Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 780207, 10 pages
doi:10.1155/2010/780207

Research Article

Application of He's Homotopy Perturbation Method for Cauchy Problem of Ill-Posed Nonlinear Diffusion Equation

Ali Zakeri, Azim Aminataei, and Qodsiyeh Jannati

Department of Mathematics, K. N. Toosi University of Technology, P.O. Box 16315-1618, 1541849611 Tehran, Iran

Correspondence should be addressed to Azim Aminataei, ataei@kntu.ac.ir

Received 18 July 2009; Revised 14 March 2010; Accepted 31 March 2010

Academic Editor: Marko Robnik

We consider a Cauchy problem of unidimensional nonlinear diffusion equation on finite interval. This problem is ill-posed and its approximate solution is unstable. We apply the He's homotopy perturbation method (HPM) and obtain the third-order asymptotic expansion. We show that if the conductivity term in diffusion equation has a specified condition, the above solution can be estimated. Finally, a numerical experiment is provided to illustrate the method.

1. Introduction

The diffusion equation, one of the classical partial differential equations (PDEs), describes the process of diffusivity propagation. It has a great deal of application in different branches of sciences which have found a considerable amount of interest in recent years. This kind of equation arises naturally in a variety of models from theoretical physics, chemistry, and biology [1–8]. For instance, diffusion equations are used to investigate heat conduction, steady states and hysteresis, spatial patterns, blood oxygenation, moving fronts, pulses, and oscillations phenomena. Without any excessive simplification, these problems are all nonlinear. Therefore one needs to use a variety of different methods from different areas of mathematics such as numerical analysis, bifurcation and stability theory, similarity solutions, perturbations, topological methods, and many others, in order to study them [9–16].

Recently HPM is widely applied to linear and nonlinear problems. The method was proposed first by He in 1997 and systematical description in 2000 which is, in fact a coupling of traditional perturbation method and homotopy in topology. The application of the HPM to nonlinear problems has been developed, because this method continuously deforms the difficult problem under study into a simple one which is easy to solve. The method yields a very rapid convergence of the solution series in the most cases. Because of this rapid convergency, HPM has become a powerful mathematical tool, when it is successfully coupled with the perturbation theory. Also, HPM was used to solve variational problems by different investigators before [17–29]. One can find the recent developments of the HPM in [30–33].

This work is concerned to the nonlinear Cauchy diffusion problem and the HPM is applied to solve it. The organization of this paper is as follows. Section 2 is devoted to introduce the statement of Cauchy problem. In Section 3, we give the concepts of HPM. In Section 4, we derive the solution of Cauchy equation of nonlinear diffusion problem by HPM. In Section 5, we present an experiment wherein its numerical results illustrate the accuracy and efficiency of the proposed method, and finally in Section 6, some conclusions are considered.

2. Statement of the Cauchy Problem

Let $\phi(x,t)$ be a smooth function in $\Omega \equiv [0,l] \times [0,T]$ where l and T are constant values, and $f(t)$, $g(t)$, $a(t)$, and $b(t)$ are known functions in $[0,T]$. Now, we assume that $u(x,t)$ satisfies the nonlinear Cauchy diffusion equation:

$$Au = \phi(x,t) \quad \text{in } \Omega_0 \equiv (0,l) \times (0,T), \tag{2.1}$$

subject to the initial conditions:

$$\begin{aligned} u(0,t) &= f(t), \quad 0 \le t \le T, \\ u_x(0,t) &= g(t), \quad 0 \le t \le T, \end{aligned} \tag{2.2}$$

where A is defined as

$$A(u(x,t)) = \partial_t u(x,t) - \partial_x \{(a(t)u(x,t) + b(t))\partial_x u(x,t)\}, \tag{2.3}$$

such that $a(t)u(x,t) + b(t)$ is positive [3–6], and $u(x,t)$ is an unknown.

