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Research Article

Oscillation Behavior of a Class of Second-Order Dynamic Equations with Damping on Time Scales

Weisong Chen,[1] Zhenlai Han,[1, 2] Shurong Sun,[1, 3] and Tongxing Li[1, 2]

[1] *School of Science, University of Jinan, Jinan, Shandong 250022, China*
[2] *School of Control Science and Engineering, Shandong University, Jinan, Shandong 250061, China*
[3] *Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO 65409-0020, USA*

Correspondence should be addressed to Zhenlai Han, hanzhenlai@163.com

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By using a Riccati transformation and inequality, we present some new oscillation theorems for the second-order nonlinear dynamic equation with damping on time scales. An example illustrating the importance of our results is also included.

1. Introduction

The theory of time scales, which has recently received a lot of attraction, was introduced by Hilger in his Ph.D. Thesis in 1990 [1] in order to unify continuous and discrete analysis. The books on the subjects of time scale, that is, measure chain, by Bohner and Peterson [2, 3] summarize and organize much of time scale calculus.

We are concerned with second-order nonlinear dynamic equations with damping

$$\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+p(t)\left(x^{\Delta}(t)\right)^{\gamma}+q(t)f(x^{\sigma}(t))=0 \tag{1.1}$$

on a time scale $\mathbb{T}$; here p and q are real-valued positive rd-continuous positive functions defined on $\mathbb{T}$, and γ is a quotient of odd positive integers. We assume that $f(x)/x^{\gamma}\geq L>0$, $x\neq 0$, $\sup\mathbb{T}=\infty$, and define $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various dynamic equations [4–13]. However, there are few papers dealing with the oscillation of dynamic equations with damping term [14–17].

Saker [18] presented several oscillation criteria for the nonlinear second-order dynamic equation

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta} + q(t)f(x(\sigma(t))) = 0, \quad t \in [a,b], \tag{1.2}$$

where $a, b \in \mathbb{T}$ and $a < b$.

Hassan [19] studied the oscillation behavior of the second-order half-linear dynamic equation

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + p(t)x^{\gamma}(t) = 0, \tag{1.3}$$

and obtained several new results.

Bohner et al. [20] established some oscillation criteria for the second-order nonlinear dynamic equation

$$x^{\Delta\Delta}(t) + q(t)x^{\Delta\sigma}(t) + p(t)(f \circ x^{\sigma}) = 0. \tag{1.4}$$

Erbe et al. [16] considered the second-order nonlinear dynamic equations with damping

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + p(t)\left(x^{\Delta\sigma}(t)\right)^{\gamma} + q(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T}, \tag{1.5}$$

and established some sufficient conditions for oscillation of (1.5).

Saker et al. [17] investigated the oscillation of second-order dynamic equations with damping term of the form

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p(t)x^{\Delta\sigma}(t) + q(t)f(x(\sigma(t))) = 0, \quad t \in \mathbb{T}, \tag{1.6}$$

and obtained some new oscillation criteria for (1.6).

Zafer [21] studied the second-order nonlinear dynamic equations on time scales

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y^{\sigma} = 0, \quad t \in \mathbb{T}, \tag{1.7}$$

and presented some oscillation and nonoscillation criteria. Obviously, (1.7) is the special situation of (1.1).

Note that in the special case when $\mathbb{T} = \mathbb{R}$, (1.1) becomes the second-order nonlinear damped differential equation

$$((x'(t))^{\gamma})' + p(t)(x'(t))^{\gamma} + q(t)f(x^{\sigma}(t)) = 0, \quad t \in \mathbb{R}, \tag{1.8}$$

and when $\mathbb{T} = \mathbb{Z}$, (1.1) becomes the second-order nonlinear damped difference equation

$$\Delta\left((\Delta x(t))^{\gamma}\right) + p(t)(\Delta x(t))^{\gamma} + q(t)f(x^{\sigma}(t)) = 0, \quad t \in \mathbb{Z}, \tag{1.9}$$

where $\Delta x(t) = x(t+1) - x(t)$.

This paper is organized as follows: in Section 2, we give some preliminaries and lemmas. In Section 3, we will establish some oscillation criteria for (1.1). In Section 4, we give an example to illustrate the main results.

