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Research Article

A Note on the (h, q)-Extension of Bernoulli Numbers and Bernoulli Polynomials

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We observe the behavior of roots of the (h, q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the q-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$. The main purpose of this paper is also to investigate the zeros of the (h, q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$. Furthermore, we give a table for the zeros of the (h, q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$.

1. Introduction

Throughout this paper $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$, and $\mathbb{C}_p$ will be denoted by the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively, compare with [1–6]. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume that $|q-1|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. If $q \in \mathbb{C}$, then we normally assume that $|q| < 1$. For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{f \mid f : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function$\}$, the p-adic q-integral (or q-Volkenborn integration) was defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \tag{1.1}$$

where $[x]_q = (1-q^x)/(1-q)$, compare with [1–8]. Thus, we note that

$$I_1(f) = \lim_{q\to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{0\le x<p^N} f(x), \quad \text{compare with } [1,2,3,4,5,6]. \tag{1.2}$$

By (1.2), we easily see that

$$I_1(f_1) = I_1(f) + f'(0), \quad \text{compare with } [1,2,3,4,5,6], \tag{1.3}$$

where $f_1(x) = f(x+1)$, $f'(0) = (d/dx)f(x)|_{x=0}$.

In (1.3), if we take $f(x) = q^{hx}e^{xt}$, then we have

$$\int_{\mathbb{Z}_p} q^{hx}e^{xt}d\mu_1(x) = \frac{h\log q + t}{q^h e^t - 1}, \quad \text{compare with } [6], \tag{1.4}$$

for $|t| \le p^{-1/(p-1)}$, $h \in \mathbb{Z}$.

Recently, many mathematicians have studied Bernoulli numbers and Bernoulli polynomials. Bernoulli polynomials possess many interesting properties and arising in many areas of mathematics and physics. For more studies in this subject we may see references [1–8]. The motivation for this study comes from the following papers. Some interesting analogues of the Bernoulli numbers and polynomials were investigated by Ryoo and Kim [6]. We begin by recalling here definitions of (h,q)-extension of Bernoulli numbers and polynomials as follows.

Definition 1.1 (see [6]). The (h,q)-extension of Bernoulli numbers $B_{n,q}^{(h)}$ and polynomials $B_{n,q}^{(h)}(x)$ is defined by means of the generating functions as follows:

$$\begin{aligned} F_q^{(h)}(t) &= \frac{h\log q + t}{q^h e^t - 1} = \sum_{n=0}^{\infty} B_{n,q}^{(h)}\frac{t^n}{n!}, \\ F_q^{(h)}(t,x) &= \frac{h\log q + t}{q^h e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(h)}(x)\frac{t^n}{n!}. \end{aligned} \tag{1.5}$$

Note that $B_{n,q}^{(h)}(0) = B_{n,q}^{(h)}$, $\lim_{q\to 1}B_{n,q}^{(h)}(x) = B_n(x)$, and $B_{n,q}^{(0)}(x) = B_n(x)$, where B_n are the nth Bernoulli numbers.

By (1.4) and (1.5), we have the following Witt formula. For $h \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p \le p^{-1/(p-1)}$, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{hx}x^n d\mu_1(x) &= B_{n,q}^{(h)}, \\ \int_{\mathbb{Z}_p} q^{hy}(x+y)^n d\mu_1(y) &= B_{n,q}^{(h)}(x). \end{aligned} \tag{1.6}$$

In this paper, we investigate the (h,q)-extension of Bernoulli numbers and Bernoulli polynomials in order to obtain some interesting results and explicit relationships. The aim of this paper to observe an interesting phenomenon of "scattering" of the zeros of the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$. The outline of this paper is as follows. In Section 2, we study the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$. In Section 3, we describe the beautiful zeros of the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$ using a numerical investigation. Also we display distribution and structure of the zeros of the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$ by using computer. By using the results of our paper, the readers can observe the regular behaviour of the roots of the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$. Finally, we carried out computer experiments for demonstrating a remarkably regular structure of the complex roots of the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$.

