Research Article

Global Behaviors of a Chemostat Model with Delayed Nutrient Recycling and Periodically Pulsed Input

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The dynamic behaviors in a chemostat model with delayed nutrient recycling and periodically pulsed input are studied. By introducing new analysis technique, the sufficient and necessary conditions on the permanence and extinction of the microorganisms are obtained. Furthermore, by using the Liapunov function method, the sufficient condition on the global attractivity of the model is established. Finally, an example is given to demonstrate the effectiveness of the results in this paper.

1. Introduction

This paper is mainly concerned with single-species chemostat-type model with nutrient recycling. Usually, nutrient recycling is regarded as an instantaneous term by neglecting the time required to regenerate nutrient from dead biomass by bacterial decomposition. The motivation for such models is given by Beretta et al. in [1], where such systems are used to model the growth of planktonic communities in lakes, where the plankton feeds on a limiting nutrient supplied at a constant rate. The basic single-species chemostat model with delayed nutrient recycling is the following differential equation:

$$\frac{dS}{dt} = D\left(S^0 - S\right) - mU(S)N + bD_1 \int_{-\infty}^{t} F(t-\tau)N(\tau)d\tau,$$

$$\frac{dN}{dt} = N[-(D + D_1) + m_1 U(S)]. \tag{1.1}$$

The chemostat models with nutrient recycling have been extensively investigated by many researchers. The studied main subjects are the persistence, permanence, and extinction of microorganisms, global stability and the existence of periodic oscillation of the systems, and so forth. Many important and interesting results can be found in [1–15] and the references cited therein. In [4], Freedman and Xu extended the single-species model proposed in [1] to two-species competition models with instantaneous and delayed nutrient recycling. They developed persistence and extinction criteria for the competing populations. In [10], by applying the method of Liapunov functionals they study the global asymptotic stability of the positive equilibria of the models in [4]. In [12], a chemostat model with distributed time delays both in material recycling and biotic species growth has been considered.

As it is well known, countless organisms live in seasonally or diurnally forced environment, in which the populations obtain food, so the effects of this forcing may be quite profound. Recently many papers studied chemostat model with variations in the supply of nutrients or the washout. The chemostat models with impulsive input perturbation have been studied in many articles see [16–22] and the references cited therein, where many important and interesting results on the persistence, permanence and extinction of microorganisms, global stability, the existence of periodic oscillation and dynamical complexity of the systems are discussed. Particular in [21], the following model of the lactic acid fermentation in membrane bioreactor with impulsive input is discussed:

$$
\begin{aligned}
\frac{dS}{dt} &= \frac{Q}{V}(S^0 - S) - \frac{\mu S x}{\delta_1(K_s + S)} + \left(\frac{K_D}{\delta_1} - m\right)x, \\
\frac{dx}{dt} &= \frac{\mu S x}{K_s + S} - \left(K_D + \frac{Q_1}{V}\right)x,
\end{aligned}
\qquad t \neq nT,
$$

$$
\begin{aligned}
\Delta S &= \frac{QS^0}{V}, \\
\Delta x &= 0.
\end{aligned}
\qquad t = nT, \ n \in N.
\tag{1.2}
$$

The model has both nutrient recycling and impulsive input substrate. Using Floquet's theory of impulsive periodic linear differential equations and small-amplitude perturbation, they obtain the biomass-free periodic solution is locally stable if some conditions are satisfied, see [21, Theorems 3.1]. In [18], the following Monod type chemostat model with nutrient recycling and impulsive input is studied:

$$
\begin{aligned}
\frac{dS(t)}{dt} &= -DS(t) - \frac{\mu_m S(t)x(t)}{\delta(K_m + S(t))} + b\gamma x(t), \\
\frac{dx(t)}{dt} &= \frac{\mu_m S(t)x(t)}{K_m + S(t)} - (D + \gamma)x(t),
\end{aligned}
\qquad t \neq nT,
$$

$$
\begin{aligned}
S(t^+) &= S(t) + DS^0, \\
x(t^+) &= x(t),
\end{aligned}
\qquad t = nT, \ n \in N.
\tag{1.3}
$$

The sufficient and necessary conditions of the permanence and extinction of the microorganism species and the sufficient condition of the global asymptotic stability of the model are established.

However, we see that few authors consider the chemostat models with delayed nutrient recycling and periodically pulsed input. Based on the ideas given in [18], we develop model (1.1) into the following form by introducing impulsive input:

$$
\begin{aligned}
\frac{dx(t)}{dt} &= -dx - mU(x)y + bd_1 \int_{-\infty}^{0} F(s)y(t+s)ds, \\
&\hspace{6cm} t \neq nT, \\
\frac{dy(t)}{dt} &= y[-(d+d_1) + m_1 U(x)], \\
x(t^+) &= x(t) + p, \\
&\hspace{4cm} t = nT, \ n \in N. \\
y(t^+) &= y(t),
\end{aligned}
\tag{1.4}
$$

For system (1.4), we will investigate the permanence, extinction and the global attractivity. We will establish the sufficient and necessary conditions for the permanence, extinction.

The rest of this paper is organized as follows. In the following section we will firstly introduce the basic assumption for system (1.4). Next, we will give a equivalent form of system (1.4) by introducing a new variable. Further, we will give several useful lemmas. In Section 3 we will state and prove a boundedness result for system (1.4). In Section 4 we will state and prove an extinction result and a permanence result for system (1.4). In Section 5 we will state and prove a global attractivity result for system (1.4). Finally, in Section 6, we will discuss an example and give some numerical simulations.

2. Preliminaries

In system (1.4), x is the concentration of a limiting nutrient and y is a measure of the population of some organism; $d > 0$ is the input and output flow, and is referred to as the wash-out rate; $m > 0$ is the maximum uptake rate of nutrient, $m_1 > 0$ is the maximum specific growth rate of the organism; $d_1 > 0$ is the death rate; $b \in (0,1)$ is the fraction of nutrient recycled after death of the species; $p > 0$ is the amount of the substrate concentration pulsed each nT, where $T > 0$ is a constant; N represents the set of all positive integers; $U(s)$ is the uptake function; Delay-kernel function $F(u)$ is a nonnegative bounded integrable function defined on $R_+ = [0, \infty)$. In this paper, we always assume that d, m, d_1, m_1, b and p are constants, and $F(u) = ae^{-au}$, where $a > 0$ is a constant.

For system (1.4), we always assume that uptake function $U(s)$ satisfies the following assumption:

(H) $U(0) = 0$, $U(s)$ is continuously differentiable for all $s \geq 0$ and $dU(s)/dt > 0$ for all $s \in [0, M_0]$, where $M_0 = p/(1 - e^{-dT})$.

