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Research Article

On the Max-Type Difference Equation $x_{n+1} = \max\{A/x_n, x_{n-3}\}$

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We show that every well-defined solution of the fourth-order difference equation $x_{n+1} = \max\{A/x_n, x_{n-3}\}$, $n \in \mathbb{N}_0$, where parameter $A \geq 0$, is eventually periodic with period four.

1. Introduction

The study of max-type difference equations attracted recently a considerable attention, see, for example, [1–27], and the references listed therein. This type of difference equations stems from, for example, certain models in automatic control theory (see [28]). In the beginning of the study of these equations experts have been focused on the investigation of the behavior of some particular cases of the following general difference equation of order $k \in \mathbb{N}$:

$$x_n = \max\left\{\frac{A_n^{(1)}}{x_{n-1}}, \frac{A_n^{(2)}}{x_{n-2}}, \dots, \frac{A_n^{(k)}}{x_{n-k}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $k \in \mathbb{N}$, $A_n^{(i)}$, $i = 1, \dots, k$, are real sequences (mostly constant or periodic ones) and where the initial values $x_{-1}, \dots, x_{-k}$ are different from zero (see, e.g., [2, 3, 6, 7, 9–12, 22–25] and the references cited therein).

The study of max-type equations of the following general form

$$x_n = \max\left\{B_n^{(0)}, B_n^{(1)}\frac{x_{n-p_1}^{r_1}}{x_{n-q_1}^{s_1}}, B_n^{(2)}\frac{x_{n-p_2}^{r_2}}{x_{n-q_2}^{s_2}}, \dots, B_n^{(k)}\frac{x_{n-p_k}^{r_k}}{x_{n-q_k}^{s_k}}\right\}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $k \in \mathbb{N}$, p_i, q_i are natural numbers such that $p_1 < p_2 < \cdots < p_k$, $q_1 < q_2 < \cdots < q_k$, $r_i, s_i \in \mathbb{R}_+$ and $B_n^{(j)}$, $j = 0, 1, \ldots, k$, are sequences of real numbers, was proposed by Stević in numerous talks, for example, in [13, 14]. For some results in this direction see [1, 4, 15–17, 19–21, 26, 27]. For some nonlinear difference equations related to (1.2) see, for example, [7, 15, 17, 18, 29–38].

Definition 1.1. A sequence $(x_n)_{n=-k}^{\infty}$ is said to be *eventually periodic with period p* if there is an index $n_0 \in \{-k, \ldots, -1, 0, 1, \ldots\}$ such that $x_{n+p} = x_n$ for all $n \geq n_0$. Specially, if $n_0 = -k$, then the sequence $(x_n)_{n=-k}^{\infty}$ is *periodic with period p*.

Motivated by some ideas due to Stević (e.g., the main lemmas there, Lemmas 3.1 and 3.2 are suggested by him), the authors of [26] considered the following second-order max-type difference equation:

$$x_{n+1} = \max\left\{\frac{1}{x_n}, Ax_{n-1}\right\}, \quad n \in \mathbb{N}_0. \tag{1.3}$$

Equation(1.3) is not difficult for handling since, by the change $y_n = x_n x_{n-1}$, it is transformed into one of the following first-order difference equations

$$y_{n+1} = \max\{1, Ay_n\} \quad \text{or} \quad y_{n+1} = \min\{1, Ay_n\}. \tag{1.4}$$

Using these equations, it is easy to see that for the case $A = 1$ every solution of (1.3) is eventually periodic with period two.

Recently, in the paper [5] it was showed that every solution of the third-order max-type difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n}, x_{n-2}\right\}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary nonzero real numbers and $A \in \mathbb{R}$, is eventually periodic with period three. The fact that all solutions of (1.5) are periodic is not a surprising fact (for an explanation see [4]).

For some recent papers on difference equations all the solutions of which are periodic see, for example, [7, 39–45] and the references cited therein.

Here we show that every well-defined solution of the following fourth-order max-type difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n}, x_{n-3}\right\}, \quad n \in \mathbb{N}_0, \tag{1.6}$$

where the parameter $A \in \mathbb{R}_+ \cup \{0\}$, is eventually periodic with period four.