2. Preliminaries

It will be convenient to make the following notations:

$$d_{+}(t) := \max\{0, d(t)\}, \qquad d_{-}(t) := \max\{0, -d(t)\}, \qquad \beta(t) := \begin{cases} \alpha(t), & 0 < \gamma \le 1, \\ \alpha^{\gamma}(t), & \gamma > 1, \end{cases}$$
$$\alpha(t) := \frac{t - t_*}{t - t_* + \mu(t)}, \qquad R(t) := e_{p/(1-p\mu)}(t, t_*). \tag{2.1}$$

Lemma 2.1. *Assume that x is Δ-differentiable. Then from Keller's chain rule [2, Theorem 1.90],*

$$\left((x(t))^{\gamma}\right)^{\Delta} = \gamma\int_{0}^{1}\left[hx^{\sigma}(t) + (1-h)x(t)\right]^{\gamma-1}x^{\Delta}(t)dh. \tag{2.2}$$

Lemma 2.2 (see [22]). *If $f(x) = -Ax^{(\gamma+1)/\gamma} + Bx$, $A > 0$, then $f(x)$ attains its maximum value at $x_0 = (\gamma B/(\gamma+1)A)^{\gamma}$, and $f(x_0) = (\gamma^{\gamma}/(\gamma+1)^{\gamma+1})(B^{\gamma+1}/A^{\gamma})$.*

Lemma 2.3. *Suppose that x is an eventually positive solution of equation* (1.1), $1 - p(t)\mu(t) > 0$, *and*

$$\int_{t_0}^{\infty}\frac{\Delta t}{R^{1/\gamma}(t)} = \infty. \tag{2.3}$$

Then there exists a $t_* > t_0$, such that for $t > t_*$,

$$\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0, \quad x^{\Delta}(t) > 0, \quad x^{\Delta\Delta}(t) < 0, \quad x(t) > (t - t_*)x^{\Delta}(t), \quad \frac{x(t)}{x^{\sigma}(t)} > \alpha(t). \tag{2.4}$$

Proof. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x^{\sigma}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. From (1.1), we have

$$\left[\left(x^{\Delta}(t)\right)^{\gamma}\right]^{\Delta} + p(t)\left(x^{\Delta}(t)\right)^{\gamma} < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.5}$$

So, we get

$$\frac{1}{1-\mu(t)p(t)}\left[\left(x^{\Delta}(t)\right)^{\gamma}\right]^{\Delta}+\frac{p(t)}{1-\mu(t)p(t)}\left(x^{\Delta}(t)\right)^{\gamma}<0, \quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{2.6}$$

Therefore,

$$\left[R(t)\left(x^{\Delta}(t)\right)^{\gamma}\right]^{\Delta}<0, \quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{2.7}$$

We claim that $x^{\Delta}(t)>0$. If not, there exist $t_1\geq t_0$ and a constant $C<0$ such that

$$R(t)\left(x^{\Delta}(t)\right)^{\gamma}\leq C<0, \tag{2.8}$$

hence

$$x^{\Delta}(t)\leq\left(\frac{C}{R(t)}\right)^{1/\gamma}. \tag{2.9}$$

Integrating the above inequality from t_1 to t, we obtain

$$x(t)\leq x(t_1)+C^{1/\gamma}\int_{t_1}^{t}\frac{1}{R^{1/\gamma}(s)}\Delta s\longrightarrow-\infty, \quad t\longrightarrow\infty, \tag{2.10}$$

which is a contradiction. Hence,

$$x^{\Delta}(t)>0. \tag{2.11}$$

Obviously, by (2.7) and (2.11), we can see that

$$\left[\left(x^{\Delta}(t)\right)^{\gamma}\right]^{\Delta}<0. \tag{2.12}$$

From (2.11) and (2.12), we have

$$x^{\Delta\Delta}(t)<0. \tag{2.13}$$

It follows from (2.13) that

$$x(t)>x(t)-x(t_*)=\int_{t_*}^{t}x^{\Delta}(s)\Delta s\geq x^{\Delta}(t)(t-t_*). \tag{2.14}$$

In view of (2.14) and $x^\sigma(t) = x(t) + \mu(t)x^\Delta(t)$, it is easy to get that

$$\frac{x(t)}{x^\sigma(t)} > \alpha(t). \tag{2.15}$$

□

3. Main Results

In this section, we will give some new oscillation criteria for (1.1).

Theorem 3.1. *Assume that* (2.3) *holds. Further, suppose that* $1 - p(t)\mu(t) > 0$, *and there exists a positive* Δ*-differentiable function* δ, *such that for all sufficiently large* t_*,

$$\limsup_{t\to\infty} \int_{t_*}^{t} \left[Lq(s)\delta^\sigma(s) - \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}(s)}{A^\gamma(s)} \right] \Delta s = \infty, \tag{3.1}$$

where $A(t) = \gamma\delta^\sigma(t)\beta(t)/(\delta(t))^{(\gamma+1)/\gamma}$, $B(t) = (\delta^\Delta(t) - p(t)\delta^\sigma(t)\alpha^\gamma(t))/\delta(t)$. *Then every solution* x *of* (1.1) *oscillates on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume $x(t) > 0$, for $t \ge t_* \ge t_0$. Consider the generalized Riccati substitution