The initial conditions in system (1.4) are given in the following form:

$$
\begin{aligned}
x(0) &= x_0 > 0, \\
y(\theta) = \varphi(\theta) \geq 0, \quad \theta \in R_-, \quad \varphi(0) > 0, \quad \sup_{\theta \in R_-} \varphi(\theta) < \infty,
\end{aligned}
\tag{2.1}
$$

where $R_- = (-\infty, 0]$. It is easy to prove that solution $(x(t), y(t))$ of system (1.4) with initial condition (2.1) is positive, that is, $x(t) > 0$ and $y(t) > 0$ in the interval of the existence.

Now, for system (1.4) we introduce a new variable $z(t)$ as follows:

$$z(t) = \int_{-\infty}^{t} F(t - \tau)y(\tau)d\tau. \tag{2.2}$$

Then, system (1.4) is equivalent to the following system:

$$
\begin{aligned}
\frac{dx(t)}{dt} &= -dx(t) - mU(x(t))y(t) + bd_1 z(t), \\
\frac{dy(t)}{dt} &= y(t)[-(d + d_1) + m_1 U(x(t))], \qquad t \neq nT, \\
\frac{dz(t)}{dt} &= -az(t) + ay(t), \\
x(t^+) &= x(t) + p, \\
y(t^+) &= y(t), \qquad t = nT, \ n \in N, \\
z(t^+) &= z(t),
\end{aligned}
\tag{2.3}
$$

where variable $z(t)$ can be interpreted as an intermediate component. Besides, initial condition (2.1) is changed into the following form

$$x(0) = x_0 > 0, \qquad y(0) = y_0 > 0, \qquad z(0) = z_0 > 0. \tag{2.4}$$

Therefore, in the rest of this paper we will mainly discuss system (2.3).

Firstly, on the positivity of solutions for system (2.3), we have the following result.

Lemma 2.1. *The solution $(x(t), y(t), z(t))$ of system* (2.3) *with initial condition* (2.4) *is positive, that is, $x(t) > 0$, $y(t) > 0$ and $z(t) > 0$ for any $t \geq 0$.*

The proof of Lemma 2.1 is simple, we hence omit it here.

We consider the following linear impulsive differential equation:

$$
\begin{aligned}
\frac{d\omega(t)}{dt} &= -d\omega(t) - \eta, \quad t \neq nT, \\
\omega(t^+) &= \omega(t) + p, \qquad t = nT, \ n \in N, \\
\omega(0^+) &= \omega_0,
\end{aligned}
\tag{2.5}
$$

where η is a constant. Clearly,

$$\omega^*(t) = -\frac{\eta}{d} + x^*(t), \quad t \in (nT, (n+1)T], \ n \in N \tag{2.6}$$

is the T-periodic solution of system (2.5), where

$$x^*(t) = \frac{pe^{-d(t-nT)}}{1 - e^{-dT}}, \quad t \in (nT, (n+1)T], \ n \in N. \tag{2.7}$$

We say that $\omega^*(t)$ is globally uniformly attractive, if for any constants $M > 0$ and $\varepsilon > 0$ there is a constant $T(M, \varepsilon) > 0$ such that for any initial time $t_0 \geq 0$ and any solution $\omega(t)$ of system (2.5) with $|\omega(t_0)| \leq M$, one has

$$|\omega(t) - \omega^*(t)| < \varepsilon \quad \forall t \geq t_0 + T(M, \varepsilon). \tag{2.8}$$

We have the following result.

Lemma 2.2. *T-periodic solution $\omega^*(t)$ of system* (2.5) *is globally uniformly attractive.*

Proof. Let $\omega(t)$ be any solution of system (2.5) with initial value $\omega(t_0)$. Define $u(t) = \omega(t) - \omega^*(t)$, then we have

$$\frac{du(t)}{dt} = -du(t), \quad t \neq nT,$$
$$u(t^+) = u(t), \quad t = nT, \ n \in N, \tag{2.9}$$
$$u(t_0) = \omega(t_0) - \omega^*(t_0).$$

The solution of system (2.9) is $u(t) = u(t_0)e^{-d(t-t_0)}$, $t \geq t_0$. For any constant $M > 0$ and $\varepsilon > 0$, when $|\omega(t_0)| \leq M$, we have

$$|u(t)| = |u(t_0)|e^{-d(t-t_0)}$$
$$\leq (M + M^*)e^{-d(t-t_0)}, \quad t \geq t_0, \tag{2.10}$$

where $M^* = \eta/d + p/(1 - e^{-dT})$. Choose $T(M, \varepsilon) = -(1/d)\ln(\varepsilon/(M + M^*))$; then for any $t_0 \geq 0$ we can obtain

$$|u(t)| \leq (M + M^*)e^{-d(t-t_0)} \leq 0 \tag{2.11}$$

for all $t \geq t_0 + T(M, \varepsilon)$. This shows that solution $\omega^*(t)$ is globally uniformly attractive. This completes the proof. $\qquad\square$

In system (2.5), when $\eta = 0$, then we obtain the subsystem of system (2.3) with $y(t) = 0$ and $z(t) = 0$ as follows:

$$\frac{dx(t)}{dt} = -dx(t), \quad t \neq nT,$$
$$x(t^+) = x(t) + p, \quad t = nT, \ n \in N. \tag{2.12}$$

Clearly, $x^*(t)$ is the positive T-periodic solution of system (2.12). Therefore, system (2.3) has the semitrivial T-periodic solution $(x^*(t), 0, 0)$. The solution of system (2.12) with initial condition $x(0^+) = x_0$ is

$$x(t) = \left(x_0 - \frac{p}{1 - e^{-dT}} \right) e^{-dt} + x^*(t), \quad t \in (nT, (n+1)T], \; n \in N. \tag{2.13}$$

From Lemma 2.2, we obtain that for any solution $x(t)$ of system (2.12), one has

$$|x(t) - x^*(t)| \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{2.14}$$

3. Boundedness

On the ultimate boundedness of all positive solutions of system (2.3) we have the following result.

Theorem 3.1. *Let $(x(t), y(t), z(t))$ be any positive solution of system (2.3) if*

$$0 < \frac{b}{a - d} < \frac{m}{am_1}, \tag{3.1}$$

then

$$\limsup_{t \to \infty} x(t) < M_0, \qquad \limsup_{t \to \infty} y(t) < \frac{m_1}{m} M_0, \qquad \limsup_{t \to \infty} z(t) < \frac{(a - d)M_0}{bd_1}, \tag{3.2}$$

where $M_0 = p/(1 - e^{-dT})$.