According to [34], we express HPM for the nonlinear problems in general case. Then, we apply this method to approximate the solution of the problem (2.1)–(2.3).

3. Description of the HPM

Suppose that A, a, b, Φ, f, and g satisfy to the above conditions. The operator A can be generally divided into two parts L and N, where L is a linear operator, and N is a nonlinear one. Therefore (2.1) can be rewritten as follows:

$$L(u) + N(u) - \phi(x,t) = 0. \tag{3.1}$$

He [35] constructed a homotopy $H : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(v_0)] + p[A(v) - \phi(x,t)] = 0, \tag{3.2}$$

or

$$H(v,p) = L(v) - L(v_0) + pL(v_0) + p\left[N(v) - \phi(x,t)\right] = 0. \tag{3.3}$$

where $p \in [0,1]$, that is called a homotopy parameter, and v_0 is an initial approximation of (2.1) which satisfies initial conditions.

Hence, it is obvious that

$$\begin{aligned} H(v,0) &= L(v) - L(v_0) = 0, \\ H(v,1) &= A(v) - \phi(x,t) = 0. \end{aligned} \tag{3.4}$$

Now, the changing process of p from 0 to 1 is just that of $H(v,p)$ from $L(v) - L(v_0)$ to $A(v) - \phi(x,t)$.

Applying the perturbation technique due to the fact that $0 \le p \le 1$ can be considered as a small parameter, we can assume that the solution of (3.2) or (3.3) can be expressed as a series in p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 \cdots, \tag{3.5}$$

when $p \to 1$; (3.2) or (3.3) corresponds to (3.1) and becomes the approximate solution of (3.1). That is,

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots. \tag{3.6}$$

The series solution (3.6) is convergent for different terms of v, and the rate of convergence depends on $A(v)$ [36–38].

4. Solution of Cauchy Equation of Nonlinear Diffusion Problem by HPM

Consider the nonlinear differential equation (2.1), with the indicated initial conditions (2.2). From (2.1) we have

$$\frac{\partial^2 u}{\partial x^2} - \left\{ \frac{1}{b(t)} \frac{\partial u}{\partial t} - \frac{a(t)}{b(t)} \left(\frac{\partial}{\partial x} \left(u \frac{\partial u}{\partial x} \right) \right) \right\} = \frac{-1}{b(t)} \Phi(x,t). \tag{4.1}$$

Then we can write (4.1) as follows:

$$L_x u - Nu = \Psi(x,t), \tag{4.2}$$

where $\Psi(x,t) = (-1/b(t))\Phi(x,t)$, $L_x u = \partial^2 u/\partial x^2$ and $Nu = (1/b(t))(\partial u/\partial t) - (a(t)/b(t))((\partial/\partial x)(u\partial u/\partial x))$ are the linear and nonlinear parts of Au, respectively.

By twice integration of (4.1) with respect to x, and applying the initial conditions (2.3), we obtain:

$$u(x,t) - xg(t) - f(t) - \int_0^x \int_0^x N_u dx\, dx = \int_0^x \int_0^x \Psi(x,t) dx\, dx = \frac{-1}{b(t)} \int_0^x \int_0^x \Phi(x,t) dx\, dx. \tag{4.3}$$

Consequently, we obtain

$$u(x,t) = xg(t) + f(t) + \int_0^x \int_0^x N_u dx\, dx - \frac{1}{b(t)} \int_0^x \int_0^x \Phi(x,t) dx\, dx. \tag{4.4}$$

By HPM, let $F(u) = u(x,t) - h(x,t) = 0$, where $h(x,t) = xg(t) + f(t) + \iint_0^x \Psi(x,t) dx\, dx$. That is, $h(x,t) = xg(t) + f(t) - (1/b(t)) \iint_0^x \Phi(x,t) dx\, dx$.

