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Research Article

Resonance and Nonresonance Periodic Value Problems of First-Order Differential Systems

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Resonance and nonresonance periodic value problems of first-order differential systems are studied. Several new existence and uniqueness of solutions for the above problems are obtained. To establish such results sufficient conditions of limit forms are given. A necessary and sufficient condition for existence of nontrivial solution is also proved.

1. Introduction

This paper is concerned with the existence and uniqueness of solutions of the nonresonance periodic boundary value problems (BVP) for the first-order differential system:

$$x'(t) + b(t)x(t) = F(t, x(t)), \quad t \in [0, 1], \tag{1.1}$$

$$x(0) = x(1), \tag{1.2}$$

where

$$F \in C([0,1] \times R^n, R^n), \quad b \in C([0,1], R) \quad \text{with} \int_0^1 b(\tau)\mathrm{d}\tau \neq 0. \tag{1.3}$$

The paper is also concerned with the existence and uniqueness of the resonance periodic boundary value problem of first-order:

$$\begin{gathered} x'(t) = f(t, x(t)), \quad t \in [0, 1], \\ x(0) = x(1), \end{gathered} \tag{1.1$_0$}$$

where

$$f \in C([0,1] \times R^n, R^n) \quad \text{with } f \neq 0 \quad \text{on } [0,1] \times R^n. \tag{1.4}$$

BVPs (1.1)-(1.2) and (1.1_0) particularly important because of numerous applications have been used in science and technology. By a solution of BVPs (1.1)-(1.2) we mean that a function $x \in C^1([0,1], R^n)$ satisfies (1.1)-(1.2). Throughout this paper if $x \in R^n$, then $\|x\|$ denotes the Euclidean norm $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$ of x on R^n. If $x, y \in R^n$, then $\langle x, y \rangle$ denotes the usual inner product.

Recently, by using fixed point methods and Leray-Schauder degree theory, for (1.1)-(1.2) the following results are obtained in [1].

Lemma 1.1 (see [1, Lemma 2.1]). *Let $F(t,x)$ and $b(t)$ be as in* (1.3) *with $b(t)$ having no zeros on* $[0,1]$. *Then BVP* (1.1)-(1.2) *is equivalent to the integral equation*

$$x(t) = \frac{1}{e^{\int_0^t b(\tau)\mathrm{d}\tau}} \left[\frac{e^{\int_0^1 b(\tau)\mathrm{d}\tau} \int_0^1 F(s,x(s)) e^{\int_0^s b(\tau)\mathrm{d}\tau} \mathrm{d}s}{e^{\int_0^1 b(\tau)\mathrm{d}\tau} - 1} + \int_0^t F(s,x(s)) e^{\int_0^s b(\tau)d\tau} \mathrm{d}s \right] \tag{1.5}$$

for $t \in [0,1]$.

Theorem 1.2 (see [1, Theorem 3.1]). *Let $F(t,x)$ and $b(t)$ be as in Lemma 1.1. If there exists a function $V \in C^1(R^n, [0,\infty))$ and nonnegative constants α and K such that for each $\lambda \in [0,1]$*

$$\lambda \|F(t,x)\| e^{\int_0^t b(\tau)\mathrm{d}\tau} \le \alpha \langle \nabla V(x), \lambda F(t,x) - b(t)x \rangle + K \tag{1.6}$$

for all $(t,x) \in [0,1] \times R^n$, *where*

$$\nabla V(x) := \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n} \right), \tag{1.7}$$

then BVP (1.1)-(1.2) *has at least one solution.*

However, we find that Lemma 1.1 is not correct since from (1.5),

$$\begin{aligned} x(0) &= \frac{e^{\int_0^1 b(\tau)\mathrm{d}\tau} \int_0^1 F(s,x(s)) e^{\int_0^s b(\tau)\mathrm{d}\tau} \mathrm{d}s}{e^{\int_0^1 b(\tau)\mathrm{d}\tau} - 1}, \\ x(1) &= \frac{1}{e^{\int_0^1 b(\tau)\mathrm{d}\tau}} \left[\frac{e^{\int_0^1 b(\tau)\mathrm{d}\tau} \int_0^1 F(s,x(s)) e^{\int_0^s b(\tau)\mathrm{d}\tau} \mathrm{d}s}{e^{\int_0^1 b(\tau)\mathrm{d}\tau} - 1} + \int_0^1 F(s,x(s)) e^{\int_0^s b(\tau)\mathrm{d}\tau} \mathrm{d}s \right]. \end{aligned} \tag{1.8}$$

