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Research Article

Spectrum of Linear Difference Operators and the Solvability of Nonlinear Discrete Problems

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Let $T \in \mathbb{N}$ with $T > 5$. Let $\mathbb{T} := \{1, \dots, T\}$. We study the Fučik spectrum Σ of the discrete problem $\Delta^2 u(t-1) + \lambda u^+(t) - \mu u^-(t) = 0$, $t \in \mathbb{T}$, $u(0) = u(T+1) = 0$, where $u^+(t) = \max\{u(t), 0\}$, $u^-(t) = \max\{-u(t), 0\}$. We give an expression of Σ via the matching-extension method. We also use such discrete spectrum theory to study nonlinear boundary value problems of difference equations at resonance.

1. Introduction

Let $T \in \mathbb{N}$ with $T > 5$. Let $\mathbb{T} := \{1, \dots, T\}$ and $\widehat{\mathbb{T}} := \{0, 1, \dots, T, T+1\}$. For $u : \mathbb{T} \to \mathbb{R}$, we define $u^+, u^- : \mathbb{T} \to \mathbb{R}$ by

$$u^+(t) = \max\{u(t), 0\}, \qquad u^-(t) = \max\{-u(t), 0\}. \tag{1.1}$$

The Fučik spectrum of the problem

$$\begin{gathered} \Delta^2 u(t-1) + \lambda u^+(t) - \mu u^-(t) = 0, \quad t \in \mathbb{T}, \\ u(0) = u(T+1) = 0 \end{gathered} \tag{1.2}$$

is defined as the set Σ of those (λ, μ) such that (1.2) has nontrivial solutions.

In the past thirty years, the Fučik spectrum of two-point boundary value problems of ordinary differential equations has been extensively studied, see Fučik [1, 2], Ruf [3], and Rynne [4] and references therein. A typical result is the following.

Theorem 1.1. *The problem*

$$u''(x) + \lambda u^+(x) - \mu u^-(x) = 0, \quad x \in (0,1),$$
$$u(0) = u(1) = 0 \tag{1.3}$$

has a nontrivial solution for $(\lambda, \mu) \in \mathbb{R}^2$ *if and only if* (λ, μ) *satisfies*

(i) *for* $k \geq 1$ *odd, either*

$$\frac{k+1}{2}\frac{1}{\sqrt{\lambda}} + \frac{k-1}{2}\frac{1}{\sqrt{\mu}} = 1, \tag{1.4}$$

or

$$\frac{k-1}{2}\frac{1}{\sqrt{\lambda}} + \frac{k+1}{2}\frac{1}{\sqrt{\mu}} = 1, \tag{1.5}$$

(ii) *for* $k \geq 2$ *even*

$$\frac{k}{2}\frac{1}{\sqrt{\lambda}} + \frac{k}{2}\frac{1}{\sqrt{\mu}} = 1. \tag{1.6}$$

Of course, the natural question is whether a similar result can be established for the discrete analogue (1.2). The purpose of this paper is to study the structure of Fučik spectrum Σ for (1.2). We give an expression of Σ via the *matching-extension* method in Sections 2 and 3. In Sections 4 and 5, we also show that Σ contains a curve $\Sigma \cap \mathrm{proj}_{\mathbb{R}^2}\mathcal{S}_2^+$ (for the definition of $\mathrm{proj}_{\mathbb{R}^2}\mathcal{S}_2^+$, see Definition 5.1 below), which is continuous, strictly decreasing, symmetric with respect to the diagonal, and for each $(\lambda, \mu) \in \Sigma \cap \mathrm{proj}_{\mathbb{R}^2}\mathcal{S}_2^+$, u has exactly one simple generalized zero in the interval $(1, T)$. Finally, in Section 6, we apply our Fučik spectrum of (1.2) to study the solvability of nonlinear problem

$$\Delta^2 u(t-1) + \lambda_1 u(t) + g(t, u(t)) = f(t), \quad t \in \mathbb{T},$$
$$u(0) = u(T+1) = 0, \tag{1.7}$$

where $g : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f : \mathbb{T} \to \mathbb{R}$, λ_1 is the first eigenvalue of the linear problem

$$\Delta^2 u(t-1) + \lambda u(t) = 0, \quad t \in \mathbb{T},$$
$$u(0) = u(T+1) = 0. \tag{1.8}$$

2. Matching Continuation

For $r \in (0, \infty)$, we denote the integer part of r by $[r]$.

Denote

$$\mathbb{N}^* = \{0\} \cup \mathbb{N}. \tag{2.1}$$

Lemma 2.1. *For fixed $\lambda, \mu \in \mathbb{R}$, the initial value problem*

$$\Delta^2 u(t-1) + \lambda u^+(t) - \mu u^-(t) = 0, \quad t \in \mathbb{N}, \tag{2.2}$$

$$u(0) = 0, \qquad u(1) = 1 \tag{2.3}$$

has a unique solution $w_+ : \mathbb{N}^* \to R$.

Proof. Equation (2.2) is equivalent to the recurrence

$$u(t+1) = 2u(t) - u(t-1) - \lambda u^+(t) + \mu u^-(t), \quad t \in \mathbb{N}, \tag{2.4}$$

which guarantees the existence and uniqueness of solution $w_+ : \mathbb{N}^* \to R$. □

Lemma 2.2. *Let $w_+(t)$ be a nontrivial solution of the initial value problem* (2.2), (2.3) *and $w_+(t_0) = 0$ for some $t_0 \in \mathbb{T}$. Then,*

$$w_+(t_0 - 1) w_+(t_0 + 1) < 0. \tag{2.5}$$

Proof. It can be easily deduced from (2.4). □

Lemma 2.3. *Let $w_+(t, \lambda, \mu)$ be the unique solution of* (2.2), (2.3) *for fixed $\lambda, \ \mu \in \mathbb{R}$. Then, (λ, μ) is an eigenvalue of* (1.2) *if and only if*

$$w_+(T+1, \lambda, \mu) = 0. \tag{2.6}$$

To find (λ, μ) satisfying (2.6), we consider the linear initial value problem

$$\Delta^2 u(t-1) + \lambda u(t) = 0, \quad t \in \mathbb{N}, \tag{2.7}$$

$$u(0) = 0, \qquad u(1) = 1. \tag{2.8}$$

The characteristic equation of (2.7) is

$$r^2 + (\lambda - 2) r + 1 = 0, \tag{2.9}$$

so

$$r = \frac{2 - \lambda \pm \sqrt{(\lambda - 2)^2 - 4}}{2}. \tag{2.10}$$

If $|\lambda - 2| > 2$, then it can be shown that there are no eigenvalues of (1.2).

In fact, in this case,

$$r_1 := \frac{2 - \lambda + \sqrt{(\lambda - 2)^2 - 4}}{2}, \qquad r_2 := \frac{2 - \lambda - \sqrt{(\lambda - 2)^2 - 4}}{2}, \tag{2.11}$$

and the general solution of (2.7) is

$$u(t) = Ar_1^t + Br_2^t. \tag{2.12}$$

It is easy to check that the function $Ar_1^x + Br_2^x$, $x \in \mathbb{R}$ has at most one zero in $\mathbb{R}$ for every $A, B \in \mathbb{R}$ with $A^2 + B^2 \neq 0$.

If $|\lambda - 2| = 2$, then it can also be shown that there are no eigenvalues of (1.2).

In fact, $|\lambda - 2| = 2$ implies that

$$r_1 = r_2 := \frac{2 - \lambda}{2}, \tag{2.13}$$

and the general solution of (2.7) is

$$u(t) = Ar_1^t + Btr_1^t. \tag{2.14}$$

It is easy to check that the function $Ar_1^x + Bxr_1^x$, $x \in \mathbb{R}$ has at most one zero in $\mathbb{R}$ for every $A, B \in \mathbb{R}$ with $A^2 + B^2 \neq 0$.