Hence, we may choose a convex homotopy such that [23]

$$H(v,p) = v(x,t) - h(x,t) - p \int_0^x \int_0^x N_v\, dx\, dx = 0, \tag{4.5}$$

where

$$F(u) = u(x,t) - h(x,t) = 0, \quad h(x,t) = xg(t) + f(t) + \int_0^x \int_0^x \Psi(x,t) dx\, dx. \tag{4.6}$$

By using (4.5), we find

$$v(x,t) = h(x,t) + p \int_0^x \int_0^x N_v dx\, dx. \tag{4.7}$$

By combining (4.1) and (4.7), we obtain

$$v(x,t) = xg(t) + f(t) - \frac{1}{b(t)} \int_0^x \int_0^x \Phi(x,t) dx\, dx + p \int_0^x \int_0^x \left(\frac{1}{b(t)} \frac{\partial}{\partial t} v(x,t) - \frac{a(t)}{b(t)} \frac{\partial}{\partial x} \left(v(x,t) \frac{\partial}{\partial x} v(x,t) \right) \right) dx\, dx, \tag{4.8}$$

or

$$\begin{aligned}
v_0(x,t) &= h(x,t) = xg(t) + f(t) - \frac{1}{b(t)}\int_0^x\int_0^x \Phi(x,t)dx\,dx,\\
v_1(x,t) &= \int_0^x\int_0^x\left\{\frac{1}{b(t)}\frac{\partial}{\partial t}v_0 - \frac{a(t)}{b(t)}\frac{\partial}{\partial x}\left(v_0\frac{\partial}{\partial x}v_0\right)\right\}dx\,dx,\\
v_2(x,t) &= \int_0^x\int_0^x\left\{\frac{1}{b(t)}\frac{\partial}{\partial t}v_1 - \frac{a(t)}{b(t)}\frac{\partial}{\partial x}\left(v_0\frac{\partial}{\partial x}v_1 + v_1\frac{\partial}{\partial x}v_0\right)\right\}dx\,dx,\\
v_3(x,t) &= \int_0^x\int_0^x\left\{\frac{1}{b(t)}\frac{\partial}{\partial t}v_2 - \frac{a(t)}{b(t)}\frac{\partial}{\partial x}\left(v_0\frac{\partial}{\partial x}v_2 + v_1\frac{\partial}{\partial x}v_1 + v_2\frac{\partial}{\partial x}v_0\right)\right\}dx\,dx,
\end{aligned} \tag{4.9}$$

where the above relations are obtained by equating the terms with identical powers of p in (4.8).

Therefore, the approximation solution is

$$u(x,t) \simeq v_0 + v_1 + v_2 + v_3. \tag{4.10}$$

In Section 5, we explain a numerical experiment. By using the HPM, an approximate solution for nonlinear diffusion equation is obtained.

5. Numerical Experiment

Let us consider the following nonlinear differential equation

$$u_t - \frac{\partial}{\partial x}\left\{\left(\frac{1}{6}e^{-t}u + (t+5)e^{-t}\right)\frac{\partial u}{\partial x}\right\} = -\frac{7}{3}t - 9, \quad (x,t) \in [0,1]\times[0,1], \tag{5.1}$$

with initial conditions:

$$u(0,t) = t, \quad 0 \le t \le 1, \qquad u_x(0,t) = 0, \quad 0 \le t \le 1. \tag{5.2}$$

If we want to use our last notation, we have

$$\Phi(x,t) = -\frac{7}{3}t - 9, \quad a(t) = \frac{1}{6}e^{-t}, \quad b(t) = (t+5)e^{-t}. \tag{5.3}$$

Obviously, the above assumptions satisfy to consideration of aforesaid conditions. In addition, the exact solution of the problem is: $u(x,t) = x^2e^t + t$.

In this experiment, we have obtained the solution of Cauchy problem at the points $x = 0.1, 0.2, 0.3, \ldots, 1$, where $t = 0.25, 0.50, 0.75$ and 1.