Proof. From condition (3.1), there is a constant $c > 0$ such that

$$bd_1 + cd - ac < 0, \quad ac - \frac{md_1}{m_1} < 0. \tag{3.3}$$

Let $(x(t), y(t), z(t))$ be any solution of system (2.3) define Liapunov function $V(t)$ as follows:

$$V(t) = x(t) + \frac{m}{m_1} y(t) + cz(t). \tag{3.4}$$

Calculating the derivative of $V(t)$ along solution $(x(t), y(t), z(t))$ of system (2.3), we have

$$\begin{aligned}
\frac{dV(t)}{dt} &= \frac{dx(t)}{dt} + \frac{m}{m_1} \frac{dy(t)}{dt} + c \frac{dz(t)}{dt} \\
&= -dV(t) + \left(ac - \frac{md_1}{m_1} \right) y + (bd_1 + cd - ac)z \\
&\leq -dV(t)
\end{aligned} \tag{3.5}$$

for all $t \geq 0$ and $t \neq nT$, and

$$V(t^+) = V(t) + p, \quad t = nT, \ n \in N. \tag{3.6}$$

From the comparison theorem of impulse differential equations, we have

$$V(t) \leq u(t), \quad \forall t \geq 0, \tag{3.7}$$

where $u(t)$ is the solution of system (2.12) with initial value $u(0^+) = V(0^+)$. From Lemma 2.2, we have $u(t) \to x^*(t)$ as $t \to \infty$. Hence, we further obtain

$$\limsup_{t \to \infty} V(t) \leq \frac{p}{1 - e^{-dT}} = M_0. \tag{3.8}$$

From this, we finally obtain

$$\limsup_{t \to \infty} x(t) < M_0, \qquad \limsup_{t \to \infty} y(t) < \frac{m_1}{m} M_0, \qquad \limsup_{t \to \infty} z(t) < \frac{(a-d)M_0}{bd_1}. \tag{3.9}$$

This completes the proof of Theorem 3.1. □

Return to the original system (1.4), we have the following corollary as a consequence of Theorem 3.1.

Corollary 3.2. *Let $(x(t), y(t))$ be any positive solution of system (1.4) if inequality (3.1) holds, then*

$$\limsup_{t \to \infty} x(t) < M_0, \qquad \limsup_{t \to \infty} y(t) < \frac{m_1}{m} M_0, \tag{3.10}$$

where $M_0 = p/(1 - e^{-dT})$.

Remark 3.3. Compare Theorem 3.1 with Lemma 2.3 given in [18] that we can see the ultimate boundedness is quite different between model with delayed nutrient recycling and the model with instantaneous nutrient recycling.

4. Extinction and Permanence

On the extinction of the microorganism species of system (2.3), we have the following result.

Theorem 4.1. *Suppose that inequality (3.1) holds and*

$$\int_0^T [-(d + d_1) + m_1 U(x^*(t))] dt \leq 0. \tag{4.1}$$

Then periodic solution $(x^*(t), 0, 0)$ of system (2.3) is globally attractive.

Proof. From condition (3.1), there is a constant $c > 0$ such that

$$bd_1 + cd - ac < 0, \quad ac - \frac{md_1}{m_1} < 0. \tag{4.2}$$

Let $(x(t), y(t), z(t))$ be any solution of system (2.3), define Liapunov function $V(t)$ as follows

$$V(t) = x(t) + \frac{m}{m_1} y(t) + cz(t). \tag{4.3}$$

Then similar to the proof of Theorem 3.1. We obtain $V(t) \leq u(t)$ for all $t \geq 0$, where $u(t)$ is the solution of system (2.12) with initial value $u(0^+) = V(0^+)$ and $u(t) \to x^*(t)$ as $t \to \infty$. Hence, there exists a function $\alpha(t) : R_+ \to R$ satisfying $\alpha(t) \to 0$ as $t \to \infty$ such that

$$V(t) \leq u(t) = x^*(t) + \alpha(t) \tag{4.4}$$

for all $t \geq 0$. From the definition of $V(t)$, we further have

$$x(t) \leq x^*(t) + \alpha(t) - \frac{m}{m_1} y(t) - cz(t). \tag{4.5}$$

From the second equation of system (2.3), we obtain

$$\frac{dy(t)}{dt} \leq y(t) \left[-(d + d_1) + m_1 U \left(x^*(t) + \alpha(t) - \frac{m}{m_1} y(t) - cz(t) \right) \right]. \tag{4.6}$$

From condition (4.1), we obtain for any $\varepsilon_0 > 0$

$$\int_0^T \left[-(d + d_1) + m_1 U \left(x^*(t) - \frac{m}{m_1} \varepsilon_0 \right) \right] dt < 0. \tag{4.7}$$

Since $\lim_{t \to \infty} \alpha(t) = 0$, we can obtain

$$\limsup_{t \to \infty} \int_t^{t+T} \left[-(d + d_1) + m_1 U \left(x^*(t) + \alpha(t) - \frac{m}{m_1} \varepsilon_0 - cz(t) \right) \right] dt$$

$$\leq \lim_{t \to \infty} \int_t^{t+T} \left[-(d + d_1) + m_1 U \left(x^*(t) + \alpha(t) - \frac{m}{m_1} \varepsilon_0 \right) \right] dt \tag{4.8}$$

$$= \int_0^T \left[-(d + d_1) + mp \left(s^*(t) - \frac{1}{m} \varepsilon_0 \right) \right] dt < 0.$$

Hence, there exist constants $\eta > 0$ and $T_0 > 0$ such that when $t \geq T_0$,

$$\int_t^{t+T} \left[-(d + d_1) + mp\left(s^*(t) + \alpha(t) - \frac{m}{m_1}\varepsilon_0 - cz(t) \right) \right] \leq -\eta. \tag{4.9}$$

and $|\alpha(t)| < 1$.