2. Basic Properties for the (h,q)-Extension of Bernoulli Numbers and Bernoulli Polynomials

Let q be a complex number with $|q| < 1$ and $h \in \mathbb{Z}$. By the meaning of (1.5), the (h,q)-extension of Bernoulli numbers $B_{n,q}^{(h)}$ and Bernoulli polynomials $B_{n,q}^{(h)}(x)$ is defined by means of the following generating function:

$$F_q^{(h)}(t) = \frac{h\log q + t}{q^h e^t - 1} = \sum_{n=0}^{\infty} B_{n,q}^{(h)} \frac{t^n}{n!}, \tag{2.1}$$

$$F_q^{(h)}(x,t) = \frac{h\log q + t}{q^h e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(h)}(x) \frac{t^n}{n!}, \tag{2.2}$$

respectively.

Here is the list of the first (h,q)-extension of Bernoulli numbers $B_{n,q}^{(h)}$.

$$\begin{gathered}
B_{0,q}^{(h)} = \frac{h\log q}{-1+q^h}, \qquad B_{1,q} = \frac{1}{-1+q^h} - \frac{hq^h\log q}{(-1+q^h)^2}, \\
B_{2,q^h}^{(h)} = -\frac{2q^h}{(-1+q^h)^2} - \frac{hq^h\log q}{(-1+q^h)^2} + \frac{2hq^{2h}\log q}{(-1+q^h)^3}, \\
B_{3,q^h}^{(h)} = -\frac{3q^h}{(-1+q^h)^2} + \frac{6q^{2h}}{(-1+q^h)^3} - \frac{hq^h\log q}{(-1+q^h)^2} + \frac{6hq^{2h}\log q}{(-1+q^h)^3} - \frac{6hq^{3h}\log q}{(-1+q^h)^4}, \ldots,
\end{gathered} \tag{2.3}$$

because

$$\frac{\partial}{\partial x} F_q^{(h)}(x,t) = tF_q^{(h)}(x,t) = \sum_{n=0}^{\infty} \frac{d}{dx} B_{n,q}^{(h)}(x) \frac{t^n}{n!}, \tag{2.4}$$

it follows the important relation

$$\frac{d}{dx}B_{n,q}^{(h)}(x) = nB_{n-1,q}^{(h)}(x). \tag{2.5}$$

We have the integral formula as follows:

$$\int_a^b B_{n-1,q}^{(h)}(x)dx = \frac{1}{n}\left(B_{n,q}^{(h)}(b) - B_{n,q}^{(h)}(a)\right). \tag{2.6}$$

Here is the list of the first (h,q)-extension of Bernoulli Polynomials $B_{n,q}^{(h)}(x)$.

$$\begin{gathered}
B_{0,q}^{(h)} = \frac{h\log q}{-1+q^h}, \\
B_{1,q}^{(h)} = \frac{1}{(-1+q^h)} - \frac{hq^h\log q}{(-1+q^h)^2} + \frac{hx\log q}{(-1+q^h)}, \\
B_{2,q}^{(h)} = \frac{2q^h}{(-1+q^h)^2} + \frac{2x}{(-1+q^h)} + \frac{2hq^{2h}\log q}{(-1+q^h)^3} - \frac{hq^h\log q}{(-1+q^h)^2} - \frac{2hq^h x\log q}{(-1+q^h)^2} + \frac{hx^2\log q}{(-1+q^h)}, \ldots.
\end{gathered} \tag{2.7}$$

Since

$$\begin{aligned}
\sum_{l=0}^{\infty} B_{l,q}^{(h)}(x+y)\frac{t^l}{l!} = \frac{h\log q + t}{q^h e^t - 1}e^{(x+y)t} &= \sum_{n=0}^{\infty} B_{n,q}^{(h)}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty} y^m \frac{t^m}{m!} \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} B_{n,q}^{(h)}(x)\frac{t^n}{n!}y^{l-n}\frac{t^{l-n}}{(l-n)!}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n}B_{n,q}^{(h)}(x)y^{l-n}\right)\frac{t^l}{l!},
\end{aligned} \tag{2.8}$$

we have the following theorem.

Theorem 2.1. *(h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$ satisfies the following relation:*

$$B_{l,q}^{(h)}(x+y) = \sum_{n=0}^{l}\binom{l}{n}B_{n,q}^{(h)}(x)y^{l-n}. \tag{2.9}$$

From (2.2)*, we can derive the following equality:*

$$\sum_{n=0}^{\infty}\left(q^h B_{n,q}^{(h)}(x+1) - B_{n,q}^{(h)}(x)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(x^n h\log q + nx^{n-1}\right)\frac{t^n}{n!}. \tag{2.10}$$

Hence, we obtain the following difference equation.