We construct a homotopy in the same form as we have described in Section 3:

$$H(v,p) = u(x,t) - h(x,t) - p\int_0^x\int_0^x\left\{\frac{e^t}{(t+5)}\frac{\partial u}{\partial t} - \frac{1}{6(t+5)}\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)\right\}dx\,dx = 0. \tag{5.4}$$

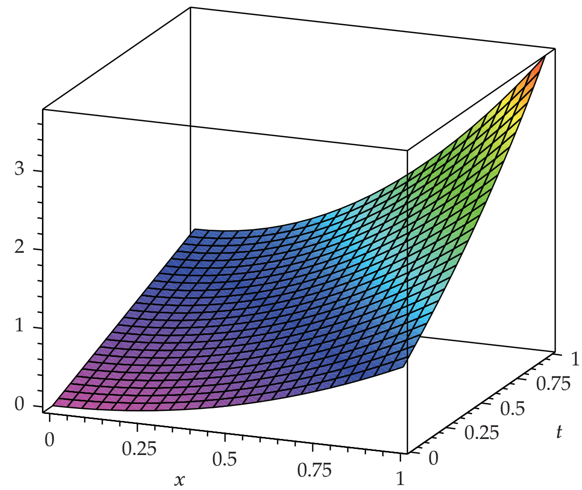

Figure 1: Exact solution of the nonlinear diffusion problem on the interval $[0, 1]$.

By substituting (3.5) into the above equation, and equating the terms with identical powers of p, we have

$$\begin{aligned}
v_0(x,t) &= \frac{1}{6(t+5)}\left(6t^2+30t+7e^t x^2 t+27e^t x^2\right),\\
v_1(x,t) &= \frac{-e^t x^2}{432(t+5)^3}\left(-129e^t x^2+7e^t x^2 t^2+6e^t x^2 t+528t^2-540t-5400+84t^3\right),\\
v_2(x,t) &= \frac{e^t x^2}{77760(t+5)^5}\Big(-12177e^{2t}x^4+705e^{2t}x^4t^2-2865e^{2t}x^4t+161e^{2t}x^4t^3\\
&\quad +6930e^t x^2t^2-255150e^t x^2 t-526500e^t x^2+16830e^t x^2 t^3\\
&\quad +1890e^t x^2 t^4+2520t^5+28440t^4+63000t^3-243000t^2-810000t\Big),\\
v_3(x,t) &= -\frac{e^t x^2}{78382080(t+5)^7}\Big(187092e^{3t}x^6t^3-131550e^{3t}x^6t^2-3508884e^{3t}x^6t+2457e^{3t}x^6t^4\\
&\quad -6173667e^{3t}x^6-49829472e^{2t}x^4t^2-219217320e^{2t}x^4t\\
&\quad -217954800e^{2t}x^4+357268e^{2t}x^4t^4+713160e^{2t}x^4t^3\\
&\quad +31164e^{2t}x^4t^5+84672e^t x^2t^6+1155168e^t x^2t^5\\
&\quad +35592480e^t x^2t^4-131997600e^t x^2t^3-805140000e^t x^2t^2\\
&\quad -578340000e^t x^2t+1360800000e^t x^2+423360t^7\\
&\quad +6894720t^6+3447360t^5+1209600t^4\\
&\quad -34020000t^3-680400000t^2\Big).
\end{aligned} \tag{5.5}$$

The exact solution, approximate solution, absolute error, relative error, L_2-norm error, maximum absolute error, and maximum relative error at some time levels are presented in Tables 1 and 2.