If $y(t) \geq \varepsilon_0$ for all $t \geq T_0$, then from (4.6) we obtain

$$\frac{dy(t)}{dt} \leq y(t) \left[-(d + d_1) + m_1 U\left(x^*(t) + \alpha(t) - \frac{m}{m_1}\varepsilon_0 - cz(t) \right) \right]. \tag{4.10}$$

For any $t \geq T_0$, we choose an integer $p \geq 0$ such that $t \in (T_0 + pT, T_0 + (p+1)T]$; then integrating (4.10) from T_0 to t, from (4.9), we can obtain

$$\begin{aligned}
y(t) &\leq y(T_0) \exp\left\{ \int_{T_0}^t \left[-(d + d_1) + m_1 U\left(x^*(t) + \alpha(t) - \frac{m}{m_1}\varepsilon_0 - cz(t) \right) \right] dt \right\} \\
&\leq y(T_0) \exp(-\eta p) \exp\left\{ \int_{T_0+pT}^t \left[-(d + d_1) + m_1 U\left(x^*(t) + \alpha(t) - \frac{m}{m_1}\varepsilon_0 - cz(t) \right) \right] dt \right\} \\
&\leq y(T_0) \exp(-\eta p) \exp\left(\left[U\left(M_0 + 1 - \frac{m}{m_1}\varepsilon_0 \right) - (d + d_1) \right] T \right),
\end{aligned} \tag{4.11}$$

where constant M_0 is given in Theorem 3.1. Since $p \to \infty$ as $t \to \infty$, from (4.11), we obtain $y(t) \to 0$ as $t \to \infty$ which leads to a contradiction. Hence, there is a $t^* \geq T_0$ such that $y(t^*) < \varepsilon_0$.

Now, we claim that there exists a constant $M_1 > 1$ such that $y(t) \leq \varepsilon_0 M_1$ for all $t \geq t^*$. In fact, if there exists a $t_1 > t^*$ such that $y(t_1) > \varepsilon_0 M_1$, then there exists a $t_2 \in (t^*, t_1)$ such that $y(t_2) = \varepsilon_0$ and $y(t) > \varepsilon_0$ for $t \in (t_2, t_1)$. Choose an integer $p \geq 0$ such that $t_1 \in [t_2 + pT, t_2 + (p+1)T)$. Since for any $t \in (t_2, t_1)$

$$\frac{dy(t)}{dt} \leq y(t) \left[-(d + d_1) + m_1 U\left(x^*(t) + \alpha(t) - \frac{m}{m_1}\varepsilon_0 - cz(t) \right) \right], \tag{4.12}$$

integrating this inequality from t_2 to t_1, from (4.9) we can obtain

$$\begin{aligned}
y(t_1) &\leq y(t_2) \exp\left\{ \int_{t_2}^{t_1} \left[-(d + d_1) + m_1 U\left(x^*(t) + \alpha(t) - \frac{m}{m_1}\varepsilon_0 - cz(t) \right) \right] dt \right\} \\
&\leq y(t_2) \exp(-\eta p) \exp\left\{ \int_{t_2+pT}^{t_1} \left[-(d + d_1) + m_1 U\left(x^*(t) + \alpha(t) - \frac{m}{m_1}\varepsilon_0 \right) \right] dt \right\} \\
&\leq \varepsilon_0 \exp\left(\left[U\left(M_0 + 1 - \frac{m}{m_1}\varepsilon_0 \right) - (d + d_1) \right] T \right).
\end{aligned} \tag{4.13}$$

Obviously, choose constant

$$M_1 = \exp\left(\left[U\left(M_0 + 1 - \frac{m}{m_1}\varepsilon_0\right) - (d + d_1)\right]T\right); \tag{4.14}$$

then from (4.13) we obtain a contradiction. Hence, we have $y(t) \leq \varepsilon_0 M_1$ for all $t \geq t^*$. Since ε_0 is arbitrary, we finally have $\lim_{t \to \infty} y(t) = 0$. Obviously, we can obtain from the third equation of system (2.3) $\lim_{t \to \infty} z(t) = 0$ and further from the first equation of system (2.3) we can obtain easily that $\lim_{t \to \infty}(x(t) - x^*(t)) = 0$. This completes the proof of Theorem 4.1.   □

On the permanence of the nutrient and the microorganism for system (2.3), we have the following result.

Theorem 4.2. *Suppose that inequality* (3.1) *holds and*

$$\int_0^T [-(d + d_1) + m_1 U(x^*(t))]dt > 0, \tag{4.15}$$

where $x^*(t)$ *is the unique positive T-periodic solution of system* (2.12). *Then system* (2.3) *is permanent.*

Proof. Let $(x(t), y(t), z(t))$ be any solution of system (2.3) with initial value (2.4). Since inequality (3.1) holds, from Theorem 3.1, for any $\varepsilon > 0$ there is a $T_1 > 0$ such that

$$x(t) \leq M_0 + \varepsilon, \quad y(t) \leq \frac{m_1}{m}(M_0 + \varepsilon), \quad \forall t \geq T_1. \tag{4.16}$$

From assumption (**H**) and the theorem of mean value, for all $t \geq 0$, there exists a $\xi(t) \in (0, x(t))$ such that $U(x(t)) = (dU(\xi(t))/ds)x(t)$. Since $dU(s)/ds$ is continuous for $s \geq 0$, there exists a constant $M_1 > 0$ such that

$$\frac{U(x(t))}{x(t)} = \frac{dU(\xi(t))}{ds} \leq M_1, \quad \forall t \geq 0. \tag{4.17}$$

From the first equation of system (2.3), we obtain

$$\frac{dx(t)}{dt} \geq -dx(t) - mU(x(t))y(t)$$

$$\geq -dx(t) - m_1(M_0 + \varepsilon)\frac{dU(\xi(t))}{ds}x(t) \tag{4.18}$$

$$\geq -(d + m_1(M_0 + \varepsilon)M_1)x(t)$$

for all $t \geq T_1$ and $t \neq nT$, and $x(t^+) = x(t) + p$, $t = nT$, $n \in N$. Using the comparison theorem of impulsive differential equation, we obtain $x(t) \geq v(t)$ for all $t \geq T_1$, where $v(t)$ is the solution of the following impulsive equation:

$$\frac{dv(t)}{dt} = -(d + m_1(M_0 + \varepsilon)M_1)v(t), \quad t \neq nT,$$

$$v(t^+) = v(t) + p, \quad t = nT, \ n \in N,$$

$$(4.19)$$

with initial condition $v(T_1^+) = x(T_1^+)$. Further from Lemma 2.2, we have

$$\lim_{t \to \infty} (v(t) - v^*(t)) = 0, \tag{4.20}$$

where $v^*(t)$ is the unique T-periodic solution of (4.19) and

$$v^*(t) = \frac{p \exp\{-(d + m_1(M_0 + \varepsilon)M_1)(t - nT)\}}{1 - \exp\{-(d + m_1(M_0 + \varepsilon)M_1)T\}} \tag{4.21}$$

for all $t \in (nT, (n+1)T]$ and $n \in N$. Therefore, we further obtain

$$\liminf_{t \to \infty} x(t) \geq \liminf_{t \to \infty} v(t) \geq \frac{p \exp(-(d + m_1(M_0 + \varepsilon)M_1)T)}{1 - \exp(-(d + m_1(M_0 + \varepsilon)M_1)T)}. \tag{4.22}$$

This shows that $x(t)$ in system (2.3) is permanent.

