2, \\ 0.05Ix(t,y), & k = 3, \end{cases} \qquad y \in (0,1),\ t > 0,\ \tau_1 = \frac{\pi}{2},\ \tau_2 = \pi,\ \tau_3 = \frac{3\pi}{2}, \\
x(0,y) = -x(2\pi,y), \quad \text{in}(0,1).
\end{gathered} \tag{5.3}$$

Let $g_0 : (0,1)\times\mathbb{R} \to \mathbb{R}$ and h: $L^2(0,1) \to \overline{\mathbb{R}}$ satisfy (4.1) and Assumptions [H3] and [H4]. The operator A is defined as follows:

$$A\phi_n = \left(-\frac{1}{n} + in\right)\phi_n, \quad n = 1,2,\ldots. \tag{5.4}$$

Then

$$T(t)\phi_n = e^{(-(1/n)+in)t}\phi_n, \tag{5.5}$$

and $T(t)$ is asymptotically stable but not exponentially stable.

Let $F = -2I$, then $A_F = A - 2I$ generates the C_0-semigroup $\{T_F(t), t \geq 0\}$ given by

$$T_F(t)\phi_n = e^{-(2+(1/n)-in)t}\phi_n. \tag{5.6}$$

Obviously, $\{T_F(t), t \geq 0\}$ is exponentially stable. By Lemma 2.5, there exists a

$$\lambda > \frac{\ln\left[(1.05)^2 \times 0.95\right]}{2\pi} \approx 0.0075; \tag{5.7}$$

then $\{S_F(t,\theta), t > \theta \geq 0\}$ is exponential stable. By Theorem 4.1, problem (5.1) has at least one optimal control pair (x^*, u^*).

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