Clearly $x(0) \neq x(1)$, that is, $x(t)$ does not satisfy (1.2). In fact, from fault of Lemma 1.1, the integral operator Tx on (1.1)-(1.2) is not appropriate in [1]. Hence we are not sure that the above Theorem 1.2 and the other results in [1] are correct.

The purpose of this paper is to establish several new existence and unique theorem for BVP (1.1)-(1.2). For the existence results, we give limit form conditions. The fixed point theorems are also applied. A new priori estimation on possible solutions of a family of BVP(1.1)-(1.2) is obtained and some ideas are from [1]. For recent development on BVP(1.1)-(1.2), except [1], we are referred to the papers [2–9] and references cited therein.

2. Existence

In the proof of the existence theorem below, we will use the following fixed point theorem.

Theorem 2.1 (Shaeffer's theorem [10, Theorem 4.4.12]). *Let X be a normed space and $T : X \to X$ be a completely continuous map. If the set*

$$S := \{x \in X : x = \lambda T x,\ \lambda \in [0,1]\} \tag{2.1}$$

is bounded, then T has at least one fixed point.

Let $X = C([0,1], R^n)$ denote the set of all continuous functions defined on $[0,1]$ and

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|_0^2\right)^{1/2}, \qquad |x_i|_0 = \max_{0\le t\le 1} |x_i(t)|, \quad i = 1, 2, \ldots, n. \tag{2.2}$$

Then X is a Banach space endowed with norm $\|\cdot\|$.

Lemma 2.2. *Assume that* (1.3) *holds.* $x \in C^1([0,1], R^n)$ *is a solution of BVP* (1.1)-(1.2) *if and only if x is a solution of the integral equation*

$$x(t) = \int_0^1 g(t,s) F(s, x(s))\mathrm{d}s, \quad t \in [0,1], \tag{2.3}$$

where Green function is

$$g(t,s) = \begin{cases} \dfrac{e^{-\int_s^t b(\tau)\mathrm{d}\tau}}{1 - e^{-\int_0^1 b(\tau)\mathrm{d}\tau}}, & 0 \le s \le t \le 1, \\[2ex] \dfrac{e^{-\int_0^1 b(\tau)\mathrm{d}\tau + \int_t^s b(\tau)\mathrm{d}\tau}}{1 - e^{-\int_0^1 b(\tau)\mathrm{d}\tau}}, & 0 \le t \le s \le 1, \end{cases} \tag{2.4}$$

Lemma 2.2 may be verified by direct computation. Also see [3, page 425] and [4, page 491].

Obviously, from (2.4) there is a constant G such that

$$\max_{0\le s,t\le 1} \left|g(t,s)\right| = G. \tag{2.5}$$

Theorem 2.3. *Assume that* (1.3) *and one of the following conditions hold.*

(i) $\|F(t,x)\|$ *is bounded on* $[0,1] \times R^n$.

(ii) *There exist function* $V \in C^1(R^n, [0,\infty))$ *and bounded function* $h(t,x) \in C([0,1]\times R^n, R^n)$ *such that for* $t \in [0,1]$ *and* $\lambda \in (0,1]$, *uniformly,*

$$\liminf_{\|x\|\to\infty} \frac{\langle \nabla V(x), \lambda F(t,x) - b(t)x\rangle + \|h(t,x)\|}{\lambda\|F(t,x)\|} > 0. \tag{2.6}$$

Then BVP(1.1)-(1.2) *has at least one solution.*

Proof. From Lemma 2.2 we see that BVP(1.1)-(1.2) is equivalent to the integral equation (2.3). Define the map $T : X \to X$ by

$$(Tx)(t) = \int_0^1 g(t,s)F(s,x(s))\mathrm{d}s, \quad t \in [0,1]. \tag{2.7}$$

By a standard argument, it is easy to prove that T is continuous and completely continuous.