Table 1

(a) Exact solution, approximate solution, absolute error, and relative error of $u(x,t)$ at the time $t = 0.25$

x	Exact solution	Approximate solution	Absolute error	Relative error
0.1	0.2628402542	0.2628402542	0.00	0.00
0.2	0.3013610167	0.3013610083	8.20×10^{-9}	2.75×10^{-8}
0.3	0.3655622875	0.3655622312	5.62×10^{-8}	1.54×10^{-7}
0.4	0.4554440667	0.4554438471	2.20×10^{-7}	4.81×10^{-7}
0.5	0.5710063542	0.5710057031	6.51×10^{-7}	1.14×10^{-6}
0.6	0.7122491501	0.7122475258	1.62×10^{-6}	2.28×10^{-6}
0.7	0.8791724543	0.8791688716	3.58×10^{-6}	4.07×10^{-6}
0.8	1.071776267	1.071769074	7.19×10^{-6}	6.71×10^{-6}
0.9	1.290060588	1.290047189	1.34×10^{-6}	1.03×10^{-5}
1	1.534025417	1.534001959	2.34×10^{-5}	1.52×10^{-5}

(b) Exact solution, approximate solution, absolute error, and relative error of $u(x,t)$ at the time $t = 0.50$

x	Exact solution	Approximate solution	Absolute error	Relative error
0.1	0.5164872127	0.5164872161	3.40×10^{-9}	6.19×10^{-9}
0.2	0.5659488508	0.5659488481	2.70×10^{-9}	4.77×10^{-9}
0.3	0.6483849144	0.6483848411	7.33×10^{-8}	1.13×10^{-7}
0.4	0.7637954034	0.7637950723	3.31×10^{-7}	4.33×10^{-7}
0.5	0.9121803178	0.9121793115	1.01×10^{-6}	1.10×10^{-6}
0.6	1.093539658	1.093537168	2.49×10^{-6}	2.27×10^{-6}
0.7	1.307873423	1.307868043	5.38×10^{-6}	4.11×10^{-6}
0.8	1.555181613	1.555171074	1.05×10^{-5}	6.77×10^{-6}
0.9	1.835464230	1.835445111	1.91×10^{-5}	1.04×10^{-5}
1	2.148721271	2.148688717	3.26×10^{-5}	1.51×10^{-5}

(c) Exact solution, approximate solution, absolute error, and relative error of $u(x,t)$ at the time $t = 0.75$

x	Exact solution	Approximate solution	Absolute error	Relative error
0.1	0.7711700002	0.7711700127	1.19×10^{-8}	1.54×10^{-8}
0.2	0.8346800007	0.8346800302	2.88×10^{-8}	3.45×10^{-8}
0.3	0.9405300015	0.9405299770	2.45×10^{-8}	2.60×10^{-8}
0.4	1.088720003	1.088719693	3.11×10^{-7}	2.85×10^{-7}
0.5	1.279250004	1.279248891	1.11×10^{-6}	8.70×10^{-7}
0.6	1.512120006	1.512117104	2.90×10^{-6}	1.91×10^{-6}
0.7	1.787330008	1.787323655	6.35×10^{-6}	3.55×10^{-6}
0.8	2.104880011	2.104867651	1.24×10^{-5}	5.87×10^{-6}
0.9	2.464770014	2.464748039	2.20×10^{-5}	8.91×10^{-6}
1	2.867000017	2.866963750	3.62×10^{-5}	1.26×10^{-5}

(d) Exact solution, approximate solution, absolute error, and relative error of $u(x,t)$ at the time $t = 1$

x	Exact solution	Approximate solution	Absolute error	Relative error
0.1	1.027182818	1.027182849	3.07×10^{-8}	2.98×10^{-8}
0.2	1.108731273	1.108731374	1.00×10^{-7}	9.04×10^{-8}
0.3	1.244645364	1.244645495	1.31×10^{-7}	1.04×10^{-7}
0.4	1.434925092	1.434925050	4.23×10^{-8}	2.94×10^{-8}
0.5	1.679570457	1.679569755	7.01×10^{-7}	4.17×10^{-7}
0.6	1.978581458	1.978579197	2.26×10^{-6}	1.14×10^{-6}
0.7	2.331958096	2.331952871	5.22×10^{-6}	2.24×10^{-6}
0.8	2.739700370	2.739690321	1.00×10^{-5}	3.66×10^{-6}
0.9	3.201808281	3.201791426	1.69×10^{-5}	5.26×10^{-6}
1	3.718281828	3.718256914	2.49×10^{-5}	6.70×10^{-6}