Next, we prove that there exists a constant $m_2 > 0$ such that $\liminf_{t \to \infty} y(t) \geq m_2$ for any solution $(x(t), y(t), z(t))$ of system (2.3) with initial value (2.4). From assumption (**H**), we can choose a constant ε_0 such that $dU(s)/ds > 0$ for all $s \in [0, M_0 + \varepsilon_0]$. According to (4.6), we can choose positive constants ε, ε_1 and ε_2, and $\varepsilon \leq \varepsilon_0$, such that

$$\int_0^T \left[-(d + d_1) + m_1 U\left(x^*(t) - \frac{mU(M_0 + \varepsilon)}{d}\varepsilon_1 - \varepsilon_2 \right) \right] dt > \varepsilon_2. \tag{4.23}$$

We first prove that there is a constant $\eta > 0$ and $\eta < \varepsilon_1$ such that

$$\limsup_{t \to \infty} y(t) \geq \eta \tag{4.24}$$

for any solution $(x(t), y(t), z(t))$ of system (2.3) with initial value (2.4).

In fact, if (4.24) is not true, then there is a solution $(x(t), y(t), z(t))$ of system (2.3) such that $\limsup_{t \to \infty} y(t) \leq \varepsilon_1$. Hence, there is a $T_0 > 0$ such that $y(t) < \varepsilon_1$, for all $t \geq T_0$. Further, from Theorem 3.1, there exists a constant $T_1 \geq T_0$ such that for all $t \geq T_1$

$$x(t) < M_0 + \varepsilon, \quad y(t) < \frac{m_1}{m}(M_0 + \varepsilon). \tag{4.25}$$

From the first equation of system (2.3), we have

$$
\begin{aligned}
\frac{dx(t)}{dt} &\geq -dx(t) - mU(x(t))y(t) \\
&\geq -dx(t) - \varepsilon_1 mU(M_0 + \varepsilon), \quad t \geq T_1, \ t \neq nT, \\
x(t^+) &= x(t) + p, \quad t = nT, \ n \in N.
\end{aligned}
\tag{4.26}
$$

From the comparison theorem of impulse differential equations and Lemma 2.2, for above $\varepsilon_2 > 0$, there is a $n_1 \in N$ and $n_1 T > T_1$ such that

$$
x(t) \geq -\frac{mU(M_0 + \varepsilon)}{d}\varepsilon_1 + x^*(t) - \varepsilon_2
\tag{4.27}
$$

for all $t \geq n_1 T$. Then from the second equation of system (2.3), we have that

$$
\frac{dy(t)}{dt} \geq y(t)\left[-(d + d_1) + m_1 U\left(x^*(t) - \frac{mU(M_0 + \varepsilon)}{d}\varepsilon_1 - \varepsilon_2\right)\right]
\tag{4.28}
$$

for all $t > n_1 T$. Integrating (4.29) from $n_1 T$ to $t \geq n_1 T$, we obtain

$$
\frac{dy(t)}{dt} \geq y(n_1 T) \exp \int_{n_1 T}^{t}\left[-(d + d_1) + m_1 U\left(x^*(t) - \frac{mU(M_0 + \varepsilon)}{d}\varepsilon_1 - \varepsilon_2\right)\right] dt.
\tag{4.29}
$$

Obviously, from (4.23) and (4.29), we obtain $y(t) \to \infty$ as $t \to \infty$, which leads to a contradiction. Therefore, (4.24) is true.

Now, we prove $y(t)$ in system (2.3) is permanent. Assume that it is not true, then there exists a sequence of initial values $\{\varphi_k\} = \{(x_0^{(k)}, y_0^{(k)}, z_0^{(k)})\}$ which satisfies initial condition (2.4) such that for solution $(x(t, \varphi_k), y(t, \varphi_k), z(t, \varphi_k))$, of system (2.3),

$$
\liminf_{t \to \infty} y(t, \varphi_k) < \frac{\eta}{k^2}, \quad \forall k = 1, 2, \ldots.
\tag{4.30}
$$

From (4.24) and (4.30) we obtain that there exist two time sequences $\{s_q^{(k)}\}$ and $\{t_q^{(k)}\}$ such that for each $k = 1, 2, \ldots,$

$$
0 < s_1^{(k)} < t_1^{(k)} < s_2^{(k)} < t_2^{(k)} < \cdots < s_q^{(k)} < t_q^{(k)} < \cdots,
$$
$$
s_q^{(k)} \longrightarrow \infty, \quad t_q^{(k)} \longrightarrow \infty \quad \text{as } q \longrightarrow \infty,
\tag{4.31}
$$

$$
y\left(t_q^{(k)}, \varphi_k\right) = \frac{\eta}{k^2}, \quad y\left(s_q^{(k)}, \varphi_k\right) = \frac{\eta}{k},
\tag{4.32}
$$

$$
\frac{\eta}{k^2} < y(t, \varphi_k) < \frac{\eta}{k} \quad \forall t \in \left(s_q^{(k)}, t_q^{(k)}\right).
\tag{4.33}
$$

From Theorem 3.1, there is a $T^{(k)} \geq T_0$ such that

$$y(t, \varphi_k) \leq \frac{m_1}{m}(M_0 + \varepsilon), \quad x(t, \varphi_k) \leq M_0 + \varepsilon \quad \forall t \geq T^{(k)}. \tag{4.34}$$

Further, from (4.31) for every k there is an integer $K^{(k)} > 0$ such that $s_q^{(k)} > T^{(k)}$ for all $q \geq K^{(k)}$. Hence, for any $t \in [s_q^{(k)}, t_q^{(k)}]$ and $q \geq K^{(k)}$, we have

$$\frac{dy(t, \varphi_k)}{dt} = y(t, \varphi_k)\left[-(d + d_1) + m_1 U(x(t, \varphi_k))\right]$$
$$\geq -\gamma y(t, \varphi_k), \tag{4.35}$$

where $\gamma = d + d_1 + m_1 U(M_0 + \varepsilon)$. Therefore for any $q \geq K^{(k)}$ and $k = 1, 2, \ldots$, integrating (4.35) on $[s_q^{(k)}, t_q^{(k)}]$, we obtain from (4.32)

$$\frac{\eta}{k^2} = y\left(t_q^{(k)}, \varphi_k\right)$$
$$\geq y\left(s_q^{(k)}, \varphi_k\right) \exp\left[-\gamma\left(t_q^{(k)} - s_q^{(k)}\right)\right] \tag{4.36}$$
$$= \frac{\eta}{k} \exp\left[-\gamma\left(t_q^{(k)} - s_q^{(k)}\right)\right].$$