Assume that $|\lambda - 2| < 2$ and set

$$2 - \lambda = 2\cos\theta. \tag{2.15}$$

Then,

$$r = \cos\theta \pm i \sin\theta = e^{\pm i\theta}. \tag{2.16}$$

Hence, a general solution of (2.7) is

$$u(t) = A\cos\theta t + B\sin\theta t. \tag{2.17}$$

From (2.8), it follows that $u(0) = A = 0$ and

$$B = \frac{1}{\sin\theta}. \tag{2.18}$$

Since $|\lambda - 2| < 2$, it follows from (2.15) that $\theta \neq k\pi$. Thus, B is well defined and

$$u(t) = \frac{1}{\sin\theta}\sin\theta t, \qquad t \in \mathbb{N}^*. \tag{2.19}$$

Lemma 2.4. *The first eigenvalue of the problem* (1.8) *is*

$$\lambda_1 = 2 - 2\cos\frac{\pi}{T+1}, \tag{2.20}$$

which is simple and the eigenfunction corresponding to λ_1 *is*

$$\varphi_1(t) = \sin\frac{\pi}{T+1}t, \quad t \in \mathbb{T}. \tag{2.21}$$

Proof. Applying the same method used in [5, Example 7.1], with obvious changes, we can get the desired result. □

Lemma 2.5. *(i) The line* $\{\lambda_1\} \times \mathbb{R}$ *and the set* $\mathbb{R} \times \{\lambda_1\}$ *are contained in* Σ.

(ii) Σ *is symmetric with respect to the straight line* $\lambda = \mu$.

Proof. (i) From the definition of λ_1 and φ_1 in (2.20) and (2.21), it follows that for any $\mu \in \mathbb{R}$,

$$\begin{gathered} \Delta^2\varphi_1(t-1) + \lambda_1(\varphi_1)^+(t) - \mu(\varphi_1)^-(t) = 0, \qquad t \in \mathbb{T}, \\ \varphi_1(0) = \varphi_1(T+1) = 0. \end{gathered} \tag{2.22}$$

This implies that $\{\lambda_1\} \times \mathbb{R} \subset \Sigma$. Similarly, $\mathbb{R} \times \{\lambda_1\} \subset \Sigma$.

(ii) Let $(\overline{\lambda}, \overline{\mu}) \in \Sigma$ and y be the corresponding eigenfunction. Then,

$$\begin{gathered} \Delta^2 y(t-1) + \overline{\lambda} y^+(t) - \overline{\mu} y^-(t) = 0, \quad t \in \mathbb{T}, \\ y(0) = y(T+1) = 0. \end{gathered} \tag{2.23}$$

Let $z(t) = -y(t)$ for $t \in \widehat{\mathbb{T}}$. Then,

$$\begin{gathered} \Delta^2 z(t-1) + \overline{\mu} z^+(t) - \overline{\lambda} z^-(t) = 0, \qquad t \in \mathbb{T}, \\ z(0) = z(T+1) = 0, \end{gathered} \tag{2.24}$$

which means that $(\overline{\mu}, \overline{\lambda}) \in \Sigma$. □

In order to construct a nontrivial solution of (2.2), (2.3), we extend u to the following function defined on real interval $[0, \infty)$:

$$\tilde{u}(x) = \frac{1}{\sin\theta}\sin\theta x, \quad x \in [0, \infty). \tag{2.25}$$

Let β_1 be the first positive zero of $\tilde{u}$, then

$$\tilde{u}(\beta_1) = 0, \quad \tilde{u}(x) > 0, \quad x \in (0, \beta_1), \tag{2.26}$$

$$\tilde{u}(x) = B_1 \sin \frac{\pi}{\beta_1} x, \tag{2.27}$$

with $B_1 := (\sin(\pi/\beta_1))^{-1}$. From (2.27) and (2.15), we have that

$$\lambda = 2 - 2\cos \frac{\pi}{\beta_1}, \tag{2.28}$$

that is,

$$\beta_1 = \frac{\pi}{\arccos((2-\lambda)/2)}. \tag{2.29}$$

Moreover, $\tilde{u}(x)$ satisfies that

$$\begin{gathered}
\Delta^2 \tilde{u}(t-1) + \lambda \tilde{u}(t) = 0, \quad t \in \{1, 2, \dots, [\beta_1]\}, \\
\tilde{u}(0) = 0, \qquad \tilde{u}(\beta_1) = 0, \\
\tilde{u}(t) > 0, \quad t \in \{1, \dots, [\beta_1 - 1]\}, \\
\tilde{u}([\beta_1]) > 0, \quad \text{if } \beta_1 > [\beta_1], \\
\tilde{u}([\beta_1]) = 0, \quad \text{if } \beta_1 = [\beta_1], \\
\tilde{u}([\beta_1] + 1) < 0,
\end{gathered} \tag{2.30}$$

where $[r]$ denotes the integer part of r.

Definition 2.6. Suppose that a function $y : \widehat{\mathbb{T}} \to \mathbb{R}$. If $y(t) = 0$, then t is a *zero* of y. If $y(t) = 0$ and $y(t-1)y(t+1) < 0$, then t is a *simple zero* of y. If $y(t-1)y(t) < 0$, then we say that y has a *node* at the point $s = t$. The nodes and simple zeros of y are called the *simple generalized zeros* of y.

Lemma 2.7. *Let*

$$\tilde{\Sigma} = \Sigma \setminus ((\{\lambda_1\} \times \mathbb{R}) \cup (\mathbb{R} \times \{\lambda_1\})). \tag{2.31}$$

Then, $(\lambda, \mu) \in \tilde{\Sigma}$ *implies that*

$$2 - 2\cos \frac{\pi}{T} < \lambda < 4, \qquad 2 - 2\cos \frac{\pi}{T} < \mu < 4. \tag{2.32}$$

Proof. From (2.20) and (2.28), we see that $(\lambda, \mu) \in \tilde{\tilde{\Sigma}}$ if and only if the eigenfunction corresponding to (λ, μ) has at least simple generalized zero in the open interval $(1, T)$. This means that $1 < \beta_1 < T$, which implies

$$2 - 2\cos\frac{\pi}{T} < \lambda < 4. \tag{2.33}$$

Similarly, we have that $2 - 2\cos(\pi/T) < \mu < 4$. □

Now, let (λ, μ) satisfy

(H) $2 - 2\cos(\pi/T) < \lambda < 4$, $2 - 2\cos(\pi/T) < \mu < 4$.

Let us consider the sequence $\{u(t, \lambda, \mu)\}$ induced by (2.4), (2.3), and let β_1 be the first positive zero of $\tilde{u}$ (see (2.27) for the definition of $\tilde{u}$). Then, we see from (2.4) that

$$u([\beta_1] + 1) = 2u([\beta_1]) - u([\beta_1] - 1) - \lambda u([\beta_1]), \tag{2.34}$$

which means that $u([\beta_1] + 1)$ is uniquely determined by $u([\beta_1])$, $u([\beta_1] - 1)$ and λ, and is independent of μ.

Now, let $v(t)$ be the unique solution of the initial value problem

$$\Delta^2 v(t-1) + \mu v(t) = 0, \quad t \in \{[\beta_1] - 1, [\beta_1], [\beta_1] + 1, \ldots\}, \tag{2.35}$$

$$v([\beta_1]) = u([\beta_1]), \qquad v([\beta_1] + 1) = u([\beta_1] + 1). \tag{2.36}$$

By a similar method to get $\tilde{u}$ and β_1, we get the following lemma.

Lemma 2.8. *Let (H) hold, and let b satisfy*

$$\mu = 2 - 2\cos\frac{\pi}{b}. \tag{2.37}$$

Then, $1 < b < T$.

Proof. From (2.37) and (H), it follows that

$$2 - 2\cos\frac{\pi}{T} < 2 - 2\cos\frac{\pi}{b} < 4. \tag{2.38}$$

This is

$$\cos\frac{\pi}{T} > cos\frac{\pi}{b} > cos\frac{\pi}{1}, \tag{2.39}$$

which implies $1 < b < T$. □

Lemma 2.9. *Let (H) hold. Then,* (2.35), (2.36) *has a solution of the form*

$$v(t) = B_2 \sin \frac{\pi(t-\alpha_2)}{b}, \tag{2.40}$$

where α_2 is the unique solution of

$$\begin{gathered} B_1 \sin\left(\frac{\pi[\beta_1]}{\beta_1}\right) = B_2 \sin \frac{\pi([\beta_1]-\alpha_2)}{b}, \\ B_1 \sin\left(\frac{\pi([\beta_1]+1)}{\beta_1}\right) = B_2 \sin \frac{\pi([\beta_1]+1-\alpha_2)}{b}, \end{gathered} \tag{2.41}$$

in $([\beta_1], [\beta_1+1])$ *if* $\beta_1 > [\beta_1]$*, and* $\alpha_2 = \beta_1$ *if* $\beta_1 = [\beta_1]$*;* B_2 *is a negative constant satisfying*

$$B_2 \sin \frac{\pi([\beta_1]+1-\alpha_2)}{b} = 2B_1 \sin \frac{\pi([\beta_1])}{\beta_1} - B_1 \sin \frac{\pi([\beta_1]-1)}{\beta_1} - \lambda B_1 \sin \frac{\pi([\beta_1])}{\beta_1}. \tag{2.42}$$

Proof. The relation (2.40) can be deduced by the similar method to get (2.27).