Now we apply Shaeffer's theorem to prove that BVP (1.1)-(1.2) has at least one solution. Hence we need to prove that the set

$$S_\lambda = \{x \in X : x = \lambda Tx,\ \lambda \in [0,1]\} \tag{2.8}$$

is a bounded set with the bound being independent of $\lambda \in [0,1]$. Then we can conclude existence of at least one fixed point $x \in X$ of T. In consequence, from Lemma 2.2, BVP (1.1)-(1.2) has at least one solution $x \in X$.

From Lemma 2.2 and (2.7), it is easy to see that $x = \lambda Tx$ is equivalent to BVP:

$$\begin{gathered} x'(t) + b(t)x(t) = \lambda F(t,x(t)), \\ x(0) = x(1). \end{gathered} \tag{2.9}$$

In view of (2.5), (2.7), and (2.9), we have

$$\|x\| = \lambda\|Tx\| \le \lambda G\int_0^1 \|F(s,x(s))\|\mathrm{d}s, \quad \lambda \in [0,1]. \tag{2.10}$$

If (i) holds, then there exists a constant M such that $\|x(t)\| \le M$. This implies that S_λ is bounded and M is independent of $\lambda \in [0,1]$. By using Theorem 2.1, we know that BVP (1.1)-(1.2) has at least one solution. If (ii) holds, suppose that $\lambda = 0$, then $\|x\| = 0$; if $\lambda \in (0,1]$, assume, for the sake of contradiction, that S_λ is unbounded. Thus there exists $\{x_n\}_{n=1}^{\infty} \in S_\lambda$ such that $\lim_{n\to\infty}\|x_n\| = \infty$ and $\lim_{n\to\infty}\|F(t,x_n(t))\| = \infty$. Hence from (2.6) there exists constant $N > 0$ satisfying that for $n > N$ and $t \in [0,1], \lambda \in (0,1]$, uniformly,

$$\frac{\langle \nabla V(x_n), \lambda F(t,x_n(t)) - b(t)x_n(t)\rangle + \|h(t,x_n(t))\|}{\lambda\|F(t,x_n(t))\|} \ge \delta > 0, \tag{2.11}$$

where δ is independent of $\lambda \in (0,1]$. Taking into account (2.5), (2.9), and (2.11), we have that for $n \geq N, t \in [0,1]$ and $\lambda \in (0,1]$,

$$\begin{aligned}
\|x_n(t)\| &= \lambda\|(Tx_n)(t)\| \leq \lambda G\int_0^1 \|F(s,x_n(s))\|\mathrm{d}s \leq G\int_0^1 \|F(s,x_n(s))\|\mathrm{d}s \\
&\leq \frac{G}{\delta}\left[\int_0^1 \langle \nabla V(x_n(s)), \lambda F(s,x_n(s)) - b(s)x_n(s)\rangle \mathrm{d}s + \int_0^1 \|h(s,x_n(s))\|\mathrm{d}s\right] \\
&= \frac{G}{\delta}\left[\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}V(x_n(s))\mathrm{d}s + \int_0^1 \|h(s,x_n(s))\|\mathrm{d}s\right] \\
&= \frac{G}{\delta}[V(x_n(1)) - V(x_n(0))] + \frac{G}{\delta}\int_0^1 \|h(s,x_n(s))\|\mathrm{d}s \\
&= \frac{G}{\delta}\int_0^1 \|h(s,x_n(s))\|\mathrm{d}s,
\end{aligned} \tag{2.12}$$

which contradicts that $h(t,x)$ is bounded on $[0,1] \times R^n$. Therefore S_λ is bounded with the bound being independent of $\lambda \in [0,1]$. This proves that BVP (1.1)-(1.2) has at least one solution. The proof of Theorem 2.3 is complete. □

In particular, choose, respectively, $V(x) = \|x\|^\alpha, \alpha \geq 2$, and $V(x) = e^{\|x\|}$; from Theorem 2.3, the following results can be obtained.