Remark 1.2. Note that if $A = 0$, then (1.6) becomes $x_{n+1} = x_{n-3}$, from which it follows that every solution is periodic with period four. Hence, in the sequel we will consider the case $A \neq 0$.

In the sequel we will frequently use the following simple lemma, given without a proof (for related results see [4, 5]).

Lemma 1.3. *Assume that* $(x_n)_{n=-3}^{\infty}$ *is a solution of* (1.6) *and there is* $k_0 \in \mathbb{N}_0 \cup \{-3, -2, -1\}$ *such that*

$$x_{k_0} = x_{k_0+4}, \qquad x_{k_0+1} = x_{k_0+5}, \qquad x_{k_0+2} = x_{k_0+6}, \qquad x_{k_0+3} = x_{k_0+7}. \tag{1.7}$$

Then this solution is eventually periodic with period four.

2. Main Results

In this subsection we give a specific form of the solutions of the difference equation (1.6) when the parameter $A > 0$ and in each case we can deduce that every solution of this equation is periodic with period four.

Depending on the positivity of four initial values of (1.6), there are the following 16 cases to be considered:

$$\begin{array}{ll}
\text{(i)} \quad x_{-3}, x_{-2}, x_{-1}, x_0 > 0, & \text{(ii)} \quad x_{-3}, x_{-2}, x_{-1}, x_0 < 0, \\
\text{(iii)} \quad x_0 < 0, \quad x_{-3}, x_{-2}, x_{-1} > 0, & \text{(iv)} \quad x_{-1} < 0, \quad x_{-3}, x_{-2}, x_0 > 0, \\
\text{(v)} \quad x_{-2} < 0, \quad x_{-3}, x_{-1}, x_0 > 0, & \text{(vi)} \quad x_{-3} < 0, \quad x_{-2}, x_{-1}, x_0 > 0, \\
\text{(vii)} \quad x_0, x_{-1} < 0, \quad x_{-3}, x_{-2} > 0, & \text{(viii)} \quad x_0, x_{-2} < 0, \quad x_{-3}, x_{-1} > 0, \\
\text{(ix)} \quad x_0, x_{-3} < 0, \quad x_{-1}, x_{-2} > 0, & \text{(x)} \quad x_{-1}, x_{-2} < 0, \quad x_0, x_{-3} > 0, \\
\text{(xi)} \quad x_{-1}, x_{-3} < 0, \quad x_0, x_{-2} > 0, & \text{(xii)} \quad x_{-2}, x_{-3} < 0, \quad x_0, x_{-1} > 0, \\
\text{(xiii)} \quad x_0, x_{-1}, x_{-2} < 0, \quad x_{-3} > 0, & \text{(xiv)} \quad x_0, x_{-1}, x_{-3} < 0, \quad x_{-2} > 0, \\
\text{(xv)} \quad x_0, x_{-2}, x_{-3} < 0, \quad x_{-1} > 0, & \text{(xvi)} \quad x_{-1}, x_{-2}, x_{-3} < 0, \quad x_0 > 0.
\end{array} \tag{2.1}$$

First, we prove another auxiliary result.

Lemma 2.1. *Assume that the parameter* $A > 0$. *Then every solution of* (1.6) *is eventually positive if initial values satisfy one of conditions (i), (iii)–(xvi).*

Proof. If $x_0 > 0$ or $x_{-3} > 0$, then

$$x_1 = \max\left\{\frac{A}{x_0}, x_{-3}\right\} > 0. \tag{2.2}$$

From this, (1.6), and by induction it follows that $x_n > 0$ for every $n \in \mathbb{N}_0$.

If $x_{-2} > 0$, then

$$x_2 = \max\left\{\frac{A}{x_1}, x_{-2}\right\} > 0. \tag{2.3}$$

From this, (1.6), and by induction it follows that $x_n > 0$ for every $n \geq 2$.

If $x_{-1} > 0$, then

$$x_3 = \max\left\{\frac{A}{x_1}, x_{-1}\right\} > 0. \tag{2.4}$$

Similar to the previous case, by induction it follows that $x_n > 0$ for every $n \geq 3$. □

Now, we can formulate and prove our main results.