If $\beta_1 > [\beta_1]$, then we define a function $F : [[\beta_1], [\beta_1+1]] \to \mathbb{R}$ by

$$F(\alpha) = \sin \frac{\pi}{\beta_1}([\beta_1]) \sin \frac{\pi([\beta_1]+1-\alpha)}{b} - \sin \frac{\pi([\beta_1]-\alpha)}{b} \sin \frac{\pi([\beta_1]+1)}{\beta_1}. \tag{2.43}$$

It is easy to check that

$$\begin{gathered} F([\beta_1+1]) = -\sin \frac{\pi([\beta_1]-[\beta_1+1])}{b} \sin \frac{\pi}{\beta_1}([\beta_1+1]) < 0, \\ F([\beta_1]) = \sin \frac{\pi}{\beta_1}([\beta_1]) \sin \frac{\pi([\beta_1+1]-[\beta_1])}{b} > 0. \end{gathered} \tag{2.44}$$

Moreover, for $\alpha \in ([\beta_1], [\beta_1+1])$,

$$\begin{aligned} \frac{\partial F}{\partial \alpha}(\alpha) = &-\frac{\pi}{b} \sin \frac{\pi}{\beta_1}([\beta_1]) \cos \frac{\pi([\beta_1+1]-\alpha)}{b} \\ &+ \frac{\pi}{b} \cos \frac{\pi([\beta_1]-\alpha)}{b} \sin \frac{\pi}{\beta_1}([\beta_1+1]) < 0, \end{aligned} \tag{2.45}$$

(here, we use the fact that $b > 1$, see Lemma 2.8). So, there is a unique $\alpha_2 \in ([\beta_1], [\beta_1+1])$, such that $F(\alpha_2) = 0$.

If $\beta_1 = [\beta_1]$, then $v(\beta_1) = 0$. This together with (2.4) imply that

$$v(\beta_1 + 1) = -v(\beta_1 - 1), \tag{2.46}$$

that is,

$$B_2 \sin \frac{\pi(\beta_1 + 1 - \alpha_2)}{b} = -B_2 \sin \frac{\pi(\beta_1 - 1 - \alpha_2)}{b}. \tag{2.47}$$

So, we may take that

$$\alpha_2 = \beta_1. \tag{2.48}$$

Finally, it follows from (2.4) and the fact $u([\beta_1] + 1) = v([\beta_1] + 1)$ (see (2.36)) that B_2 satisfies (2.42). □

Now, for $x \in \mathbb{R}$, let

$$\tilde{v}(x) := B_2 \sin \frac{\pi(x - \alpha_2)}{b}, \tag{2.49}$$

which is an extension of the function $v : (\mathbb{N}^* \cap [\,[\beta_1 - 1], \infty)) \to \mathbb{R}$ defined in (2.40).

Denote

$$\beta_2 := \alpha_2 + b, \tag{2.50}$$

Then $\tilde{v}(x)$ satisfies

$$\begin{gathered}
\Delta^2 \tilde{v}(t-1) + \mu \tilde{v}(t) = 0, \quad t \in \{[\beta_1 + 1], \dots, [\beta_2]\}, \\
\tilde{v}([\beta_1]) = \tilde{u}([\beta_1]), \qquad \tilde{v}([\beta_1 + 1]) = \tilde{u}([\beta_1 + 1]), \qquad \tilde{v}(\beta_2) = 0, \\
\tilde{v}(t) < 0, \quad t \in \{[\beta_1 + 1], \dots, [\beta_2 - 1]\}, \\
\tilde{v}([\beta_2]) < 0, \quad \text{if } \beta_2 > [\beta_2], \qquad \tilde{v}([\beta_2]) = 0, \quad \text{if } \beta_2 = [\beta_2], \\
\tilde{v}([\beta_2] + 1) > 0.
\end{gathered} \tag{2.51}$$

Definition 2.10. We say that α_2 defined in Lemma 2.8 is the *feasible initial phase* of the family of functions $\sin(\pi(x-\alpha)/b)$. If α_2 is the feasible initial phase of the family of functions $\sin(\pi(x-\alpha)/b)$, then we say that $\tilde{v}(x)$

$$\tilde{v}(t) := B_2 \sin \frac{\pi(t - \alpha_2)}{b}, \quad t \in [[\beta_1], [\beta_2 + 1]] \tag{2.52}$$

is the *matching extension* of $\tilde{u}(x)$, $x \in [0, [\beta_1 + 1]]$.

Remark 2.11. Notice that $\tilde{v}(x) \not\equiv \tilde{u}(x)$ for $x \in [[\beta_1], [\beta_1 + 1]]$ in general. However, $\tilde{v}(t) \equiv \tilde{u}(t)$ for $t \in \{[\beta_1], [\beta_1 + 1]\}$.

Obviously, we may repeat the above process to "*extend*" $\tilde{u}(t)$ to a sequence defined on $\mathbb{N}$ in a unique manner. Moreover, this sequence coincides with the sequence $w_+(t)$ in Lemma 2.1.

Let $\alpha_1 := 0$. Recall that β_1 is defined by (2.29)

$$\beta_1 = \frac{\pi}{\arccos((2-\lambda)/2)}, \tag{2.53}$$

and $\alpha_2 = \beta_1$ if $\beta_1 = [\beta_1]$, and α_2 is the unique solution of

$$\sin \frac{\pi}{\beta_1}([\beta_1]) \sin \frac{\pi([\beta_1] + 1 - \alpha)}{b} - \sin \frac{\pi([\beta_1] - \alpha)}{b} \sin \frac{\pi([\beta_1] + 1)}{\beta_1} = 0. \tag{2.54}$$

Recall

$$\beta_2 := \alpha_2 + b, \tag{2.55}$$

with

$$b = \frac{\pi}{\arccos((2-\mu)/2)}, \tag{2.56}$$

Repeating the above process, we may take $\alpha_k = \beta_{k-1}$ if $\beta_{k-1} = [\beta_{k-1}]$, and α_k is the unique solution of

$$F_k(\alpha) = 0, \tag{2.57}$$

where

$$\begin{aligned} F_k(\alpha) := {} & \sin \frac{\pi([\beta_{k-1}] - \alpha_{k-1})}{\beta_{k-1} - \alpha_{k-1}} \sin \frac{\pi([\beta_{k-1} + 1] - \alpha)}{\beta_k - \alpha_k} \\ & - \sin \frac{\pi([\beta_{k-1}] - \alpha)}{\beta_k - \alpha_k} \sin \frac{\pi([\beta_{k-1} + 1] - \alpha_{k-1})}{\beta_{k-1} - \alpha_{k-1}}. \end{aligned} \tag{2.58}$$

Put

$$\beta_k = \begin{cases} \alpha_k + \beta_1 & \text{if } k \text{ is odd}, \\ \alpha_k + b & \text{if } k \text{ is even}. \end{cases} \tag{2.59}$$

Recall

$$B_1 := \left(\sin\frac{\pi}{\beta_1}\right)^{-1}, \tag{2.60}$$

and define

$$\begin{aligned} &B_k \sin\frac{\pi([\beta_{k-1}]+1-\alpha_k)}{\beta_k-\alpha_k} \\ &\quad = 2B_{k-1}\sin\frac{\pi([\beta_{k-1}])}{\beta_{k-1}-\alpha_{k-1}} - B_{k-1}\sin\frac{\pi([\beta_{k-1}]-1)}{\beta_{k-1}-\alpha_{k-1}} - \nu B_{k-1}\sin\frac{\pi([\beta_{k-1}])}{\beta_{k-1}-\alpha_{k-1}}. \end{aligned} \tag{2.61}$$

with

$$\nu = \begin{cases} \mu & \text{if } k \text{ is odd}, \\ \lambda & \text{if } k \text{ is even}. \end{cases} \tag{2.62}$$