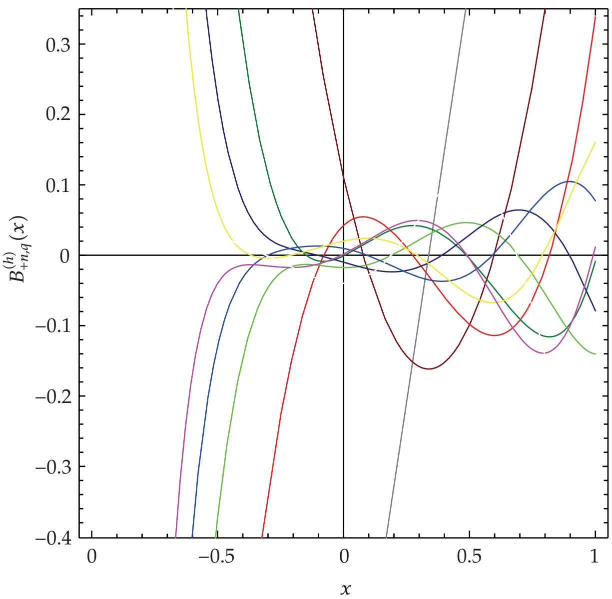

Figure 1: Curve of $B^{(3)}_{n,1/2}(x)$.

Theorem 2.2. *For any positive integer n, we obtain*

$$q^h B^{(h)}_{n,q}(x+1) - B^{(h)}_{n,q}(x) = x^n h \log q + n x^{n-1}. \tag{2.11}$$

3. Distribution and Structure of the Zeros

In this section, we assume that $q \in \mathbb{C}$, with $|q| < 1$. We observed the behavior of real roots of the (h,q)-extension of Bernoulli polynomials $B^{(h)}_{n,q}(x)$. We display the shapes of the (h,q)-extension of Bernoulli polynomials $B_{n,q}(x)$ and we investigate the zeros of the (h,q)-extension of Bernoulli polynomials $B^{(h)}_{n,q}(x)$. For $n = 1,\ldots,10$, we can draw a plot of the (h,q)-extension of Bernoulli polynomials $B^{(h)}_{n,q}(x)$, respectively. This shows the ten plots combined into one. We display the shape of $B^{(h)}_{n,q}(x)$, $-1 \le x \le 1$, $q = 1/2$ (Figure 1). We investigate the beautiful zeros of the (h,q)-extension of Bernoulli polynomials $B^{(h)}_{n,q}(x)$ by using a computer. We plot the zeros of the (h,q)-extension of Bernoulli polynomials $B^{(3)}_{n,q}(x)$ for $n = 15, 20, 25, 30$ and $x \in \mathbb{C}$ (Figure 2).

Our numerical results for approximate solutions of real zeros of $B^{(h)}_{n,1/2}(x)$ are displayed (Tables 1 and 2).

We plot the zeros of (h,q)-extension of Bernoulli polynomials $B^{(h)}_{n,q}(x)$ for $n = 30$, $q = 1/2$, $h = 5, 7, 9, 11$, and $x \in \mathbb{C}$ (Figure 3). We plot the zeros of (h,q)-extension of Bernoulli polynomials $B^{(h)}_{n,q}(x)$ for $n = 30$, $q = 9/10, 99/100$, and $x \in \mathbb{C}$ (Figure 4).

We observe a remarkably regular structure of the complex roots of the (h,q)-extension of Bernoulli polynomials $B^{(h)}_{n,q}(x)$. We hope to verify a remarkably regular structure of the complex roots of the (h,q)-extension of Bernoulli polynomials $B^{(h)}_{n,q}(x)$ (Table 1). This numerical investigation is especially exciting because we can obtain an interesting