Consequently,

$$t_q^{(k)} - s_q^{(k)} \geq \frac{\ln k}{\gamma} \quad \forall q \geq K^{(k)}, \ k = 1, 2, \ldots. \tag{4.37}$$

Since for any integer $k, q \geq K^{(k)}, t \in [s_q^{(k)}, t_q^{(k)}]$ and $q \geq K^{(k)}$ we have

$$\frac{dx(t, \varphi_k)}{dt} = -dx(t, \varphi_k) - my(t, \varphi_k)U(s(t, \varphi_k)) + bd_1 z(t, \varphi_k)$$
$$\geq -dx(t, \varphi_k) - my(t, \varphi_k)U(x(t, \varphi_k)) \tag{4.38}$$
$$\geq -dx(t, \varphi_k) - m\varepsilon_1 U(M_0 + \varepsilon).$$

By the comparison theorem it follows that $x(t, \varphi_k) \geq \overline{x}(t)$, forall $t \in [s_q^{(k)}, t_q^{(k)}]$, where $\overline{x}(t)$ is the solution of the following system with initial condition $\overline{x}(s_q^{(k)}) = x(s_q^{(k)}, \varphi_k)$:

$$\frac{d\omega(t)}{dt} = -d\omega(t) - m\varepsilon_1 U(M_0 + \varepsilon), \quad t \neq nT,$$
$$\omega(t^+) = \omega(t) + p, \quad t = nT, \ n \in N. \tag{4.39}$$

By Lemma 2.2, system (4.39) has a unique positive T-periodic solution

$$\omega^*(t) = x^*(t) - \frac{mU(M_0 + \varepsilon)}{d}\varepsilon_1, \tag{4.40}$$

and $\omega^*(t)$ is globally uniformly attractive. Hence, for above ε_2, there is a constant $T^* > 0$, and T^* is independent of any n and $q \geq K^{(k)}$, such that

$$\overline{x}(t) \geq x^*(t) - \frac{mU(M_0 + \varepsilon)}{d}\varepsilon_1 - \varepsilon_2 \quad \forall t \geq s_q^{(k)} + T^*. \tag{4.41}$$

Choose an integer $N_0 > 0$, such that when $k \geq N_0$ and $q \geq K^{(k)}$ we have $t_q^{(k)} - s_q^{(k)} \geq T^* + T$. Hence, for any $q \geq K^{(k)}$, $k \geq N_0$ and $t \in [t_q^{(k)} - T, t_q^{(k)}]$, we have

$$
\begin{aligned}
\frac{dy(t, \varphi_k)}{dt} &= y(t, \varphi_k)\left[-(d + d_1) + m_1 U(x(t, \varphi_k))\right] \\
&\geq y(t, \varphi_k)\left[-(d + d_1) + m_1 U\left(x^*(t) - \frac{mU(M_0 + \varepsilon)}{d}\varepsilon_1 - \varepsilon_2\right)\right].
\end{aligned}
\tag{4.42}
$$

Integrating (4.42) from $t_q^{(k)} - T$ to $t_q^{(k)}$, by (4.33) we obtain

$$
\begin{aligned}
\frac{\eta}{k^2} &= y\left(t_q^{(k)}, \varphi_k\right) \\
&\geq y\left(t_q^{(k)} - T, \varphi_k\right) \exp \int_{t_q^{(k)} - T}^{t_q^{(k)}} \left[-(d + d_1) + mp\left(x^*(t) - \frac{p(M_0 + \varepsilon)}{d}\varepsilon_1 - \varepsilon_2\right)\right] dt \\
&\geq \frac{\eta}{k^2} \exp(\varepsilon_2) \\
&> \frac{\eta}{k^2},
\end{aligned}
\tag{4.43}
$$

which is contradictory. This contradiction shows that there exists a constant $m_2 > 0$ such that

$$\liminf_{t \to \infty} y(t) \geq m_2 \tag{4.44}$$

for any positive solution $(x(t), y(t), z(t))$ of system (2.3). Choose $\varepsilon_3 > 0$ such that $m_2 - \varepsilon_3 > 0$, then there exists a $T_3 > 0$ such that $y(t) \geq m_2 - \varepsilon_3$ for all $t > T_3$. From the third equation of system (2.3), we have

$$\dot{z}(t) \geq -az(t) + (m_2 - \varepsilon_3), \quad \forall t > T_3. \tag{4.45}$$

By the comparison theorem it follows that $z(t) \geq \overline{z}(t)$, for all $t > T_3$, where $\overline{z}(t)$ is the solution of the following system with initial condition $\overline{z}(T_3) = z(T_3) > 0$:

$$\dot{u}(t) = -au(t) + (m_2 - \varepsilon_3). \tag{4.46}$$

By simple calculation we have

$$\overline{z}(t) \geq \frac{m_2 - \varepsilon_3}{a}, \quad \forall t > T_3. \tag{4.47}$$

Hence, we further obtain

$$\liminf_{t \to \infty} z(t) \geq \frac{m_2 - \varepsilon_3}{a}. \tag{4.48}$$

Let

$$\underline{m} = \min \left\{ \frac{p \exp(-(d + m_1(M_0 + \varepsilon)M_1)T)}{1 - \exp(-(d + m_1(M_0 + \varepsilon)M_1)T)}, \ m_2, \ \frac{m_2 - \varepsilon_3}{a} \right\}; \tag{4.49}$$

then we have

$$\liminf_{t \to \infty} x(t) \geq \underline{m}, \quad \liminf_{t \to \infty} y(t) \geq \underline{m}, \quad \liminf_{t \to \infty} z(t) \geq \underline{m}. \tag{4.50}$$

This completes the proof of Theorem 4.2. □

Returning to the original system (1.4), as a consequence of Theorems 4.1 and 4.2 we have the following corollary.