Theorem 2.2. *Assume that the parameter $A > 0$. Then every solution of* (1.6) *with positive initial values is eventually periodic with period four.*

Proof. From (1.6), we see that

$$x_1 = \max\left\{\frac{A}{x_0}, x_{-3}\right\}. \tag{2.5}$$

We consider the following two cases.

(a_1) $x_1 = A/x_0$. In this case $A/x_0 \geq x_{-3}$, and we see that

$$x_2 = \max\left\{\frac{A}{x_1}, x_{-2}\right\} = \max\{x_0, x_{-2}\}. \tag{2.6}$$

Now, there exists two subcases.

(a_{11}) $x_2 = x_0$, which occurs when $x_0 \geq x_{-2}$. We have

$$x_3 = \max\left\{\frac{A}{x_2}, x_{-1}\right\} = \max\left\{\frac{A}{x_0}, x_{-1}\right\}. \tag{2.7}$$

(a_{111}) If $x_{-1} \geq A/x_0$, then $x_3 = x_{-1}$, and

$$\begin{aligned} x_4 &= \max\left\{\frac{A}{x_3}, x_0\right\} = \max\left\{\frac{A}{x_{-1}}, x_0\right\} = x_0, \\ x_5 &= \max\left\{\frac{A}{x_4}, x_1\right\} = \max\left\{\frac{A}{x_0}, \frac{A}{x_0}\right\} = \frac{A}{x_0}, \\ x_6 &= \max\left\{\frac{A}{x_5}, x_2\right\} = \max\{x_0, x_0\} = x_0. \end{aligned} \tag{2.8}$$

Hence, $x_3 = x_{-1}$, $x_4 = x_0$, $x_5 = x_1$, and $x_6 = x_2$, which implies that $(x_n)_{n=-3}^{\infty}$ is an eventually (from x_{-1}) periodic solution with period four. In this case we see that the solution has the following form:

$$\left(x_{-3}, x_{-2}, x_{-1}, x_0, \frac{A}{x_0}, x_0, x_{-1}, x_0, \frac{A}{x_0}, x_0, \ldots\right). \tag{2.9}$$

(a_{112}) If $A/x_0 \geq x_{-1}$, then $x_3 = A/x_0$, and

$$\begin{aligned} x_4 &= \max\left\{\frac{A}{x_3}, x_0\right\} = \max\{x_0, x_0\} = x_0, \\ x_5 &= \max\left\{\frac{A}{x_4}, x_1\right\} = \max\left\{\frac{A}{x_0}, \frac{A}{x_0}\right\} = \frac{A}{x_0}, \\ x_6 &= \max\left\{\frac{A}{x_5}, x_2\right\} = \max\{x_0, x_0\} = x_0, \\ x_7 &= \max\left\{\frac{A}{x_6}, x_3\right\} = \max\left\{\frac{A}{x_0}, \frac{A}{x_0}\right\} = \frac{A}{x_0}. \end{aligned} \tag{2.10}$$

Hence, $x_4 = x_0$, $x_5 = x_1$, $x_6 = x_2$, and $x_7 = x_3$, which implies that $(x_n)_{n=-3}^{\infty}$ is an eventually (from x_0) periodic solution with period four (in this case minimal period is two). This solution takes the form

$$\left(x_{-3}, x_{-2}, x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{x_0}, x_0, \frac{A}{x_0}, x_0, \frac{A}{x_0}, \ldots\right). \tag{2.11}$$

(a_{12}) $x_2 = x_{-2}$, which occurs when $x_{-2} \geq x_0$, and

$$x_3 = \max\left\{\frac{A}{x_2}, x_{-1}\right\} = \max\left\{\frac{A}{x_{-2}}, x_{-1}\right\}. \tag{2.12}$$

(a_{121}) If $x_{-1} \geq A/x_{-2}$, then $x_3 = x_{-1}$, and

$$x_4 = \max\left\{\frac{A}{x_3}, x_0\right\} = \max\left\{\frac{A}{x_{-1}}, x_0\right\}. \tag{2.13}$$

(a_{1211}) If $x_0 \geq A/x_{-1}$, then $x_4 = x_0$, and

$$x_5 = \max\left\{\frac{A}{x_4}, x_1\right\} = \max\left\{\frac{A}{x_0}, \frac{A}{x_0}\right\} = \frac{A}{x_0}. \tag{2.14}$$