Finally, it is easy to check that

$$\begin{aligned} &F_k([\beta_{k-1}]) > 0, \qquad F_k([\beta_{k-1}+1]) < 0, \\ &\frac{\partial F_k}{\partial\alpha}(\alpha) = -\frac{\pi}{\beta_k-\alpha_k}\sin\frac{\pi([\beta_{k-1}]-\alpha_{k-1})}{\beta_{k-1}-\alpha_{k-1}}\cos\frac{\pi([\beta_{k-1}+1]-\alpha)}{\beta_k-\alpha_k} \\ &\qquad + \frac{\pi}{\beta_k-\alpha_k}\cos\frac{\pi([\beta_{k-1}]-\alpha)}{\beta_k-\alpha_k}\sin\frac{\pi([\beta_{k-1}+1]-\alpha_{k-1})}{\beta_{k-1}-\alpha_{k-1}} \\ &\quad < 0. \end{aligned} \tag{2.63}$$

So, there exists a unique $\alpha_k \in ([\beta_{k-1}], [\beta_{k-1}+1])$ which satisfies (2.57).

To sum up, we may define

$$\Gamma_k(x) := B_k \sin\frac{\pi(x-\alpha_k)}{\beta_k-\alpha_k}, \quad x \in [[\beta_{k-1}], [\beta_k+1]], \tag{2.64}$$

which can be thought as the "*matching extension*" of Γ_{k-1}. Notice that α_k is the left zero of Γ_k on $[[\beta_{k-1}], [\beta_k+1]]$, and β_k is the right zero of Γ_k on $[[\beta_{k-1}], [\beta_k+1]]$.

Clearly, we may start the discuss from the IVP

$$\begin{aligned} &\Delta^2 u(t-1) + \lambda u(t) = 0, \quad t \in \mathbb{T}, \\ &u(0) = 0, \qquad u(1) = -1 \end{aligned} \tag{2.65}$$

which has unique solution $w_-(t,\lambda,\mu)$. Then, we may uniquely determine the kth segmental arc $\widehat{\Gamma}_k$ on $[[\delta_{k-1}], [\delta_k+1]]$ and the feasible initial phase γ_k via matching extension method, and accordingly, we get the left zero γ_k and the right zero δ_k of $\widehat{\Gamma}_k$ on $[[\delta_{k-1}], [\delta_k+1]]$.

3. The Main Result

The main result of this paper is the following discrete analogue of Theorem 1.1.

Theorem 3.1. $(\lambda, \mu) \in \mathbb{R}^2$ *is an eigenvalue of* (1.2) *if and only if* (λ, μ) *satisfies*

(i) *for* $k \geq 2$ *even*

$$\frac{k}{2}\frac{\pi}{\arccos((2-\lambda)/2)} + \frac{k}{2}\frac{\pi}{\arccos((2-\mu)/2)} + \sum_{j=1}^{k-1}(\alpha_{j+1} - \beta_j) = T + 1 \tag{3.1}$$

(ii) *for* $k \geq 1$ *odd, either*

$$\frac{k+1}{2}\frac{\pi}{\arccos((2-\lambda)/2)} + \frac{k-1}{2}\frac{\pi}{\arccos((2-\mu)/2)} + \sum_{j=1}^{k-1}(\alpha_{j+1} - \beta_j) = T + 1 \tag{3.2}$$

or

$$\frac{k-1}{2}\frac{\pi}{\arccos((2-\lambda)/2)} + \frac{k+1}{2}\frac{\pi}{\arccos((2-\mu)/2)} + \sum_{j=1}^{k-1}(\gamma_{j+1} - \delta_j) = T + 1. \tag{3.3}$$

Proof. (i) Let $k \geq 1$ be even. Let

$$\Gamma_m := \begin{cases} \left\{\left(x, B_1 \sin\frac{\pi}{\beta_1}(x-\alpha_m)\right) \mid x = [\alpha_m], \dots, [\beta_m + 1]\right\}, & m = 2j-1,\ j \in \left\{1, \dots, \frac{k}{2}\right\}, \\ \left\{\left(x, B_2 \sin\frac{\pi}{b}(x-\alpha_m)\right) \mid x = [\alpha_m], \dots, [\beta_m + 1]\right\}, & m = 2j,\ j \in \left\{1, \dots, \frac{k-2}{2}\right\}, \\ \left\{\left(x, B_2 \sin\frac{\pi}{b}(x-\alpha_m)\right) \mid x = [\alpha_m], \dots, [\beta_m]\right\}, & m = k. \end{cases} \tag{3.4}$$

Here,

$$\beta_1 = \frac{\pi}{\arccos((2-\lambda)/2)}, \qquad b = \frac{\pi}{\arccos((2-\mu)/2)}. \tag{3.5}$$

Then, a nontrivial solution of (1.2) can be constructed via matching extension method to Γ_1 and $\Gamma_2, \dots, \Gamma_{k-1}$ and Γ_k, respectively. Moreover,

$$T + 1 = \frac{k}{2}\frac{\pi}{\arccos((2-\lambda)/2)} + \frac{k}{2}\frac{\pi}{\arccos((2-\mu)/2)} + \sum_{j=1}^{k-1}(\alpha_{j+1} - \beta_j). \tag{3.6}$$

(ii) The case that k is odd can be treated by the similar way. □

Example 3.2. We may use *Mathematica 5.0* to give a numerical example for Theorem 3.1.

Take $k = 2$, $T = 3$.

Take $\beta_1 = 2.2$, then $\lambda \doteq 1.7153703202231987$.

Take $b = 1.8198598$, then $\mu \doteq 2.3097214213167683$.

Using the relation

$$\sin\frac{\pi}{\beta_1}([\beta_1])\sin\frac{\pi([\beta_1+1]-\alpha)}{b} = \sin\frac{\pi([\beta_1]-\alpha)}{b}\sin\frac{\pi}{\beta_1}([\beta_1+1]), \tag{3.7}$$

we may find

$$\alpha_2 \doteq 2.1801402269418726. \tag{3.8}$$

Since

$$2.2 + 1.8198598 + (2.1801402269418726 - 2.2) \doteq 4, \tag{3.9}$$

we may think that (1.7153703202231987, 2.3097214213167683) is an eigenvalue of the problem

$$\begin{gathered} \Delta^2 u(t-1) + \lambda u^+(t) - \mu u^-(t) = 0, \quad t \in \{1,2,3\}, \\ u(0) = u(4) = 0. \end{gathered} \tag{3.10}$$

Now, by the recurrence relation

$$\begin{gathered} u(t+1) = 2u(t) - u(t-1) - 1.7153703202231987u^+(t) + 2.309739789282126u^-(t), \\ u(0) = 0, \quad u(1) = 1, \end{gathered} \tag{3.11}$$

it is easily to compute that

$$u(4) \doteq -4.6600427960896695 \times 10^{-8} \approx 0. \tag{3.12}$$

This is a desired numerical result.