Corollary 4.3. *Suppose that inequality* (3.1) *holds. Then,*

(a) *solution* $(x^*(t), 0)$ *of system* (1.4) *is globally attractive if and only if*

$$\int_0^T [-(d + d_1) + m_1 U(x^*(t))] dt \leq 0; \tag{4.51}$$

(b) *system* (1.4) *is permanent if and only if*

$$\int_0^T [-(d + d_1) + m_1 U(s^*(t))] dt > 0. \tag{4.52}$$

Remark 4.4. From [18, Theorem 3.2], we find that the results on permanence is similar between model with delayed nutrient recycling and the model with instantaneous nutrient recycling.

Remark 4.5. According to Corollary 4.3, the sufficient and necessary conditions on the permanence and extinction of the microorganisms are obtained for a chemostat models with delayed nutrient recycling and periodically pulsed input. Obviously, it is an very good extension of the corresponding results given in [18].

5. Global Attractivity

Now, we discuss the global attractivity of all positive solutions of system (2.3), we have the following result.

Theorem 5.1. *Suppose that inequalities* (3.1) *and* (4.15) *hold. If*

$$\alpha d_1 - \beta d < 0, \tag{5.1}$$

where $\alpha = \sup_{\theta \in [0, M_0]} U'(\theta)$ *and* $\beta = \inf_{\theta \in [0, M_0]} U'(\theta)$, *then for any two positive solutions* $(x_1(t), y_1(t), z_1(t))$ *and* $(x_2(t), y_2(t), z_2(t))$ *of system* (2.3),

$$\lim_{t \to \infty} (x_1(t) - x_2(t)) = 0, \qquad \lim_{t \to \infty} (y_1(t) - y_2(t)) = 0,$$

$$\lim_{t \to \infty} (z_1(t) - z_2(t)) = 0. \tag{5.2}$$

Proof. From inequality (3.1) and (5.1), there exist two constants $c > 0$ and $c_1 > 0$ such that

$$bd_1 + cd - ac < 0, \quad ac - \frac{md_1}{m_1} < 0,$$

$$d - c_1 m_1 \alpha > 0, \quad c_1 m \beta - \frac{md_1}{m_1} > 0. \tag{5.3}$$

Further, we can choose a constant $\varepsilon_0 > 0$ such that

$$d - c_1 m_1 \alpha_{\varepsilon_0} > 0, \quad \frac{m - \varepsilon_0}{c_1} \left(c_1 m \beta_{\varepsilon_0} - \frac{md_1}{m_1} \right) > 0, \tag{5.4}$$

where $\alpha_{\varepsilon_0} = \sup_{\theta \in [0, M_0 + \varepsilon_0]} U'(\theta)$ and $\beta_{\varepsilon_0} = \inf_{\theta \in [0, M_0 + \varepsilon_0]} U'(\theta)$.
    Let

$$V(t) = x(t) + \frac{m}{m_1} y(t) + cz(t), \tag{5.5}$$

then system (2.3) is equivalent to the following system:

$$\frac{dV(t)}{dt} = -dV(t) + \left(ac - \frac{md_1}{m_1}\right)y(t) + (bd_1 + dc - ac)z(t),$$

$$\frac{dy(t)}{dt} = y(t)\left[-(d + d_1) + m_1U\left(V(t) - \frac{m}{m_1}y(t) - cz(t)\right)\right], \qquad t \neq nT,$$

$$\frac{dz(t)}{dt} = ay(t) - az(t), \tag{5.6}$$

$$V(t^+) = V(t) + p,$$

$$y(t^+) = y(t), \qquad\qquad t = nT, \ n \in N.$$

$$z(t^+) = z(t),$$

Let $(x_1(t), y_1(t), z_1(t))$ and $(x_2(t), y_2(t), z_2(t))$ be any two positive solutions of system (2.3); from Theorems 3.1 and 4.1 we have

$$\underline{m} \leq \lim_{t \to \infty} \inf x_i(t) \leq \lim_{t \to \infty} \sup x_i(t) \leq M_0,$$

$$\underline{m} \leq \lim_{t \to \infty} \inf y_i(t) \leq \lim_{t \to \infty} \sup y_i(t) \leq \frac{m_1}{m}M_0, \tag{5.7}$$

$$\underline{m} \leq \lim_{t \to \infty} \inf z_i(t) \leq \lim_{t \to \infty} \sup z_i(t) \leq \frac{(a - d)M_0}{bd_1}$$

for $i = 1, 2$. Hence, there exists a $T > 0$ such that

$$\underline{m} - \varepsilon_0 \leq x_i(t) \leq M_0 + \varepsilon_0,$$

$$\underline{m} - \varepsilon_0 \leq y_i(t) \leq \frac{m_1}{m}M_0 + \varepsilon_0, \tag{5.8}$$

$$\underline{m} - \varepsilon_0 \leq z_i(t) \leq \frac{(a - d)M_0}{bd_1} + \varepsilon_0$$

for all $t \geq T$. From above inequalities and the theorem of mean value, we can obtain

$$|y_1(t) - y_2(t)| \geq (\underline{m} - \varepsilon_0)|\ln y_1(t) - \ln y_2(t)|,$$

$$U(x_1(t)) - U(x_2(t)) = \frac{dU(\xi(t))}{ds}(x_1(t) - x_2(t)), \tag{5.9}$$

for all $t \geq T$, where $\xi(t)$ is situated between $x_1(t)$ and $x_2(t)$.

Let

$$V_i(t) = x_i(t) + \frac{m}{m_1}y_i(t) + cz_i(t), \quad i = 1, 2, \tag{5.10}$$

then $(V_i(t), y_i(t), z_i(t))$ is the solution of system (5.6). Define the Liapunov function as follows:

$$W(t) = |V_1(t) - V_2(t)| + c_1 |\ln y_1(t) - \ln y_2(t)| + c|z_1(t) - z_2(t)|. \tag{5.11}$$