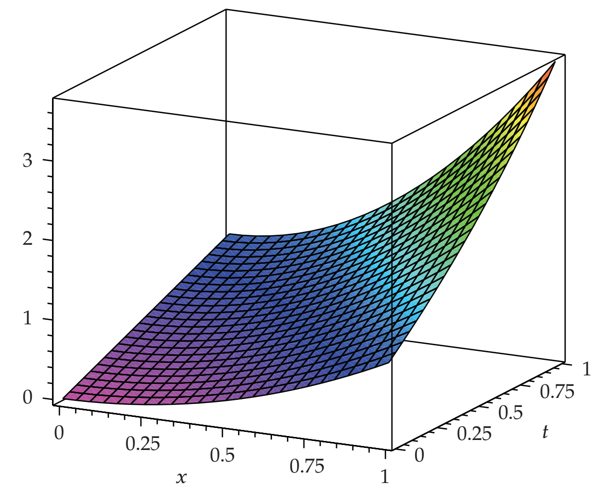

Figure 2: Approximate solution of the nonlinear diffusion problem on the interval $[0, 1]$.

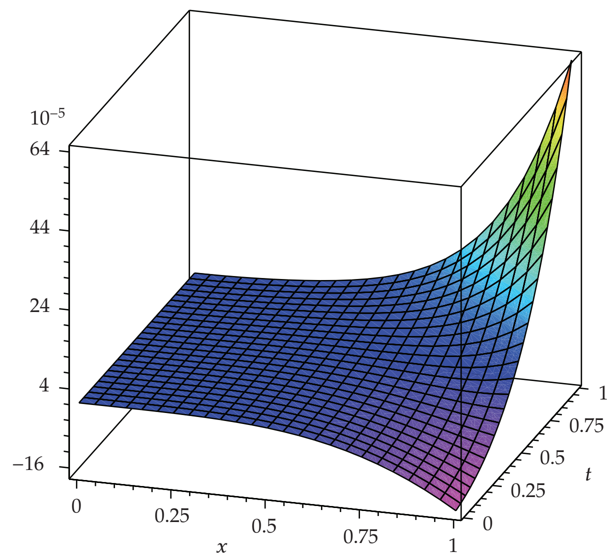

Figure 3: Absolute error of the nonlinear diffusion problem on the interval $[0, 1]$.

Table 2: L_2-norm error $\|u_e(x,t) - u_a(x,t)\|_2$, maximum absolute error, and maximum relative error at the times $t = 0.25, 0.50, 0.75$, and 1.

t	$\|u_e(x,t) - u_a(x,t)\|_2$	Maximum absolute error	Maximum relative error
0.25	0.001934504062	0.00002345900000	0.00001529179356
0.5	0.002320886866	0.00003255400000	0.00001515040617
0.75	0.002486049062	0.00003626700000	0.00001264980809
1	0.002172409395	0.00002491464924	0.00000670058118

In the tables fortunately, we do not have any diametrical sharp changes in our error bounds and it has no common difficulties that may appear in numerical approaches like Runge's phenomenon [39]. This means that our method works steady at all time levels. Also, the computed relative errors magnitude is acceptable and it makes our approach admissible.

Figure 1 represents the exact solution of the nonlinear diffusion problem on the interval $[0, 1]$. As it is illustrated in Figure 2, the approximate solution gives the solution in function form. We would like to emphasize that we have presented the results in tables at some points, in order to compare our computed values with exact solution easily.

In addition, it is possible to draw the absolute error graph because it is yield in function form too. We drew absolute error function in Figure 3, to show how little its magnitude is.