Corollary 2.4. *In Theorem 2.3, replace* (2.6) *by the condition* (2.13) *or* (2.14) *below, then Theorem 2.3 is still valid, where*

$$\liminf_{\|x\|\to\infty} \frac{\alpha\|x\|^{\alpha-2}\langle x, \lambda F(t,x) - b(t)x\rangle + \|h(t,x)\|}{\lambda\|F(t,x)\|} > 0, \tag{2.13}$$

$$\liminf_{\|x\|\to\infty} \frac{\left(e^{\|x\|}/\|x\|\right)\langle x, \lambda F(t,x) - b(t)x\rangle + \|h(t,x)\|}{\lambda\|F(t,x)\|} > 0. \tag{2.14}$$

Proof. We only prove the corollary for (2.13). For (2.14) the proof is similar. We omit it.

Let $V(x) = \|x\|^\alpha$. Since

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^\alpha &= \alpha\|x(t)\|^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\| = \alpha\|x(t)\|^{\alpha-2}\left(\sum_{i=1}^n x_i(t)x_i'(t)\right) \\
&= \alpha\|x(t)\|^{\alpha-2}\langle x(t), x'(t)\rangle = \alpha\|x(t)\|^{\alpha-2}\langle x(t), \lambda F(t,x(t)) - b(t)x(t)\rangle,
\end{aligned} \tag{2.15}$$

it follows that $\langle \nabla V(x(t)), \lambda F(t,x(t)) - b(t)x(t)\rangle = \alpha\|x\|^{\alpha-2}\langle x(t), \lambda F(t,x(t)) - b(t)x(t)\rangle$. The proof is complete. □

Next, we give a necessary and sufficient condition for BVP (1.1)-(1.2) to have trivial solution.

Theorem 2.5. *Assume that* (1.3) *holds. A necessary and sufficient condition for BVP* (1.1)-(1.2) *to have trivial solution is*

$$F(t,0) \equiv 0, \quad \text{for } t \in [0,1]. \tag{2.16}$$

Proof.

Sufficiency 1. Suppose that (2.16) holds. From Lemma 2.2 we have

$$x(t) = \int_0^1 g(t,s)F(s,0)\mathrm{d}s \equiv 0, \quad \text{for } t \in [0,1], \tag{2.17}$$

which implies that $x(t) \equiv 0$ on $[0,1]$ is a solution of BVP (1.1)-(1.2).

Necessity 1. Suppose that BVP (1.1)-(1.2) has solution $x(t) \equiv 0, t \in [0,1]$. Noting (1.1)-(1.2), it is obvious that (2.16) holds. The proof is complete.

□

For the resonance BVP (1.1_0) where

$$f \in C([0,1] \times R^n, R^n) \quad \text{with } f \neq 0 \quad \text{on } [0,1] \times R^n, \tag{2.18}$$

we are unable to transform into BVP as an equivalent integral equation. However, consider the equivalent form of (1.1_0):

$$\begin{gathered} x'(t) + b(t)x(t) = f(t,x(t)) + b(t)x(t), \\ x(0) = x(1), \end{gathered} \tag{2.19}$$

where

$$b \in C([0,1],R) \quad \text{with } \int_0^1 b(\tau)\mathrm{d}\tau \neq 0. \tag{2.20}$$

Since Lemma 2.2 holds for BVP (2.19), we apply Lemma 2.2 and Theorems 2.3 and 2.5 to the following result.

Theorem 2.6. *Assume that* (2.18) *and* (2.20) *hold and one of the following conditions is satisfied.*

$(\mathrm{i})_0$ $\|f(t,x) + b(t)x\|$ *is bounded on* $[0,1] \times R^n$.