4. Some Properties of α_2

Proposition 4.1. *If $\lambda = \mu$ and $\beta_i = [\beta_i]$, then $\beta_j = \alpha_{j+1}$ for $j = 1, 2, \ldots$.*

Proof. In fact, if $\lambda = \mu$ and $\beta_i = [\beta_i]$, then

$$\beta_1 = b = \frac{\pi}{\arccos((2-\lambda)/2)}, \tag{4.1}$$

which implies that

$$\alpha_m, \beta_m \in \mathbb{N}^*. \tag{4.2}$$

□

Proposition 4.2. *Let (H) hold. Assume that λ is such that*

$$\beta_1 = [\beta_1] + \frac{1}{2}. \tag{4.3}$$

Then, for every $\mu > 1$, we have that

$$\alpha_2 = [\beta_1] + \frac{1}{2}. \tag{4.4}$$

Proof. By $\beta_1 = [\beta_1] + 1/2$, it follows that

$$\sin \frac{\pi [\beta_1]}{\beta_1} = -\sin \frac{\pi([\beta_1] + 1)}{\beta_1}. \tag{4.5}$$

This together with $F(\alpha_2) = 0$ imply that

$$\sin \frac{\pi([\beta_1] - \alpha_2)}{\beta_2 - \alpha_2} = -\sin \frac{\pi([\beta_1] + 1 - \alpha_2)}{\beta_2 - \alpha_2}, \tag{4.6}$$

and accordingly,

$$\sin \frac{\pi([\beta_1] - \alpha_2)}{\beta_2 - \alpha_2} + \sin \frac{\pi([\beta_1] + 1 - \alpha_2)}{\beta_2 - \alpha_2} = 2\sin \frac{\pi([\beta_1] + 1/2 - \alpha_2)}{\beta_2 - \alpha_2} \cos \frac{\pi/2}{\beta_2 - \alpha_2} = 0. \tag{4.7}$$

By (H), we have that $\beta_2 - \alpha_2 > 1$. Therefore,

$$[\beta_1] + \frac{1}{2} - \alpha_2 = 0. \tag{4.8}$$

□

Proposition 4.3. *Let (H) hold. Then,*

(i) $\beta_1 > [\beta_1] + 1/2$ *implies* $[\beta_1] + 1/2 < \alpha_2 < \beta_1$,

(ii) $\beta_1 < [\beta_1] + 1/2$ *implies* $[\beta_1] + 1/2 > \alpha_2 > \beta_1$.

Proof. (i) By the definition of F, (see (2.43)), we have that

$$F(\beta_1) = \sin \frac{\pi}{\beta_1}([\beta_1]) \sin \frac{\pi([\beta_1 + 1] - \beta_1)}{b} - \sin \frac{\pi([\beta_1] - \beta_1)}{b} \sin \frac{\pi([\beta_1 + 1])}{\beta_1}. \tag{4.9}$$

Since $\beta_1 > [\beta_1] + 1/2$, we have that

$$\sin\left(\frac{\pi}{\beta_1}[\beta_1]\right) > -\sin\left(\frac{\pi}{\beta_1}[\beta_1 + 1]\right), \tag{4.10}$$

which together with (4.9) implies that

$$F(\beta_1) \leq \sin\frac{\pi}{\beta_1}([\beta_1])\left[\sin\frac{\pi([\beta_1 + 1] - \beta_1)}{b} + \sin\frac{\pi([\beta_1] - \beta_1)}{b}\right] < 0. \tag{4.11}$$

Since $[\beta_1 + 1] - \beta_1 < \beta_1 - [\beta_1]$. Thus,

$$F(\beta_1) < F(\alpha_2) = 0. \tag{4.12}$$

From the fact that F is decreasing on $([\beta_1], [\beta_1] + 1)$, it follows that $\beta_1 > \alpha_2$.

On the other hand, we have from (2.43) that

$$F\left([\beta_1] + \frac{1}{2}\right) = \sin\frac{\pi}{\beta_1}([\beta_1])\sin\frac{\pi}{2b} + \sin\frac{\pi}{2b}\sin\frac{\pi}{\beta_1}([\beta_1 + 1]). \tag{4.13}$$

Combining this with the fact that $[\beta_1 + 1] - \beta_1 < \beta_1 - [\beta_1]$, it concludes that $F([\beta_1] + 1/2) > 0$. So, $[\beta_1] + 1/2 < \alpha_2$.

(ii) $\beta_1 < [\beta_1] + 1/2$ yields that

$$\sin\left(\frac{\pi[\beta_1]}{\beta_1}\right) < -\sin\left(\frac{\pi[\beta_1 + 1]}{\beta_1}\right), \tag{4.14}$$

which together with (4.9) and the fact that $[\beta_1 + 1] - \beta_1 > \beta_1 - [\beta_1]$ imply that

$$F(\beta_1) \geq \sin\frac{\pi}{\beta_1}([\beta_1])\left[\sin\frac{\pi([\beta_1 + 1] - \beta_1)}{b} + \sin\frac{\pi([\beta_1] - \beta_1)}{b}\right] > 0. \tag{4.15}$$

Since F is decreasing on $([\beta_1], [\beta_1] + 1)$, it follow that $\beta_1 < \alpha_2$.

Finally, we may use (4.13) and the fact $[\beta_1 + 1] - \beta_1 > \beta_1 - [\beta_1]$ to deduce that

$$F\left([\beta_1] + \frac{1}{2}\right) < F(\alpha_2) = 0, \tag{4.16}$$

which implies that $[\beta_1] + 1/2 > \alpha_2$. $\square$

5. Some Properties of $\Sigma \cap \mathrm{proj}_{\mathbb{R}^2} \mathcal{S}_2^+$

Definition 5.1. For $k \in \mathbb{N}$ and $\nu \in \{+,-\}$, let S_k^ν denote the set of functions $u : \widehat{\mathbb{T}} \to \mathbb{R}$ satisfying

(1) u has only simple generalized zeros in $[1,T]$;

(2) u has exactly $k-1$ simple generalized zeros in $[1,T]$;

(3) $\nu u(1) > 0$.

Let

$$\mathcal{S}_2^+ := \mathbb{R}^2 \times S_2^+. \tag{5.1}$$

An immediate consequence of Lemma 2.2 is the following.

Lemma 5.2. *If u of a nontrivial solution of* (1.2)*, then $u \in S_k^\nu$ for some $k \in \mathbb{N}$ and $\nu \in \{+,-\}$.*

In the rest of this section, one will pay one's attention to the study of $\Sigma \cap S_2^+$.
Let (H) hold and define a function $G : (1,T) \times ([\beta_1],[\beta_1]+1) \to \mathbb{R}$ by

$$G(b,\alpha) = \sin\frac{\pi}{\beta_1}\left([\beta_1]\right)\sin\frac{\pi\left([\beta_1+1]-\alpha\right)}{b} - \sin\frac{\pi\left([\beta_1]-\alpha\right)}{b}\sin\frac{\pi}{\beta_1}\left([\beta_1+1]\right). \tag{5.2}$$

Then, from Lemma 2.8, one has that for each $b \in (\pi/\arccos((2-\mu)/2),T)$, there exists a unique $\alpha_2(b) \in ([\beta_1],[\beta_1+1])$ such that

$$G(b,\alpha_2(b)) = 0. \tag{5.3}$$

Lemma 5.3. *Let (H) hold. Then,*

(i) *$\alpha_2(b)$ is continuous on $(\pi/\arccos((2-\mu)/2),T)$,*

(ii) *$|d\alpha_2/db| < 1/b$.*

Proof. (i) From (5.2), we see that $G(b,\alpha)$ is continuous and

$$\begin{aligned}\frac{\partial G}{\partial \alpha} = &-\frac{\pi}{b}\sin\left(\frac{\pi}{\beta_1}[\beta_1]\right)\cos\frac{\pi\left([\beta_1+1]-\alpha\right)}{b}\\ &+\frac{\pi}{b}\cos\frac{\pi\left([\beta_1]-\alpha\right)}{b}\sin\left(\frac{\pi}{\beta_1}[\beta_1+1]\right).\end{aligned} \tag{5.4}$$

This together with the facts that $\alpha \in ([\beta_1],[\beta_1+1])$ and $b > 1$ imply that $\partial G/\partial \alpha_2 < 0$. So, the implicit function theorem yields that $\alpha_2(b)$ is continuous on $(\pi/\arccos((2-\mu)/2),T)$.