Hence, $x_2 = x_{-2}$, $x_3 = x_{-1}$, $x_4 = x_0$, and $x_5 = x_1$, which implies that $(x_n)_{n=-3}^{\infty}$ is an eventually (from x_{-2}) periodic solution with period four. It can be written in the form

$$(x_n)_{n=-3}^{\infty} = \left(x_{-3}, x_{-2}, x_{-1}, x_0, \frac{A}{x_0}, x_{-2}, x_{-1}, x_0, \frac{A}{x_0}, \dots\right). \tag{2.15}$$

(a_{1212}) If $A/x_{-1} \geq x_0$, then $x_4 = A/x_{-1}$, and

$$\begin{aligned}
x_5 &= \max\left\{\frac{A}{x_4}, x_1\right\} = \max\left\{x_{-1}, \frac{A}{x_0}\right\} = \frac{A}{x_0},\\
x_6 &= \max\left\{\frac{A}{x_5}, x_2\right\} = \max\{x_0, x_{-2}\} = x_{-2},\\
x_7 &= \max\left\{\frac{A}{x_6}, x_3\right\} = \max\left\{\frac{A}{x_{-2}}, x_{-1}\right\} = x_{-1},\\
x_8 &= \max\left\{\frac{A}{x_7}, x_4\right\} = \max\left\{\frac{A}{x_{-1}}, \frac{A}{x_{-1}}\right\} = \frac{A}{x_{-1}}.
\end{aligned} \tag{2.16}$$

Hence, $x_5 = x_1$, $x_6 = x_2$, $x_7 = x_3$, and $x_8 = x_4$, which implies that $(x_n)_{n=-3}^{\infty}$ is an eventually (from x_1) periodic solution with period four. Moreover, it can be written as follows:

$$(x_n)_{n=-3}^{\infty} = \left(x_{-3}, x_{-2}, x_{-1}, x_0, \frac{A}{x_0}, x_{-2}, x_{-1}, \frac{A}{x_{-1}}, \frac{A}{x_0}, x_{-2}, x_{-1}, \frac{A}{x_{-1}}, \dots\right). \tag{2.17}$$

(a_{122}) If $A/x_{-2} \geq x_{-1}$, then $x_3 = A/x_{-2}$, and

$$\begin{aligned}
x_4 &= \max\left\{\frac{A}{x_3}, x_0\right\} = \max\{x_{-2}, x_0\} = x_{-2},\\
x_5 &= \max\left\{\frac{A}{x_4}, x_1\right\} = \max\left\{\frac{A}{x_{-2}}, \frac{A}{x_0}\right\} = \frac{A}{x_0},\\
x_6 &= \max\left\{\frac{A}{x_5}, x_2\right\} = \max\{x_0, x_{-2}\} = x_{-2},\\
x_7 &= \max\left\{\frac{A}{x_6}, x_3\right\} = \max\left\{\frac{A}{x_{-2}}, \frac{A}{x_{-2}}\right\} = \frac{A}{x_{-2}},\\
x_8 &= \max\left\{\frac{A}{x_7}, x_4\right\} = \max\{x_{-2}, x_{-2}\} = x_{-2}.
\end{aligned} \tag{2.18}$$

As above, the solution is eventually (from x_1) periodic with period four and it has the form

$$\left(x_{-3}, x_{-2}, x_{-1}, x_0, \frac{A}{x_0}, x_{-2}, \frac{A}{x_{-2}}, x_{-2}, \frac{A}{x_0}, x_{-2}, \frac{A}{x_{-2}}, x_{-2}, \frac{A}{x_0}, \dots\right). \tag{2.19}$$

(a_2) $x_1 = x_{-3}$. In this case $x_{-3} \geq A/x_0$, and we see that

$$x_2 = \max\left\{\frac{A}{x_1}, x_{-2}\right\} = \max\left\{\frac{A}{x_{-3}}, x_{-2}\right\}. \tag{2.20}$$

There again exist two subcases.