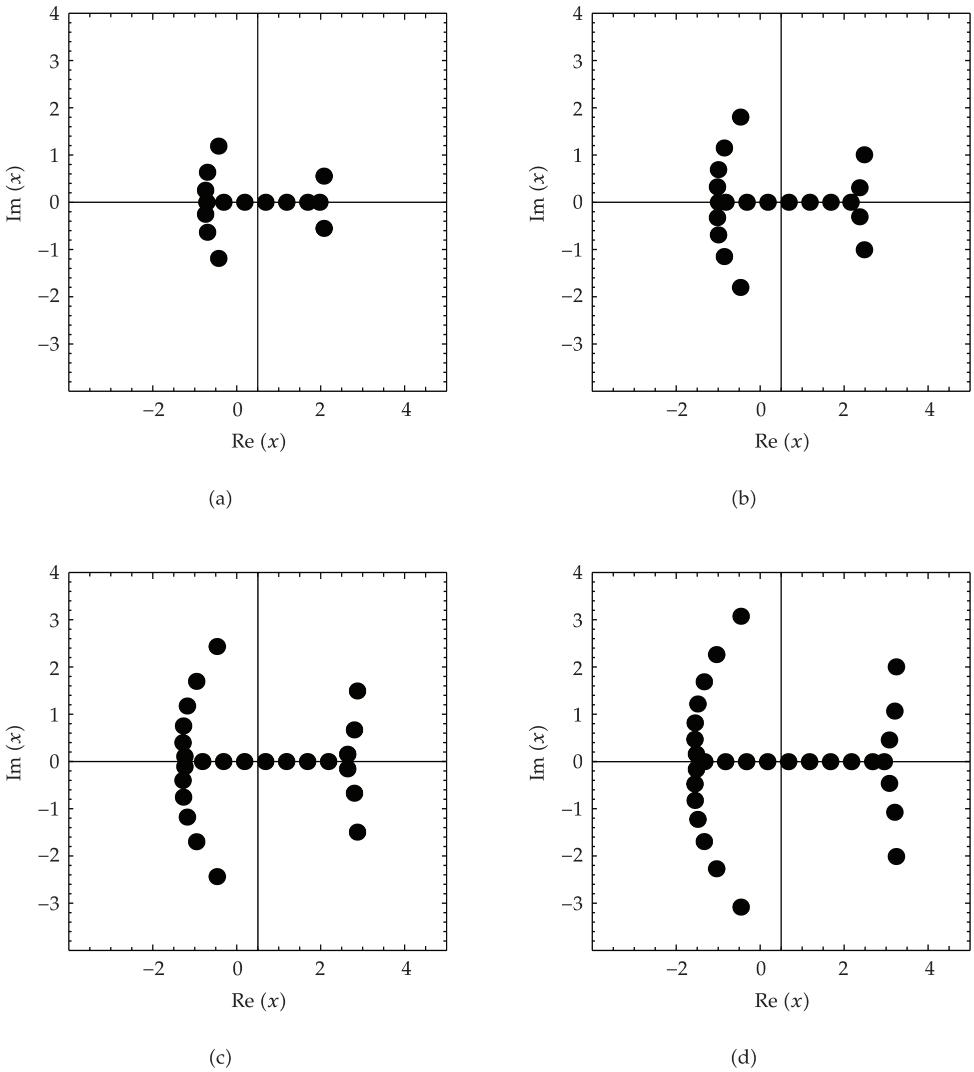

Figure 2: Zeros of $B_{n,1/2}^{(3)}(x)$ for $n = 15, 20, 25, 30$.

phenomenon of scattering of the zeros of the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$. These results are used not only in pure mathematics and applied mathematics, but also used in mathematical physics and other areas. Next, we calculated an approximate solution satisfying the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$. The results are given in Table 2.

Stacks of zeros of $B_{n,q}^{(h)}(x)$ for $q = 1/3,\ 1 \leq n \leq 30$ from a 3D structure are presented (in Figure 5).

Figure 6 presents the distribution of real zeros of the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(3)}(x)$ for $q = 1/2,\ 1 \leq n \leq 30$.

Figure 7 presents the distribution of real zeros of the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(3)}(x)$ for $q = 9/10,\ 1 \leq n \leq 30$.