(ii) From (5.2),

$$\begin{aligned}\frac{\partial G}{\partial b} = &-\frac{\pi\left([\beta_1+1]-\alpha\right)}{b^2}\sin\left(\frac{\pi}{\beta_1}[\beta_1]\right)\cos\frac{\pi\left([\beta_1+1]-\alpha\right)}{b}\\ &+\frac{\pi\left([\beta_1]-\alpha\right)}{b^2}\cos\frac{\pi\left([\beta_1]-\alpha\right)}{b}\sin\left(\frac{\pi}{\beta_1}[\beta_1+1]\right).\end{aligned} \tag{5.5}$$

Denote

$$\begin{aligned}
A(b,\alpha_2) &= \sin\left(\frac{\pi}{\beta_1}[\beta_1]\right)\cos\frac{\pi([\beta_1+1]-\alpha_2)}{b},\\
B(b,\alpha_2) &= \cos\frac{\pi([\beta_1]-\alpha_2)}{b}\sin\left(\frac{\pi}{\beta_1}[\beta_1+1]\right).
\end{aligned} \tag{5.6}$$

Then,

$$A(b,\alpha_2) > 0, \qquad B(b,\alpha_2) < 0, \tag{5.7}$$

and subsequently,

$$\begin{aligned}
\left|\frac{d\alpha_2}{db}(b)\right| &= \left|-\frac{\partial G(b,\alpha_2)/\partial b}{\partial G(b,\alpha_2)/\partial \alpha_2}\right|\\
&= \left|-\frac{-(\pi([\beta_1+1]-\alpha_2)/b^2)A(b,\alpha_2)+(\pi([\beta_1]-\alpha_2)/b^2)B(b,\alpha_2)}{-(\pi/b)A(b,\alpha_2)+(\pi/b)B(b,\alpha_2)}\right|\\
&= \left|\frac{(\pi([\beta_1+1]-\alpha_2)/b^2)A(b,\alpha_2)+(\pi([\beta_1]-\alpha_2)/b^2)(-B(b,\alpha_2))}{-(\pi/b)A(b,\alpha_2)-(\pi/b)(-B(b,\alpha_2))}\right|\\
&< \left|\frac{(\pi([\beta_1+1]-\alpha_2)/b^2)A(b,\alpha_2)-(\pi([\beta_1]-\alpha_2)/b^2)(-B(b,\alpha_2))}{-(\pi/b)A(b,\alpha_2)-(\pi/b)(-B(b,\alpha_2))}\right|\\
&= \left|\frac{(\pi([\beta_1+1]-\alpha_2)/b^2)A(b,\alpha_2)+(\pi(-[\beta_1]+\alpha_2)/b^2)(-B(b,\alpha_2))}{-(\pi/b)A(b,\alpha_2)-(\pi/b)(-B(b,\alpha_2))}\right|\\
&\le \frac{1}{b},
\end{aligned} \tag{5.8}$$

since $\max\{\pi([\beta_1+1]-\alpha_2)/b^2, \pi(-[\beta_1]+\alpha_2)/b^2\} \le \pi/b^2$. □

Theorem 5.4. *Let (H) hold. Then, for each* $\mu \in (2-2\cos(\pi/T),4)$*, there exists at most one* $\lambda \in (2-2\cos(\pi/T),4)$ *such that*

$$\frac{\pi}{\arccos((2-\lambda)/2)} + \frac{\pi}{\arccos((2-\mu)/2)} + (\alpha_2-\beta_1) = T+1. \tag{5.9}$$

Proof. Set

$$g(b) := b + (\alpha_2(b)-\beta_1). \tag{5.10}$$

Then, by Lemma 5.3,

$$g'(b) = 1 + \alpha_2'(b) > 1 - \frac{1}{b} > 0, \tag{5.11}$$

which implies that g is strictly increasing in b, and accordingly, the function

$$\widehat{g}(\mu) := \frac{\pi}{\arccos((2-\mu)/2)} + \left(\alpha_2\left(\frac{\pi}{\arccos((2-\mu)/2)}\right) - \beta_1\right) \tag{5.12}$$

is strictly decreasing and continuous in μ. Therefore, the function

$$\lambda = 2 - 2\cos\left(\frac{\pi}{T+1-\widehat{g}(\mu)}\right) \tag{5.13}$$

is strictly decreasing and continuous in μ. □

Remark 5.5. Let λ_2 be the second eigenvalue of (2.18), (2.19). Then, $(\lambda_2, \lambda_2) \in \Sigma \cap \mathrm{proj}_{\mathbb{R}^2} \mathcal{S}_2^+$.

Since

$$\left\{\frac{\pi}{\arccos((2-\mu)/2)} \mid \mu \in \left(2 - 2\cos\frac{\pi}{T}, 4\right)\right\} = (1, T), \tag{5.14}$$

there exists an open interval $I \subset (2 - 2\cos(\pi/T), 4)$ such that

$$\left\{\frac{\pi}{\arccos((2-\mu)/2)} \mid \mu \in I\right\} = (2, T-1). \tag{5.15}$$

Theorem 5.6. *For every $\mu \in I$, there exists a unique $\lambda \in (2 - 2\cos(\pi/T), 4)$, such that* (5.9) *holds.*

Proof. Using the fact that $-1 < \alpha_2 - \beta_1 < 1$, it concludes that for $\mu \in I$,

$$1 < \frac{\pi}{\arccos((2-\mu)/2)} + \left(\alpha_2\left(\frac{\pi}{\arccos((2-\mu)/2)}\right) - \beta_1\right) < T, \quad \mu \in I, \tag{5.16}$$

which implies that

$$T > (T+1) - \left\{\frac{\pi}{\arccos((2-\mu)/2)} + \left(\alpha_2\left(\frac{\pi}{\arccos((2-\mu)/2)}\right) - \beta_1\right)\right\} > 1, \quad \mu \in I. \tag{5.17}$$

Combining this with the fact that

$$\left\{\frac{\pi}{\arccos((2-\lambda)/2)} \mid \lambda \in \left(2 - 2\cos\frac{\pi}{T}, 4\right)\right\} = (1, T), \tag{5.18}$$

it follows that (5.9) has at least one solution for every $\mu \in I$. The uniqueness can be deduced from Theorem 5.4. □

Remark 5.7. Espinoza [6] proved that the discrete analogue of Fučik spectrum of the Laplacian with Dirichlet boundary condition contains a curve which is continuous, strictly decreasing, and symmetric with respect to the diagonal. However, Espinoza [6] gave no information about the construction of the Fučik spectrum. Our main results, Theorems 5.4 and 3.1, are established for ordinary difference equation only. However, much more information about the construction of the Fučik spectrum is contained in these theorems.

6. Applications to Nonlinear Problems

Let λ_1 be the first eigenvalue of (1.8). Then,

$$\lambda_1 = 2 - 2\cos\frac{\pi}{T+1}. \tag{6.1}$$

The eigenfunction corresponding to λ_1 is $\sin(\pi t/(T+1))$.

Let us consider the nonlinear problem

$$\begin{gathered} \Delta^2 u(t-1) + \lambda_1 u(t) + g(t, u(t)) = f(t), \quad t \in \mathbb{T}, \\ u(0) = u(T+1) = 0, \end{gathered} \tag{6.2}$$

where g and f satisfy

(A1) $f : \mathbb{T} \to \mathbb{R}$;

(A2) $g : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exist a function $p : \mathbb{T} \to \mathbb{R}$ and a constant $q > 0$, such that

$$\left|g(t,s)\right| \le p(t) + q|s|, \quad s \in \mathbb{R},\ t \in \mathbb{T}, \tag{6.3}$$

(A3) there exist a function functions $a, A : \mathbb{T} \to \mathbb{R}$ and constants $r < 0 < R$, such that

$$\begin{aligned} g(t,s) &\ge A(t), \quad s \ge R,\ t \in \mathbb{T}, \\ g(t,s) &\le a(t), \quad s \le r,\ t \in \mathbb{T}. \end{aligned} \tag{6.4}$$

Obviously, (A3) implies

$$\liminf_{s \to \pm\infty} g(t,s)s \ge 0. \tag{6.5}$$

Denote

$$g_{+\infty}(t) := \liminf_{s \to +\infty} g(t,s), \qquad g^{-\infty}(t) := \limsup_{s \to -\infty} g(t,s). \tag{6.6}$$

Lemma 6.1 (Sturm separation theorem; [7, Theorem 6.5]). *Two linearly independent solutions of*

$$\Delta^2 y(t-1) + q(t)y(t) = 0, \quad t \in \mathbb{T} \tag{6.7}$$

cannot have a common zero. If a nontrivial solution of (6.7) *has a zero at* t_1 *and a generalized zeros at* $t_2 > t_1$*, then any second linear independent solution has a generalized zero in* $(t_1, t_2]$*. If a nontrivial solution of* (6.7) *has a generalized zero at* t_1 *and a generalized zero at* $t_2 > t_1$*, then any second linearly independent solution has a generalized zero in* $[t_1, t_2]$*.*

Definition 6.2 (see [7, Definition 6.3]). Let $q : \mathbb{T} \to \mathbb{R}$. We say that (6.7) is *disconjugate* on $\widehat{\mathbb{T}}$ provided that no nontrivial solution of (6.7) has two or more generalized zeros on $\widehat{\mathbb{T}}$.