6. Conclusions

In this study, we consider the Cauchy problem of unidimensional nonlinear diffusion equation. This problem is inherently ill-posed and unsteady. If the analytical solution exists, it needs some rigid and sophisticated computation in practice. We investigate this problem with a very modern acclaimed powerful method called HPM. Our simple rapid exact approach yields good results as we have reported in Section 5. We have computed an approximate solution with acceptable error bounds which are at least of order 10^{-5}. That makes our technique remarkable and convenient. We have used *Maple 11 Packages* on common home PC for all of our computations.

References

[1] H. Mehrer, *Diffusion Solids Fundamentals Methods Materials Diffusion Controlled Processes*, Springer, Berlin, Germany, 2007.

[2] J. L. Vázquez, *The Porous Medium Equation*, Oxford Mathematical Monographs, The Clarendon Press, Oxford, UK, 2007.

[3] J. M. Burgers, *The Nonlinear Diffusion Equation*, Reidel Publishing Company, 1973.

[4] V. Kolokoltsov, *Semi Classical Analysis for Diffusion and Stochastic Processes*, Springer, Berlin, Germany, 2000.

[5] Z. Chen, G. Huan, and Y. Ma, *Computational Methods for Multiphase Flows in Porous Media*, Computational Science & Engineering, Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 2006.

[6] R. E. Cunningham and R. J. J. Williams, *Diffusion in Gasses and Porous Media*, Plenum Press, New York, NY, USA, 1980.

[7] W. Jäger, R. Rannacher, and J. Warnatz, *Reactive Flows, Diffusion and Transport*, Springer, Berlin, Germany, 2007.

[8] J. Kärger and P. Heitjans, *Diffusion Condesed Matter*, Springer, Berlin, Germany, 2005.

[9] A. Aminataei, M. Sharan, and M. P. Singh, "A numerical solution for the nonlinear convective facilitated-diffusion reaction problem for the process of blood oxygenation in the lungs," *Journal of National Academy Mathematics*, vol. 3, pp. 182–187, 1985.

[10] A. Aminataei, M. Sharan, and M. P. Singh, "A numerical model for the process of gas exchange in the pulmonary capillaries," *Indian Journal of Pure and Applied Mathematics*, vol. 18, pp. 1040–1060, 1987.

[11] A. Aminataei, M. Sharan, and M. P. Singh, "Two-layer model for the process of blood oxygenation in the pulmonary capillaries-parabolic profiles in the core as well as in the plasma layer," *Applied Mathematical Modelling*, vol. 12, no. 6, pp. 601–609, 1988.

[12] A. Aminataei, "A numerical two layer model for blood oxygenation in lungs," *Amirkabir Journal of Science and Technology*, vol. 12, no. 45, pp. 63–85, 2001.

[13] A. Aminataei, "Comparison of explicit and implicit approaches to numerical solution of uni dimensional equation of diffusion," *Journal of Science, Al-Zahra University*, vol. 15, no. 2, pp. 1–20 & 57, 2002.