$(\mathrm{ii})_0$ *There exist functions* $V(x)$ *and* $h(t,x)$ *as in Theorem* 2.3 *such that for* $t \in [0,1]$ *and* $\lambda \in (0,1]$, *uniformly,*

$$\liminf_{\|x\| \to \infty} \frac{\langle \nabla V(x), \lambda f(t,x) - (1-\lambda)b(t)x \rangle + \|h(t,x)\|}{\lambda \| f(t,x) + b(t)x \|} > 0. \tag{2.21}$$

Then BVP(2.19)*-(2.15) has at least one solution.*

Proof. BVP (2.19) is in the form (1.1)-(1.2) with $F(t,x) = f(t,x)+b(t)x$. It is not difficult to see that (i) and (ii) reduce, respectively, to $(\text{i})_0$ and $(\text{ii})_0$ for these special cases. Hence the result follows from Theorem 2.3. The proof of Theorem 2.6 is complete. □

Theorem 2.7. *Assume that* (2.18) *and* (2.20) *hold. A necessary and sufficient condition for BVP* (2.19) *to have trivial solution is*

$$f(t,0) \equiv 0, \quad \text{for} \quad t \in [0,1]. \tag{2.22}$$

Proof. The result can be obtained from Lemma 2.2 in a similar way of the proof of Theorem 2.5. □

It is possible that in Theorems 2.3 and 2.6 the solutions of BVP (1.1)-(1.2) or BVP (2.19) include the trivial solution. Thus by Theorem 2.5 we have the following result.

Corollary 2.8. *Assume that all the conditions of Theorem 2.3 hold and*

$$F(t,0) \not\equiv 0, \quad t \in [0,1]. \tag{2.23}$$

Then BVP (1.1)-(1.2) *has at least one nontrivial solution.*

Proof. By Theorem 2.3, (1.1)-(1.2) has at least one solution. From (2.23) and Theorem 2.5, the solution is nontrivial. The proof is complete. □

Similarly, we have the following result.

Corollary 2.9. *Assume that all the conditions of Theorem 2.6 hold and*

$$f(t,0) \not\equiv 0, \quad t \in [0,1]. \tag{2.24}$$

Then BVP (2.19) *has at least one nontrivial solution.*

3. Uniqueness

In this section, we will establish uniqueness results of the solutions for BVP (1.1)-(1.2).

Consider the Banach space X defined in Section 2. Assume that $F(t,x)$ satisfies Lipschitz condition with respect to x; that is, there exists constant L such that

$$\|F(t,x) - F(t,y)\| \le L\|x-y\|, \quad t \in [0,1] \tag{3.1}$$

holds for any (t,x) and $(t,y) \in [0,1] \times R^n$.

Theorem 3.1. *Assume that* (1.3) *and* (3.1) *hold and* $GL < 1$ *where* G *is defined by* (2.5). *Then BVP* (1.1)-(1.2) *has exactly one solution. Moreover, if* (2.16) *holds, BVP* (1.1)-(1.2) *has only unique trivial solution; if* (2.23) *holds, BVP* (1.1)-(1.2) *has only unique nontrivial solution.*

Proof. First, to show that (1.1)-(1.2) has uniqueness of the solution, by Lemma 2.2 we need only to prove that (2.3) has exactly one solution.

Let x and $y \in X$. Consider the maps

$$(Tx)(t) = \int_0^1 g(t,s)F(s,x(s))\mathrm{d}s, \quad t \in [0,1],$$
$$(Ty)(t) = \int_0^1 g(t,s)F(s,y(s))\mathrm{d}s, \quad t \in [0,1]. \tag{3.2}$$

Thus

$$(Tx)(t) - (Ty)(t) = \int_0^1 g(t,s)\left[F(s,x(s)) - F(s,y(s))\right]\mathrm{d}s, \quad t \in [0,1]. \tag{3.3}$$

In view of (2.5) and (3.1), we have

$$\left\|(Tx)(t) - (Ty)(t)\right\| \le \int_0^1 \left|g(t,s)\right| \left\|F(s,x(s)) - F(s,y(s))\right\| \mathrm{d}s \le GL\|x-y\|, \tag{3.4}$$

which implies that T is contractive mapping. By the fixed point theorem of Banach, the map T has unique fixed point. In view of Lemma 2.2, BVP (1.1)-(1.2) has exactly one solution.