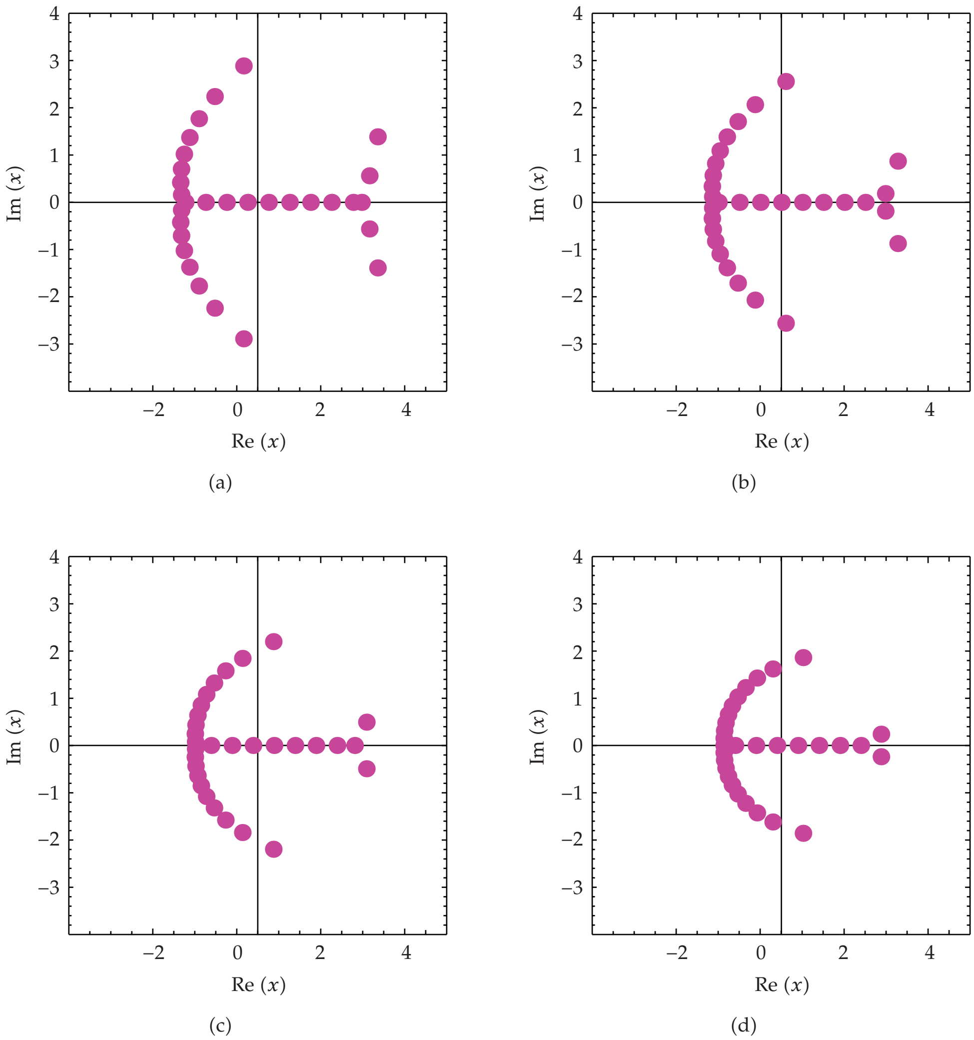

Figure 3: Zeros of $B_{30,1/2}(x)$ for $h = 5, 7, 9, 11$.

Figure 8 presents the distribution of real zeros of the Bernoulli polynomials $B_n(x)$ for $1 \leq n \leq 30$.

4. Direction for Further Research

In [7], we observed the behavior of complex roots of the Bernoulli polynomials $B_n(x)$, using numerical investigation. Prove that $B_n(x)$, $x \in \mathbb{C}$, has $\mathrm{Re}(x) = 1/2$ reflection symmetry in addition to the usual $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions. The obvious corollary is that the zeros of $B_n(x)$ will also inherit these symmetries.

$$\text{If } B_n(x_0) = 0, \text{ then } B_n(1 - x_0) = 0 = B_n(x_0^*) = B_n(1 - x_0^*), \tag{4.1}$$