(a_{21}) $x_2 = A/x_{-3}$, which occurs when $A/x_{-3} \geq x_{-2}$. So,

$$x_3 = \max\left\{\frac{A}{x_2}, x_{-1}\right\} = \max\{x_{-3}, x_{-1}\}. \tag{2.21}$$

(a_{211}) If $x_{-3} \geq x_{-1}$, then $x_3 = x_{-3}$, and

$$\begin{aligned} x_4 &= \max\left\{\frac{A}{x_3}, x_0\right\} = \max\left\{\frac{A}{x_{-3}}, x_0\right\} = x_0, \\ x_5 &= \max\left\{\frac{A}{x_4}, x_1\right\} = \max\left\{\frac{A}{x_0}, x_{-3}\right\} = x_{-3}, \\ x_6 &= \max\left\{\frac{A}{x_5}, x_2\right\} = \max\left\{\frac{A}{x_{-3}}, \frac{A}{x_{-3}}\right\} = \frac{A}{x_{-3}}, \\ x_7 &= \max\left\{\frac{A}{x_6}, x_3\right\} = \max\{x_{-3}, x_{-3}\} = x_{-3}. \end{aligned} \tag{2.22}$$

Then we see that the solution is

$$\left(x_{-3}, x_{-2}, x_{-1}, x_0, x_{-3}, \frac{A}{x_{-3}}, x_{-3}, x_0, x_{-3}, \frac{A}{x_{-3}}, x_{-3}, x_0, x_{-3}, \frac{A}{x_{-3}}, \ldots\right), \tag{2.23}$$

and $(x_n)_{n=-3}^{\infty}$ is an eventually (from x_0) periodic solution with period four.

(a_{212}) If $x_{-1} \geq x_{-3}$, then $x_3 = x_{-1}$, and

$$\begin{aligned} x_4 &= \max\left\{\frac{A}{x_3}, x_0\right\} = \max\left\{\frac{A}{x_{-1}}, x_0\right\} = x_0, \\ x_5 &= \max\left\{\frac{A}{x_4}, x_1\right\} = \max\left\{\frac{A}{x_0}, x_{-3}\right\} = x_{-3}, \\ x_6 &= \max\left\{\frac{A}{x_5}, x_2\right\} = \max\left\{\frac{A}{x_{-3}}, \frac{A}{x_{-3}}\right\} = \frac{A}{x_{-3}}. \end{aligned} \tag{2.24}$$

So, the solution takes the following form which is an eventually (from x_{-1}) periodic solution with period four:

$$\left(x_{-3}, x_{-2}, x_{-1}, x_0, x_{-3}, \frac{A}{x_{-3}}, x_{-1}, x_0, x_{-3}, \frac{A}{x_{-3}}, \ldots\right). \tag{2.25}$$

(a_{22}) $x_2 = x_{-2}$, which occurs when $x_{-2} \geq A/x_{-3}$. So,

$$x_3 = \max\left\{\frac{A}{x_2}, x_{-1}\right\} = \max\left\{\frac{A}{x_{-2}}, x_{-1}\right\}. \tag{2.26}$$

(a_{221}) If $x_{-1} \geq A/x_{-2}$, then $x_3 = x_{-1}$, and

$$x_4 = \max\left\{\frac{A}{x_3}, x_0\right\} = \max\left\{\frac{A}{x_{-1}}, x_0\right\}. \tag{2.27}$$

(a_{2211}) If $x_0 \geq A/x_{-1}$, then $x_4 = x_0$.

Therefore $(x_n)_{n=-3}^{\infty}$ is a periodic solution with period four and the solution takes the form

$$(x_{-3}, x_{-2}, x_{-1}, x_0, x_{-3}, x_{-2}, x_{-1}, x_0, \ldots). \tag{2.28}$$

(a_{2212}) If $A/x_{-1} \geq x_0$, then $x_4 = A/x_{-1}$, and

$$\begin{aligned}
x_5 &= \max\left\{\frac{A}{x_4}, x_1\right\} = \max\{x_{-1}, x_{-3}\} = x_{-3},\\
x_6 &= \max\left\{\frac{A}{x_5}, x_2\right\} = \max\left\{\frac{A}{x_{-3}}, x_{-2}\right\} = x_{-2},\\
x_7 &= \max\left\{\frac{A}{x_6}, x_3\right\} = \max\left\{\frac{A}{x_{-2}}, x_{-1}\right\} = x_{-1},\\
x_8 &= \max\left\{\frac{A}{x_7}, x_4\right\} = \max\left\{\frac{A}{x_{-1}}, \frac{A}{x_{-1}}\right\} = \frac{A}{x_{-1}}.
\end{aligned} \tag{2.29}$$