Next, if (2.16) holds, from Theorem 2.5, (1.1)-(1.2) has only unique trivial solution; if (2.23) holds, (1.1)-(1.2) has only unique nontrivial solution. The proof of Theorem 3.1 is complete. □

Now, consider BVP (1.1_0). Assume that $f(t,x)$ satisfies Lipschitz condition with respect to x; that is, there exists a constant l such that for any $x, y \in X$

$$\left\|f(t,x) - f(t,y)\right\| \le l\|x-y\|, \quad t \in [0,1]. \tag{3.5}$$

Similarly, the following result may be obtained.

Theorem 3.2. *Assume that* (3.5) *and* (2.18) *hold and there exists function $b(t)$ satisfying* (2.20) *such that $Gl < 1$. Then BVP* (2.19) *has exactly one trivial solution if* (2.22) *holds; BVP* (2.19) *has only unique nontrivial solution if* (2.24) *holds.*

To provide an exact estimations of $g(t,s)$ in (2.4), we have the estimations as follows:

$$\frac{e^{-\int_0^1 b_+(\tau)\mathrm{d}\tau}}{\left|1 - e^{\int_0^1 b(\tau)\mathrm{d}\tau}\right|} \le \left|g(t,s)\right| \le \frac{e^{-\int_0^1 b_-(\tau)\mathrm{d}\tau}}{\left|1 - e^{\int_0^1 b(\tau)\mathrm{d}\tau}\right|} := G_1, \quad 0 \le s \le t \le 1,$$
$$\frac{e^{-\int_0^1 b(\tau)\mathrm{d}\tau + \int_0^1 b_-(\tau)\mathrm{d}\tau}}{\left|1 - e^{\int_0^1 b(\tau)\mathrm{d}\tau}\right|} \le \left|g(t,s)\right| \le \frac{e^{-\int_0^1 b(\tau)\mathrm{d}\tau + \int_0^1 b_+(\tau)\mathrm{d}\tau}}{\left|1 - e^{\int_0^1 b(\tau)\mathrm{d}\tau}\right|} := G_2, \quad 0 \le s \le t \le 1, \tag{3.6}$$

where

$$b_+(t) = \max\{b(t), 0\}, \qquad b_-(t) = \max\{-b(t), 0\}. \tag{3.7}$$

In particular, for $b(t) \geq 0, t \in [0, 1]$,

$$\frac{e^{-\int_0^1 b(\tau)\mathrm{d}\tau}}{1 - e^{-\int_0^1 b(\tau)\mathrm{d}\tau}} \leq |g(t,s)| \leq \frac{1}{1 - e^{-\int_0^1 b(\tau)\mathrm{d}\tau}} := G_3, \quad 0 \leq s,\ t \leq 1. \tag{3.8}$$

and for $b(t) \leq 0, t \in [0, 1]$,

$$\frac{1}{e^{-\int_0^1 b(\tau)\mathrm{d}\tau} - 1} \leq |g(t,s)| \leq \frac{e^{-\int_0^1 b(\tau)\mathrm{d}\tau}}{e^{-\int_0^1 b(\tau)\mathrm{d}\tau} - 1} := G_4, \quad 0 \leq s,\ t \leq 1 \tag{3.9}$$

Applying Theorems 3.1 and 3.2, respectively, we obtain more exact results as follows.