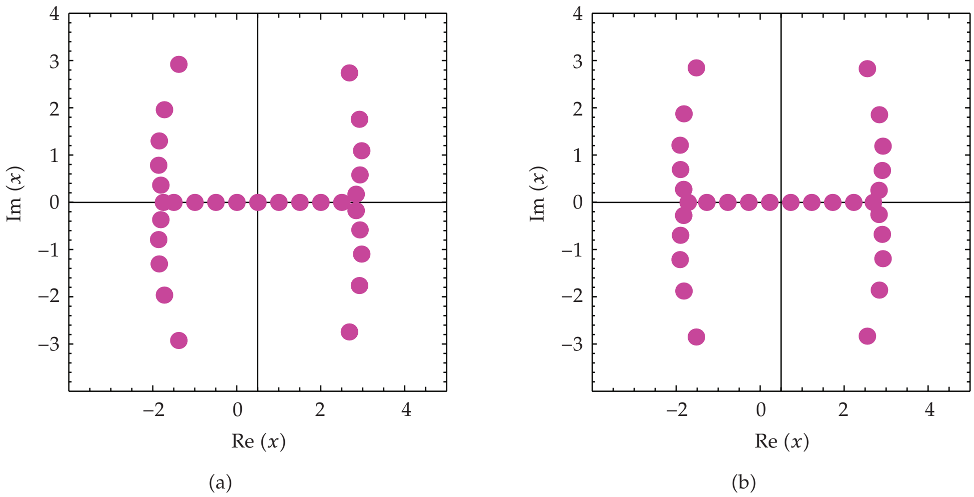

Figure 4: Zeros of $B^{(3)}_{n,30}(x)$ for $q = 9/10, 99/100$.

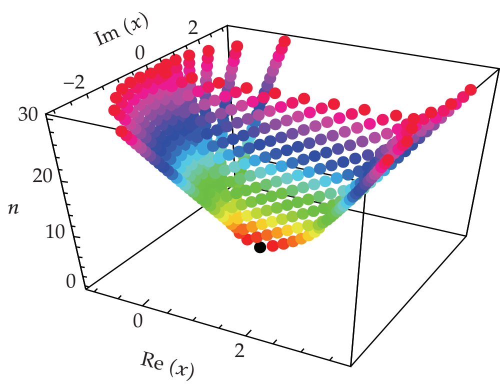

Figure 5: Stacks of zeros of $B^{(h)}_{n,q}(x)$, $1 \le n \le 30$.

Table 1: Numbers of real and complex zeros of $B^{(h)}_{n,q}(x)$.

degree n	$h = 3$		$h = 5$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	4	0	2	2
5	3	2	3	2
6	4	2	4	2
7	5	2	5	2
8	6	2	4	4
9	3	6	5	4
10	4	6	6	4
11	5	6	5	6
12	6	6	4	8

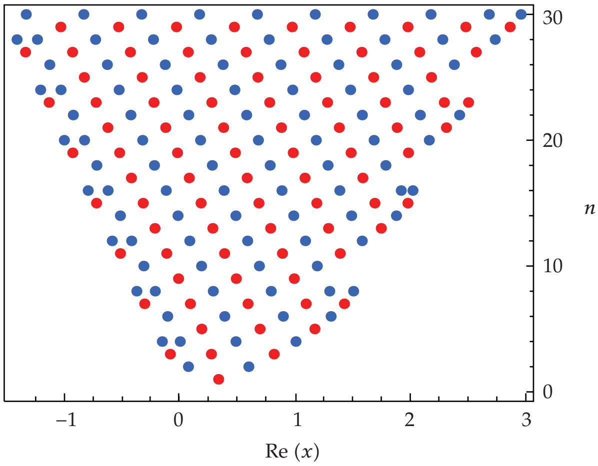

Figure 6: Real of zeros of $B_{n,q}^{(3)}(x)$, $q = 1/2$, $1 \le n \le 30$.

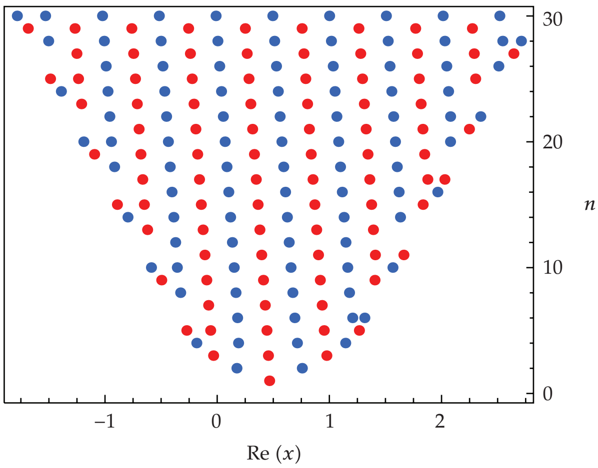

Figure 7: Real of zeros of $B_{n,q}^{(3)}(x)$, $q = 9/10$, $1 \le n \le 30$.