Therefore, again $(x_n)_{n=-3}^{\infty}$ is an eventually (from x_1) periodic solution with period four and the solution takes the form

$$\left(x_{-3}, x_{-2}, x_{-1}, x_0, x_{-3}, x_{-2}, x_{-1}, \frac{A}{x_{-1}}, x_{-3}, x_{-2}, x_{-1}, \frac{A}{x_{-1}}, \ldots\right). \tag{2.30}$$

(a_{222}) If $A/x_{-2} \geq x_{-1}$, then $x_3 = A/x_{-2}$, and

$$x_4 = \max\left\{\frac{A}{x_3}, x_0\right\} = \max\{x_{-2}, x_0\}. \tag{2.31}$$

(a_{2221}) If $x_{-2} \geq x_0$, then $x_4 = x_{-2}$, and

$$\begin{aligned}
x_5 &= \max\left\{\frac{A}{x_4}, x_1\right\} = \max\left\{\frac{A}{x_{-2}}, x_{-3}\right\} = x_{-3}, \\
x_6 &= \max\left\{\frac{A}{x_5}, x_2\right\} = \max\left\{\frac{A}{x_{-3}}, x_{-2}\right\} = x_{-2}, \\
x_7 &= \max\left\{\frac{A}{x_6}, x_3\right\} = \max\left\{\frac{A}{x_{-2}}, \frac{A}{x_{-2}}\right\} = \frac{A}{x_{-2}}, \\
x_8 &= \max\left\{\frac{A}{x_7}, x_4\right\} = \max\{x_{-2}, x_{-2}\} = x_{-2}.
\end{aligned} \tag{2.32}$$

Thus, the solution is in the following form which is eventually (from x_1) periodic with period four:

$$\left(x_{-3}, x_{-2}, x_{-1}, x_0, x_{-3}, x_{-2}, \frac{A}{x_{-2}}, x_{-2}, x_{-3}, x_{-2}, \frac{A}{x_{-2}}, x_{-2}, \ldots\right). \tag{2.33}$$

(a_{2222}) If $x_0 \geq x_{-2}$, then $x_4 = x_0$, and

$$\begin{aligned}
x_5 &= \max\left\{\frac{A}{x_4}, x_1\right\} = \max\left\{\frac{A}{x_0}, x_{-3}\right\} = x_{-3}, \\
x_6 &= \max\left\{\frac{A}{x_5}, x_2\right\} = \max\left\{\frac{A}{x_{-3}}, x_{-2}\right\} = x_{-2}, \\
x_7 &= \max\left\{\frac{A}{x_6}, x_3\right\} = \max\left\{\frac{A}{x_{-2}}, \frac{A}{x_{-2}}\right\} = \frac{A}{x_{-2}}.
\end{aligned} \tag{2.34}$$

Thus, the solution is of the following form

$$\left(x_{-3}, x_{-2}, x_{-1}, x_0, x_{-3}, x_{-2}, \frac{A}{x_{-2}}, x_0, x_{-3}, x_{-2}, \frac{A}{x_{-2}}, \ldots\right). \tag{2.35}$$

Therefore $(x_n)_{n=-3}^{\infty}$ is eventually (from x_0) periodic with period four. The proof is completed. □

From Lemma 1.3 and Theorem 2.2 we obtain the following result.