[14] A. Aminataei, "Blood oxygenation in the pulmonary circulation. a review," *The European Journal of Scientific Research*, vol. 10, no. 2, pp. 55–71, 2005.
[15] A. Aminataei, "A numerical simulation of the unsteady convective-diffusion equation," *The Journal of Damghan University of Basic Sciences*, vol. 1, no. 2, pp. 73–87, 2008.
[16] E. A. Saied and M. M. Hussein, "New classes of similarity solutions of the inhomogeneous nonlinear diffusion equations," *Journal of Physics A*, vol. 27, no. 14, pp. 4867–4874, 1994.
[17] J.-H. He, "The homotopy perturbation method nonlinear oscillators with discontinuities," *Applied Mathematics and Computation*, vol. 151, no. 1, pp. 287–292, 2004.
[18] J.-H. He, "Homotopy perturbation method for solving boundary value problems," *Physics Letters A*, vol. 350, no. 1-2, pp. 87–88, 2006.
[19] A. M. Siddiqui, R. Mahmood, and Q. K. Ghori, "Herschel-Bulkley fluid model for blood flow through a stenosed artery," *Physics Letters A*, vol. 352, no. 45, pp. 404–416, 2006.
[20] P. D. Ariel, "Optimal homotopy perturbation method for strongly nonlinear differential equations," *Nonlinear Science Letters A*, vol. 1, no. 2, pp. 43–52, 2010.
[21] V. Marinca and N. Herisanu, "Homotopy perturbation method and the natural convection flow of a third grade fluid through a circular tube," *Nonlinear Science Letters A*, vol. 1, no. 3, pp. 273–280, 2010.
[22] M. El-Shahed, "Application of he's homotopy perturbation method to volterra's integro-differential equation," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 6, no. 2, p. 163, 2005.
[23] M. Ghasemi, M. Tavassoli Khanjani, and A. Davari, "Numerical solution of two-dimensional nonlinear differential equation by homotopy perturbation method," *Applied Mathematics and Computation*, vol. 189, p. 341, 2007.
[24] M. Dehghan and F. Shakeri, "Solution of a partial differential equation subject to temperature overspecification by He' s homotopy perturbation method," *Physica Scripta*, vol. 75, no. 6, pp. 778–787, 2007.
[25] M. A. Jafari and A. Aminataei, "Numerical solution of problems in calculus of variations by homotopy perturbation method," *AIP Conference Proceedings*, vol. 1048, pp. 282–285, 2008.
[26] A. Aminataei and Q. Jannati, "Using homotopy perturbation method in the numerical solution of partial differential equation," *ICASTOR Journal of Mathematical Sciences*, vol. 2, no. 2, pp. 45–56, 2008.
[27] A. Zakeri and Q. Jannati, "An inverse problem for parabolic partial differential equations with nonlinear conductivity term," *Scholarly Research Exchange*, vol. 2009, Article ID 468570, 2009.
[28] M. A. Jafari and A. Aminataei, "Homotopy perturbation method for computing eigenelements of Sturm-Liouville two point boundary value problem," *Applied Mathematical Sciences*, vol. 3, no. 31, pp. 1519–1524, 2009.
[29] M. A. Jafari and A. Aminataei, "Application of homotopy perturbation method in the solution of Fokker-Planck equation," *Physica Scripta*, vol. 80, no. 5, Article ID 055001, 2009.
[30] J.-H. He, "An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering," *International Journal of Modern Physics B*, vol. 22, no. 21, pp. 3487–3578, 2008.
[31] J.-H. He, "Recent development of the homotopy perturbation method," *Topological Methods in Nonlinear Analysis*, vol. 31, no. 2, pp. 205–209, 2008.
[32] M. A. Noor and S. T. Mohyud-Din, "Homotopy perturbation method for solving nonlinear higher-order boundary value problems," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 9, no. 4, pp. 395–408, 2008.
[33] A. Yildirim, "Exact solutions of nonlinear differential-difference equations by he's homotopy perturbation method," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 9, no. 2, pp. 111–114, 2008.
[34] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1964.
[35] J.-H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.
[36] J.-H. He, "Some asymptotic methods for strongly nonlinear equations," *International Journal of Modern Physics B*, vol. 20, no. 10, pp. 1141–1199, 2006.
[37] S. J. Liao, *Beyond Perturbation*, CRC Press, Boca Raton, Fla, USA, 2003.
[38] A. H. Nayfeh, *Introduction to Perturbation Techniques*, Wiley-Interscience, New York, NY, USA, 1981.
[39] J.-P. Berrut and L. N. Trefethen, "Barycentric Lagrange interpolation," *SIAM Review*, vol. 46, no. 3, pp. 501–517, 2004.