Table 2: Approximate solutions of $B_{n,q}^{(3)}(x) = 0$, $q = 1/2$, $x \in \mathbb{R}$.

degree n	x
1	0.338041204
2	0.077277108, 0.598805301
3	−0.079078401, 0.27548817, 0.81771384
4	−0.14859649, 0.00639164, 0.48798387, 1.00638579
5	0.1922804, 0.6960320, 1.16991832
6	−0.1019335, 0.3908601, 0.8972294, 1.3100253
7	−0.300737, 0.094132, 0.592060, 1.094167, 1.4250444
⋮	⋮

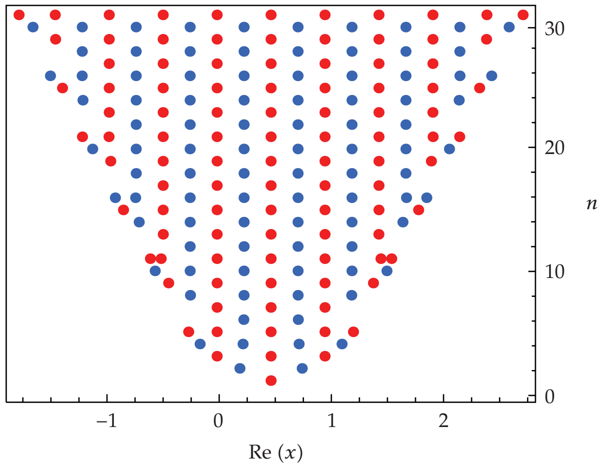

Figure 8: Real of zeros of $B_n(x)$, $1 \leq n \leq 30$.

where $*$ denotes complex conjugation (see [7]). Finally, we shall consider the more general problems. Prove that $B_n(x) = 0$ has n distinct solutions. If $B_{2n+1}(x)$ has $\mathrm{Re}(x) = 1/2$ and $\mathrm{Im}(x) = 0$ reflection symmetries, and $2n+1$ nondegenerate zeros, then $2n$ of the distinct zeros will satisfy (4.1). If the remaining one zero is to satisfy (4.1) too, it must reflect into itself, and therefore it must lie at $1/2$, the center of the structure of the zeros, that is,

$$B_n\left(\frac{1}{2}\right) = 0 \quad \forall \text{odd } n. \tag{4.2}$$

Prove that $B_{n,q}^{(h)}(x) = 0$ has n distinct solutions, that is, all the zeros are nondegenerate. Find the numbers of complex zeros $C_{B_{n,q}^{(h)}(x)}$ of $B_{n,q}^{(h)}(x)$, $\mathrm{Im}(x) \neq 0$. Since n is the degree of the polynomial $B_{n,q}^{(h)}(x)$, the number of real zeros $R_{B_{n,q}^{(h)}(x)}$ lying on the real plane $\mathrm{Im}(x) = 0$ is then $R_{B_{n,q}^{(h)}(x)} = n - C_{B_{n,q}^{(h)}(x)}$, where $C_{B_{n,q}^{(h)}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{B_{n,q}^{(h)}(x)}$ and $C_{B_{n,q}^{(h)}(x)}$. Find the equation of envelope curves bounding the real zeros lying on the plane. We prove that $B_{n,q}^{(h)}(x)$, $x \in \mathbb{C}$, has $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions. If $B_{n,q}^{(h)}(x) = 0$, then $B_{n,q}^{(h)}(x^*) = 0$, where $*$ denotes complex conjugate (see Figures 2, 3, and 4). Observe that the structure of the zeros of the Bernoulli polynomials $B_n(x)$ resembles the structure of the zeros of the q-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ as $q \to 1$ (see Figures 3, 4, and 5). In order to study the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$, we must understand the structure of the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$. Therefore, using computer, a realistic study for the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$ plays an important part. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the (h,q)-extension of Bernoulli polynomials $B_{n,q}^{(h)}(x)$ to appear in mathematics and physics. For related topics, the interested reader is referred to [3–8].

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