Theorem 2.3. *Assume the parameter $A > 0$ and that initial values of* (1.6) *satisfy one of conditions (i), (iii)–(xvi) in Lemma 2.1. Then every such solution of* (1.6) *is eventually periodic with period four.*

Proof. If initial values of (1.6) satisfy one of conditions (i), (iii)–(xvi) in Lemma 2.1, then by the same lemma it follows that the corresponding solution is eventually positive. This means that there is $k_1 \in \mathbb{N}_0 \cup \{-3, -2, -1\}$ such that $x_n > 0$ for every $n \geq k_1$. In particular, we have that $x_{k_1}, x_{k_1+1}, x_{k_1+2}, x_{k_1+3} > 0$. Since equation (1.6) is autonomous if $(x_n)_{n=-3}^{\infty}$ is a solution of (1.6), then $y_n = x_{n+k_1+3}$ is also a solution of (1.6) but such that $y_{-3}, y_{-2}, y_{-1}, y_0 > 0$. Hence, the problem is reduced to the case when all the initial values are positive. Applying Theorem 2.2, the result follows. □

In the next theorem we study those solutions of (1.6) such that $x_{-3}, x_{-2}, x_{-1}, x_0 < 0$. We would like to thank Professor Stević for giving us the elegant proof below which drastically reduced our original proof.

Theorem 2.4. *Assume that the parameter $A > 0$ and all the initial values are negative $x_{-3}, x_{-2}, x_{-1}, x_0 < 0$. Then every solution of* (1.6) *is eventually periodic with period four.*

Proof. Since $x_{-3} < 0$, $x_{-2} < 0$, $x_{-1} < 0$, $x_0 < 0$, and $A > 0$, by induction we have $x_n < 0$ for each $n \in \mathbb{N}$. By the change $x_n = -\sqrt{A}/z_n$, (1.6) becomes

$$z_{n+1} = \left(\min\left\{z_n, \frac{1}{z_{n-3}}\right\}\right)^{-1} = \max\left\{\frac{1}{z_n}, z_{n-3}\right\}, \tag{2.36}$$

where $z_n > 0$ for every $n \geq -3$. Hence by Theorem 2.2 the result follows. □

Since all the cases are reduced to the case when all initial values are positive, it is of interest to investigate this case in more details. The next theorem describes all eventually constant solutions of (1.6). Our idea stems from [46] (see also [47–50]).

Theorem 2.5. *Assume that the parameter $A > 0$. Then all positive solutions to* (1.6) *which are eventually equal to the positive equilibrium have the following form:*

$$(1, c, b, 1, 1, 1, \ldots), \quad \textit{for some } b, c \in (0, 1]. \tag{2.37}$$

Proof. First note that by the change $x_n = \sqrt{A} y_n$ the problem is reduced to the case $A = 1$. Assume that $x_n = 1$ for $n \geq k$ and $x_{k-1} = a \neq 1$. Then $1 = x_k = \max\{1/a, x_{k-4}\}$, $k \geq 1$, which implies $x_{k-4} = 1$ and $a > 1$. On the other hand, we have $1 = x_{k+3} = \max\{1/x_{k+2}, x_{k-1}\} = \max\{1, a\}$, so that $a < 1$, which is a contradiction. Hence $k \leq 0$, that is, $(x_n) = (d, c, b, 1, 1, 1, \ldots)$. Since $x_1 = 1$, we get $x_{-3} = 1$ (case $k = 1$ above). Since $1 = x_2 = \max\{x_1, x_{-2}\} = \max\{1, c\}$ and $1 = x_3 = \max\{x_2, x_{-1}\} = \max\{1, b\}$, it follows that $b, c \in (0, 1]$, as claimed. □

3. Conclusions and Future Works

We finish this article with some comments which can motivate further works.

Case $A < 0$

The case when the parameter A is negative is not treated in this article. By similar calculations as above, we have managed to show that all the solutions of (1.6) are eventually periodic with period four in many subcases. However, there are too many subcases and calculations without some new ideas, so we decided not to present any result when $A < 0$. We conjecture that every solution of (1.6), when $A < 0$, is eventually periodic with the same period four. What is more interesting is to find a reasonably short proof of the conjecture without using tiresome calculations similar to those in Theorem 2.2.

Periodicity

We also want to mention that in [4] we proved that every solution of the equation

$$x_{n+1} = \max\left\{ \frac{A}{x_{n-k}}, x_{n-l} \right\}, \quad n \in \mathbb{N}_0, \tag{3.1}$$

where $k, l \in \mathbb{N}_0$, is periodic. It is of some interest to find the minimal period of the equation, as well as to get a general result concerning this problem.

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