Lemma 6.3 (Sturm comparison theorem; [7, Theorem 8.12]). *Assume that* $q_1(t) \geq q_2(t)$ *on* $\mathbb{T}$*. If* $\Delta^2 y(t-1) + q_1(t)y(t) = 0$ *is disconjugate on* $\widehat{\mathbb{T}}$*, then* $\Delta^2 y(t-1) + q_2(t)y(t) = 0$ *is disconjugate on* $\widehat{\mathbb{T}}$*.*

Let us denote by D the Hilbert space

$$D := \{u : \mathbb{T} \longrightarrow \mathbb{R}\}, \tag{6.8}$$

with the inner product

$$\langle u, v \rangle_D := \sum_{t=1}^{T} u(t)v(t), \quad u, v \in D, \tag{6.9}$$

and the norm

$$\|u\|_D := \langle u, v \rangle_D^{1/2}, \quad u, v \in D. \tag{6.10}$$

Let us denote by D^* the Hilbert space

$$D^* := \left\{u : \widehat{\mathbb{T}} \longrightarrow \mathbb{R} \mid u(0) = u(T+1) = 0\right\}, \tag{6.11}$$

with the inner product

$$\langle u, v \rangle_{D^*} := \sum_{t=1}^{T} u(t)v(t), \quad u, v \in D^*, \tag{6.12}$$

and the norm

$$\|u\|_D := \langle u, v \rangle_{D^*}^{1/2}, \quad u, v \in D. \tag{6.13}$$

For $u \in D$, define $j : D \to D^*$ by

$$j(u(1), u(2), \ldots, u(T)) = (0, u(1), u(2), \ldots, u(T), 0), \tag{6.14}$$

then j is a natural isomorphism.

Obviously,

$$\langle u, v\rangle_D = \langle ju, jv\rangle_{D^*}, \quad (u, v) \in D. \tag{6.15}$$

So, in the rest of the paper, one will use $\langle u, v\rangle$, $\|u\|$ to denote the inner product and the norm in D (or D^*), respectively.

Let

$$u(t) = \sum_{j=1}^{T} a_j \sin \frac{j\pi t}{T+1} \tag{6.16}$$

be the fourier series of $u \in D^*$. Then, one will write

$$u(t) = u^0(t) + \widetilde{u}(t), \tag{6.17}$$

where

$$u^0(t) = a_1 \sin \frac{\pi t}{T+1}, \qquad \widetilde{u}(t) = \sum_{j=2}^{T} a_j \sin \frac{j\pi t}{T+1}. \tag{6.18}$$

Using the same method to prove [7, Lemma 1.4] with obvious changes, we obtain the following.

Lemma 6.4. *Assume that for each $n \in \mathbb{N}$, one has $0 \le \chi_n(t)$ for $t \in \mathbb{T}$, and for each $t \in \mathbb{T}$, $\chi_n(t) \to 0$ as $n \to \infty$. Then, there exists a constant $\rho > 0$, such that for all $u \in D^*$,*

$$\sum_{t=1}^{T} \left[\Delta^2 u(t-1) + \lambda_1 u^2(t) + \chi_n(t) u(t)\right]\left[u^0(t) - \widetilde{u}(t)\right] \ge \rho \|\widetilde{u}\|^2. \tag{6.19}$$

Let ψ_1 be such that

$$\Delta^2 \psi_1(t-1) + \lambda_\diamond \psi_1^+(t) - \mu_\diamond \psi_1^-(t) = 0, \quad t \in \mathbb{T}, \tag{6.20}$$

$$\psi_1(0) = \psi_1(T+1) = 0, \tag{6.21}$$

then $\psi_1 > 0$ on $\mathbb{T}$.

Assume that

(C1) Let $\chi_+, \chi_- \in D$ be such that there exist two point $(\lambda_\diamond, \mu_\diamond) \in \Sigma \cap S_1^{\pm}$ and $(\lambda^\diamond, \mu^\diamond) \in \Sigma \cap S_2^{+}$ with

$$\lambda_\diamond \leq \chi_+(t), \quad \mu_\diamond \leq \chi_-(t), \quad \forall t \in \mathbb{T}, \tag{6.22}$$

and the strict inequalities $\lambda_\diamond < \chi_+(t)$, $\mu_\diamond < \chi_-(t)$ hold for some $t \in \mathbb{T}$

$$\lambda^\diamond \geq \chi_+(t), \quad \mu^\diamond \geq \chi_-(t), \quad \forall t \in \mathbb{T}, \tag{6.23}$$

and the strict inequalities $\lambda^\diamond > \chi_+(t)$, $\mu^\diamond > \chi_-(t)$ hold for some $t \in \mathbb{T} \setminus \{\widehat{\tau}\}$, with $\widehat{\tau}$ is the unique generalized zero of ψ_2 on $(0, T+1)$, and

$$\begin{gathered} \Delta^2 \psi_2(t-1) + \lambda^\diamond \psi_2^+(t) - \mu^\diamond \psi_1^-(t) = 0, \quad t \in \mathbb{T}, \\ \psi_2(0) = \psi_2(T+1) = 0, \end{gathered} \tag{6.24}$$

with $\psi_2(1) > 0$.

Theorem 6.5. *Let (A1)–(A3) hold. Assume that there existence $(\lambda^*, \mu^*) \in \Sigma \cap S_2^+$, such that*

$$\begin{aligned} \limsup_{s \to +\infty} \frac{g(t,s)}{s} &\leq \lambda^* - \lambda_1, \\ \limsup_{s \to -\infty} \frac{g(t,s)}{s} &\leq \mu^* - \lambda_1. \end{aligned} \tag{6.25}$$

Suppose that the strict inequalities in (6.25) *hold on some $t_0 \in \mathbb{T} \setminus \{\widehat{\tau}\}$. Then, BVP* (6.2) *has at least one solutions provided*

$$\sum_{t=1}^{T} g^{-\infty}(t) \sin \frac{\pi t}{T+1} < \sum_{t=1}^{T} f(t) \sin \frac{\pi t}{T+1} < \sum_{t=1}^{T} g_{+\infty}(t) \sin \frac{\pi t}{T+1}. \tag{6.26}$$

Remark 6.6. Let us consider the nonlinear boundary value problem

$$\begin{gathered} \Delta^2 u(t-1) + \lambda_1 u(t) + g(u(t)) = f(t), \quad t \in \{1,2,3\}, \\ u(0) = u(4) = 0, \end{gathered} \tag{6.27}$$

where $f : \{1,2,3\} \to \mathbb{R}$ is a fixed function,

$$g(s) = \begin{cases} 1.1s, & s \geq 0 \\ 1.7s, & s < 0. \end{cases} \tag{6.28}$$

Obviously, f and g satisfy (A1)–(A3). From Example 3.2, it follows that (6.25) hold. Since $g_{-\infty} = -\infty$ and $g_{+\infty} = +\infty$, it follows that (6.26) holds.

To prove Theorem 6.5, we need the following.

Lemma 6.7. *Let (C1) holds. Then, the Dirichlet problem*

$$\Delta^2 u(t-1) + \chi_+(t)u^+(t) - \chi_-(t)u^-(t) = 0, \quad t \in \mathbb{T}, \tag{6.29}$$

$$u(0) = u(T+1) = 0, \tag{6.30}$$

has only the trivial solution.

Proof. Suppose on the contrary that u is a nontrivial solution of (6.29), (6.30) with

$$u(1) > 0. \tag{6.31}$$

We claim that the number of generalized zeros of u in $(0, T+1)$ is 0 or 1.

Suppose on the contrary that the number of generalized zeros of u in $(0, T+1)$ is large than 1. Let $t_1, t_2 \in \mathbb{T}$ be the first two positive generalized zeros of u with $t_1 < t_2$.

We divide the proof into three cases.

Case 1 ($t_1 < \hat{\tau}$). In this case, we claim that

$$\Delta^2 \psi(t-1) + \lambda^{\diamond}\psi(t) = 0 \tag{6.32}$$

is disconjugate on $[0, t_1]$.

Suppose on the contrary that there exists a solution ψ^* of (6.32) which is linearly independent of ψ_2 and has two consecutive generalized zeros in $[0, t_1]$. Then, from Lemma 6.1, ψ_2 has a generalized zero in $(0, t_1]$. This is impossible. Therefore, the claim is true.