Corollary 3.3. *Assume that* (1.3) *and* (3.1) *hold with* $\overline{G}L < 1$, *where* $\overline{G} = \max\{G_1, G_2\}$. *Then* (1.1)-(1.2) *has exactly one solution. In particular, if* $b(t) \geq 0$ *and* $G_3L < 1$, (1.1)-(1.2) *has exactly one solution; if* $b(t) \leq 0$ *and* $G_4L < 1$, (1.1)-(1.2) *has exactly one solution.*

Corollary 3.4. *Assume that* (2.18), (2.20), *and* (3.5) *hold with* $\overline{G}l < 1$. *Then* (2.19) *has exactly one solution. In particular, if* $b(t) \leq 0$ *and* $G_4 l < 1$, (2.19) *has exactly one solution.*

Remark 3.5. Since when there is at least a $(x, t) \in [0, 1] \times R^n$ such that $f \neq 0$ and (2.19) have the same solutions, it follows that Theorems 2.6, 2.7, and 3.2 and Corollaries 2.9 and 3.4 are also true for (1.1_0).

4. Examples and Remarks

In this section, we will present examples which highlight the theory of this paper. We also compare our results with known ones.

Example 4.1. Consider BVP (1.1)-(1.2) with $b(t) = 1$ and $F(t, x) = x^3 + K$, where $x = (x_1, x_2, \ldots, x_n)^T$ and $K = (k_1, k_2, \ldots, k_n)^T$. Let $V(x) = \|x\|^\alpha, \alpha \geq 2, h(t, x) \equiv 0$. We have

$$\begin{aligned}
\langle \nabla V(x), \lambda F(t,x) - b(t)x \rangle &= \alpha \|x\|^{\alpha-2} \langle x, \lambda\left(x^3 + K\right) - x \rangle \\
&= \alpha \|x\|^{\alpha-2} \sum_{i=1}^{n} \left[x_i\left(\lambda\left(x_i^3 + k_i\right) - x_i\right)\right] \\
&= \alpha \|x\|^{\alpha-2} \sum_{i=1}^{n} \left[\lambda\left(x_i^4 + k_i x_i\right) - x_i^2\right] \\
\|F(t,x)\| &= \left\|x^3 + K\right\| = \sum_{i=1}^{n} \left[\left(x_i^3 + k_i\right)^2\right]^{1/2}.
\end{aligned} \tag{4.1}$$

Note that $\|x\| \to \infty$ is equivalent to $|x|_0 := \max_{1\le i\le n}\{|x_i|\} \to \infty$. Therefore, in view of (2.6),

$$\liminf_{|x|_0\to\infty} \frac{\alpha\|x\|^{\alpha-2}\sum_{i=1}^{n}\left[\lambda\left(x_i^4+k_i x_i\right)-x_i^2\right]}{\lambda\sum_{i=1}^{n}\left[\left(x_i^3+k_i\right)^2\right]^{1/2}} = \infty. \tag{4.2}$$

By Theorems 2.3 and 2.5, (1.1)-(1.2) has at least one solution and all the solutions are nontrivial if $K \neq 0$; the solutions include nontrivial solution if $K = 0$. The results of [1–5] do not apply to the example.

Example 4.2. Consider BVP of scalar differential equation:

$$\begin{gathered} x'(t) + b(t)x(t) = k\cos^3 x(t), \\ x(0) = x(1), \end{gathered} \tag{4.3}$$

where $b(t)$ satisfies (1.3) and $0 < k < 1$ with $3kG < 1$, G defined in (2.5). It is easy to see that the map T corresponding to BVP (4.3) is

$$(Tx)(t) = \int_0^1 g(t,s)k\cos^3 x(s)\mathrm{d}s, \tag{4.4}$$

and it is contractive. By Theorem 3.1, (4.3) has only unique nontrivial solution. The known results in the literature are unable to obtain the conclusion.

Remark 4.3. The results obtained in the previous sections are new even for functions F and f in (1.1)-(1.2) and (1.1_0) to be scalar. The conditions given in Theorem 2.3 and Corollary 2.4 are limit forms. In general, it is easy to verify those than the inequality forms for higher-dimensional space particularly.

Remark 4.4. From the results of this paper, one easily sees that for both cases of nonresonance (1.1)-(1.2) and resonance (1.1_0) their results have not almost difference, that is, under the sense that for any results of (1.1)-(1.2) one can establish correspondingly those of (1.1_0).

Remark 4.5. The results of this paper can be generalized impulsive periodic boundary problems of first-order; see [2–5].

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