Calculating the derivative of $W(t)$, from (5.9) we have

$$
\begin{aligned}
\frac{dW(t)}{dt} &= \text{sign}(V_1(t) - V_2(t)) \\
&\quad \times \left[-d(V_1(t) - V_2(t)) + \left(ac - \frac{md_1}{m_1} \right)(y_1(t) - y_2(t)) + (bd_1 + dc - ac)(z_1(t) - z_2(t)) \right] \\
&\quad + c_1 m_1 \, \text{sign}(y_1(t) - y_2(t)) \\
&\quad \times \left[U\left(V_1(t) - \frac{m}{m_1} y_1(t) - cz_1(t) \right) - U\left(V_2(t) - \frac{m}{m_1} y_2(t) - cz_2(t) \right) \right] \\
&\quad + \text{sign}(z_1(t) - z_2(t))[ac(y_1(t) - y_2(t)) - ac(z_1(t) - z_2(t))] \\
&\leq -d|V_1(t) - V_2(t)| + \left(\frac{md_1}{m_1} - ac \right)|y_1(t) - y_2(t)| \\
&\quad + (ac - bd_1 - dc)|z_1(t) - z_2(t)| \\
&\quad + c_1 m_1 \, \text{sign}(y_1(t) - y_2(t)) \frac{dU(\xi(t))}{ds} \\
&\quad \times \left[(V_1(t) - V_2(t)) - \frac{m}{m_1}(y_1(t) - y_2(t)) - c(z_1(t) - z_2(t)) \right] \\
&\quad + ac|y_1(t) - y_2(t)| - ac|z_1(t) - z_2(t)| \\
&\leq -(d - c_1 m_1 \alpha_{\varepsilon_0})|V_1(t) - V_2(t)| \\
&\quad - \frac{m - \varepsilon_0}{c_1}\left(c_1 m \beta_{\varepsilon_0} - \frac{md_1}{m_1} \right)c_1 |\ln y_1(t) - \ln y_2(t)| \\
&\quad - \left(d - c_1 m_1 \alpha_{\varepsilon_0} + \frac{bd_1}{c} \right)c|z_1(t) - z_2(t)| \\
&\leq -\rho W(t),
\end{aligned}
\tag{5.12}
$$

where

$$\rho = \min\left\{ d - c_1 m_1 \alpha_{\varepsilon_0}, \ \frac{m - \varepsilon_0}{c_1}\left(c_1 m \beta_{\varepsilon_0} - \frac{md_1}{m_1} \right) \right\}. \tag{5.13}$$

From (5.4), we obtain $\rho > 0$. On the other hand, we directly obtain

$$W(t^+) = W(t), \quad \forall t = nT, \ n \in N. \tag{5.14}$$

Hence, for any $t > T$ we have

$$W(t) \leq W(T) \exp(-\rho t). \tag{5.15}$$

Consequently, $\lim_{t \to \infty} W(t) = 0$. From this, we finally obtain

$$\lim_{t \to \infty} (x_1(t) - x_2(t)) = 0, \qquad \lim_{t \to \infty} (y_1(t) - y_2(t)) = 0,$$
$$\lim_{t \to \infty} (z_1(t) - z_2(t)) = 0. \tag{5.16}$$

This completes the proof of Theorem 5.1.   □

As a direct consequence of Theorem 5.1, we have the following corollary on global attractivity of all positive solutions for the original system (1.4).

Corollary 5.2. *Suppose that all the conditions of Theorem 5.1 hold. Then for any two positive solutions $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ of system* (1.4),

$$\lim_{t \to \infty} (x_1(t) - x_2(t)) = 0, \qquad \lim_{t \to \infty} (y_1(t) - y_2(t)) = 0. \tag{5.17}$$

6. Numerical Examples

In this section, we will give an example to demonstrate the effectiveness of our main results.
Consider system (1.4) with the following parameters:

$$U(x) = 12 \frac{x \exp(-0.2x)}{0.5 + x}, \qquad F(t) = 2e^{-2t}, \qquad d = 0.5, \qquad d_1 = 0.2,$$
$$m = 0.9, \qquad m_1 = 0.8, \qquad b = 0.4, \qquad p = 0.5, \qquad T = 1, \qquad a = 2. \tag{6.1}$$

By calculating, we obtain

$$\frac{b}{a - d} = \frac{4}{15}, \qquad \frac{m}{am_1} = \frac{9}{8},$$
$$\int_0^1 [-(d + d_1) + m_1 U(x^*(t))] dt = 0.1351 > 0,$$
$$x^*(t) = \frac{0.5e^{-0.5(t-n)}}{1 - e^{-0.5}}, \qquad t \in (n, n+1], \ n \in N,$$
$$\alpha d_1 - \beta d = 2.4947 > 0. \tag{6.2}$$

Therefore, inequalities (4.15) hold, but (5.1) does not hold. From Theorem 4.1, we obtain that system (1.4) is permanent. But, from the numerical simulation (see Figures 1 and 2) we see that there exists a unique positive T-periodic solution $(x^*(t), y^*(t))$ of system (1.4) such that any solution $(x(t), y(t))$ of system (1.4) with initial value $(x_0, \varphi(\theta))$ tends to $(x^*(t), y^*(t))$ as
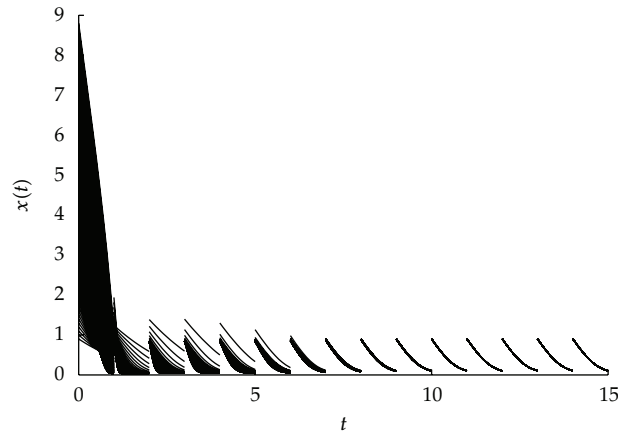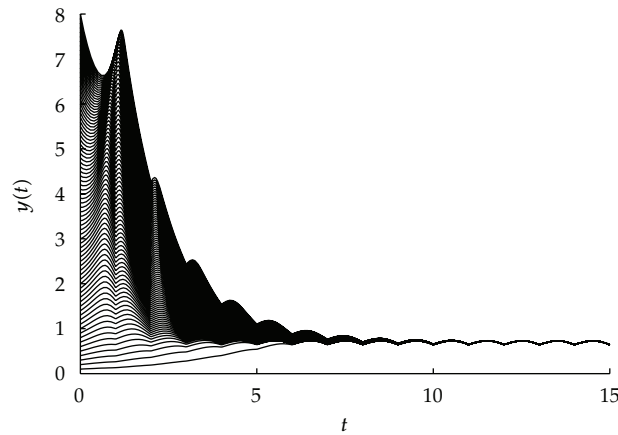
Figure 1: Time series of $x(t)$.

Figure 2: Time series of $y(t)$.

$t \to \infty$. Therefore, we can guess that if only inequality (4.15) holds then system (1.4) has a unique positive T-periodic solution which is globally attractive.

Open Problem. From the above numerical simulation, it prompts us to be able to put forward the following important and interesting open problem, that is, when inequality (3.1) holds then system (1.4) is also globally attractive as long as it is permanent.

Acknowledgment

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