Now, from the above claim and Lemma 6.3,

$$\Delta^2 u(t-1) + \chi_+(t)u(t) = 0 \tag{6.33}$$

is disconjugate on $[0, t_1]$. However, this contradicts the fact that t_1 is a generalized zero of u.

Case 2 ($t_1 > \hat{\tau}$). The case can be treated by the same method as in Case 1 with obvious changes.

Case 3 ($t_1 = \hat{\tau}$).

Subcase 3.1 ($t_1 = \hat{\tau}$ and $\psi_2(\hat{\tau}) = 0$). It is easy to check that

$$\Delta^2 \psi(t-1) - \mu^{\diamond}\psi(t) = 0 \tag{6.34}$$

is disconjugate on $[\hat{\tau}, t_2]$. The fact and Lemma 6.3 yield

$$\Delta^2 v(t-1) - \chi_-(t)v(t) = 0, \quad t \in \{\hat{\tau}, \dots, t_2 - 1\} \tag{6.35}$$

is disconjugate on $[\hat{\tau}, t_2]$, and subsequently, u cannot have two generalized zero in $[\hat{\tau}, t_2]$. This is a contradiction.

Subcase 3.2 ($t_1 = \hat{\tau}$ and $\psi_2(\hat{\tau}) < 0$). We note that the general solution of (6.35) has the form

$$v(t) = c_1 \sin(\theta t + c_2), \quad t \in \{\hat{\tau} - 1, \dots, t_2\}, \tag{6.36}$$

with

$$\theta := \arccos\frac{2-\mu^\diamond}{2}, \quad \mu^\diamond \in (0,\ 4). \tag{6.37}$$

Denote by $d(\mu^\diamond)$ the distance between any two consecutive zeros of $\sin(\theta t + c_2)$. Then,

$$d(\mu^\diamond) = \frac{\pi}{\theta}, \tag{6.38}$$

$$(T+1) - \hat{\tau} < d(\mu^\diamond) < (T+1) - (\hat{\tau} - 1). \tag{6.39}$$

Combining (6.36) with (6.39) and using the definition of generalized zero, it follows that $v(t)$ has at most one generalized zero in $[\hat{\tau} - 1, t_2]$, which implies that (6.35) is disconjugate on $[\hat{\tau} - 1, t_2]$.

This fact and Lemma 6.3 yield

$$\Delta^2 v(t-1) - \chi_-(t)v(t) = 0, \quad t \in \{\hat{\tau}, \dots, t_2 - 1\} \tag{6.40}$$

is disconjugate on $[\hat{\tau}-1, t_2]$, and subsequently, u cannot have two generalized zero in $[\hat{\tau}-1, t_2]$. This is a contradiction.

Therefore, u has at most one generalized zero in $(0, T+1)$.

If the number of generalized zeros is in $(0, T+1)$ is 0, then

$$u(t) > 0, \quad t \in \mathbb{T}. \tag{6.41}$$

Multiplying (6.29) by $\psi_1(t)$ and (6.20) by $u(t)$ and subtracting, and then taking the summation from 1 to T, it follows that

$$0 = \sum_{t=1}^{T} \left[\chi_+(t)u^+(t)\psi_1(t) - \lambda_\diamond u(t)\psi_1(t)\right] = \sum_{t=1}^{T} \left(\chi_+(t) - \lambda_\diamond\right) u(t)\psi_1(t) > 0, \tag{6.42}$$

a contradiction.

If u has a unique generalized zeros τ_u in $(0, T+1)$, then from Lemma 6.3 and the similar method to deal with Case 1, it follows that

$$\tau_u = \hat{\tau}. \tag{6.43}$$

Thus,

$$\begin{aligned} \Delta^2 u(t-1) + \chi_+(t)u^+(t) &= 0, \quad t \in \{1, \dots, \hat{\tau} - 1\}, \\ \Delta^2 u(t-1) - \chi_-(t)u^-(t) &= 0, \quad t \in \{\hat{\tau}, \dots, T+1\}. \end{aligned} \tag{6.44}$$

This together with the facts

$$\begin{aligned} \Delta^2 \psi_2(t-1) + \lambda^\diamond \psi_2^+(t) &= 0, \quad t \in \{1, \dots, \hat{\tau} - 1\}, \\ \Delta^2 \psi_2(t-1) - \mu^\diamond \psi_2^-(t) &= 0, \quad t \in \{\hat{\tau}, \dots, T+1\} \end{aligned} \tag{6.45}$$

imply that

$$\begin{aligned} 0 &= \sum_{t=1}^{\hat{\tau}-1} \left[\chi_+(t)u^+(t)\psi_2^+(t) - \lambda^\diamond u^+(t)\psi_2^+(t)\right] \\ &\quad + \sum_{t=\hat{\tau}}^{T} \left[\chi_-(t)u^-(t)\psi_2^-(t) - \mu^\diamond u^-(t)\psi_2^-(t)\right] \\ &< 0, \end{aligned} \tag{6.46}$$

a contradiction.

Therefore, $u(t) = 0$ on $\mathbb{T}$. □

Proof of Theorem 6.5. The idea is the same as in the proof of [7, Theorem 1.9]. Let $\delta < \min\{\lambda^\diamond - \lambda_1, \mu^\diamond - \lambda_1\}$ and define the homotopy family

$$\begin{gathered} \Delta^2 u(t-1) + \lambda_1 u(t) + (1-r)\delta u(t) + r g(t, u(t)) = r h(t), \quad r \in [0,1], \\ u(0) = u(T+1) = 0. \end{gathered} \tag{6.47}$$

We will show that there exists $R > 0$ such that (6.47), (6.29) has no solution u with

$$\|u\| = R. \tag{6.48}$$

Similar to the proof of [7, Remark 1.2], we may prove that there exist two positive constants q_1 and q_2, such that

$$g(t,s) = \gamma(t,s)s + h(t,s), \quad t \in \mathbb{T}, \ s \in \mathbb{R}, \tag{6.49}$$

where $0 \le \gamma(t,s) \le q_1$, $|h(t,s)| \le q_2$ for all $s \in \mathbb{R}$ and $t \in \mathbb{T}$.

Let us define

$$\hat{g}_1(t,s) = \begin{cases} \min\{g(t,s),1\}, & s \geq 1 \\ \min\{g(t,s),-1\}, & s \leq -1, \end{cases}$$
$$g^*(t,s) = g(t,s) - \hat{g}_1(t,s),$$
$$\gamma(t,s) = \begin{cases} \dfrac{g^*(t,s)}{s}, & |s| \geq 1, \\ g^*\left(t,\dfrac{s}{|s|}\right)s, & 0 < |s| < 1, \\ 0, & s = 0. \end{cases} \tag{6.50}$$

Assume to the contrary that there exists a sequence $\{(r_n,u_n)\} \subseteq [0,1] \times \mathfrak{D}^*$ with $\|u_n\| \to \infty$, such that

$$\Delta^2 u_n(t-1) + \lambda_1 u_n(t) + (1-r_n)\delta u_n(t) + r_n g(t,u_n(t)) = r_n h(t), \quad t \in \mathbb{T},$$
$$u_n(0) = u_n(T+1) = 0. \tag{6.51}$$

Set

$$\chi_n(t) = (1-r_n)\delta + r_n\gamma(t,u_n(t)). \tag{6.52}$$

Then we may assume that $\chi_n \to \chi$ in $\mathfrak{D}^*$. Moreover, it follows from (6.5), (6.25), and (6.49) that $\chi(t) \geq 0$ on $t \in \mathbb{T}$, and

$$\chi(t) \leq \lambda^\diamond - \lambda_1 \quad \text{on } \{t \in \mathbb{T} \mid \varphi_2(t) < 0\},$$
$$\chi(t) \leq \mu^\diamond - \lambda_1 \quad \text{on } \{t \in \mathbb{T} \mid \varphi_2(t) < 0\}, \tag{6.53}$$

with the strict inequalities on some $t_0 \in \mathbb{T} \setminus \{\hat{\tau}\}$. Now, by the standard arguments, see the proof of [7, Theorems 1.5 and 1.9], we may get the desired contradiction. □

Remark 6.8. Theorem 6.5 improves the main results of Rodriguez [8] and R. Ma and H. Ma [9, 10], where the nonlinearities are not *jumping* at infinity.

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