$$w(t) = \delta(t)\frac{(x^\Delta(t))^\gamma}{x^\gamma(t)}, \quad t \ge t_* \ge t_0, \tag{3.2}$$

then $w(t) > 0$, and by the product rule and then the quotient

$$\begin{aligned} w^\Delta(t) &= \delta^\Delta(t)\frac{(x^\Delta(t))^\gamma}{x^\gamma(t)} + \delta^\sigma(t)\left(\frac{(x^\Delta(t))^\gamma}{x^\gamma(t)}\right)^\Delta \\ &= \delta^\Delta(t)\frac{(x^\Delta(t))^\gamma}{x^\gamma(t)} + \delta^\sigma(t)\frac{\left((x^\Delta(t))^\gamma\right)^\Delta}{(x^\sigma(t))^\gamma} - \delta^\sigma(t)\frac{(x^\Delta(t))^\gamma((x(t))^\gamma)^\Delta}{(x(t))^\gamma(x^\sigma(t))^\gamma}. \end{aligned} \tag{3.3}$$

Using (1.1) and (3.2), we find

$$w^\Delta(t) \le w(t)\frac{\delta^\Delta(t)}{\delta(t)} - p(t)\delta^\sigma(t)\left(\frac{x^\Delta(t)}{x(t)}\right)^\gamma\left(\frac{x(t)}{x^\sigma(t)}\right)^\gamma - Lq(t)\delta^\sigma(t) - \delta^\sigma(t)\frac{(x^\Delta(t))^\gamma((x(t))^\gamma)^\Delta}{(x(t))^\gamma(x^\sigma(t))^\gamma}. \tag{3.4}$$

If $0 < \gamma \le 1$, from Lemma 2.1, we get

$$\left((x(t))^\gamma\right)^\Delta \ge \gamma(x^\sigma(t))^{\gamma-1}x^\Delta(t), \tag{3.5}$$

hence

$$w^{\Delta}(t) \le w(t)\frac{\delta^{\Delta}(t)}{\delta(t)} - p(t)\delta^{\sigma}(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}\left(\frac{x(t)}{x^{\sigma}(t)}\right)^{\gamma} - Lq(t)\delta^{\sigma}(t) - \gamma\delta^{\sigma}(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma+1}\frac{x(t)}{x^{\sigma}(t)}. \tag{3.6}$$

In view of Lemma 2.3 and (3.2), we obtain

$$w^{\Delta}(t) \le -Lq(t)\delta^{\sigma}(t) + \left(\frac{\delta^{\Delta}(t)}{\delta(t)} - p(t)\frac{\delta^{\sigma}(t)}{\delta(t)}\alpha^{\gamma}(t)\right)w(t) - \frac{\gamma\delta^{\sigma}(t)\alpha(t)}{(\delta(t))^{(\gamma+1)/\gamma}}(w(t))^{(\gamma+1)/\gamma}. \tag{3.7}$$

If $\gamma > 1$, from Lemma 2.1, we get

$$\left((x(t))^{\gamma}\right)^{\Delta} \ge \gamma(x(t))^{\gamma-1}x^{\Delta}(t), \tag{3.8}$$

hence

$$w^{\Delta}(t) \le w(t)\frac{\delta^{\Delta}(t)}{\delta(t)} - p(t)\delta^{\sigma}(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}\left(\frac{x(t)}{x^{\sigma}(t)}\right)^{\gamma} - Lq(t)\delta^{\sigma}(t) - \gamma\delta^{\sigma}(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma+1}\left(\frac{x(t)}{x^{\sigma}(t)}\right)^{\gamma}. \tag{3.9}$$

In view of Lemma 2.3, we have

$$w^{\Delta}(t) \le -Lq(t)\delta^{\sigma}(t) + \left(\frac{\delta^{\Delta}(t)}{\delta(t)} - p(t)\frac{\delta^{\sigma}(t)}{\delta(t)}\alpha^{\gamma}(t)\right)w(t) - \frac{\gamma\delta^{\sigma}(t)\alpha^{\gamma}(t)}{(\delta(t))^{(\gamma+1)/\gamma}}(w(t))^{(\gamma+1)/\gamma}. \tag{3.10}$$

Therefore,

$$w^{\Delta}(t) \le -Lq(t)\delta^{\sigma}(t) + \left(\frac{\delta^{\Delta}(t)}{\delta(t)} - p(t)\frac{\delta^{\sigma}(t)}{\delta(t)}\alpha^{\gamma}(t)\right)w(t) - \frac{\gamma\delta^{\sigma}(t)\beta(t)}{(\delta(t))^{(\gamma+1)/\gamma}}(w(t))^{(\gamma+1)/\gamma}. \tag{3.11}$$

From Lemma 2.3, we get

$$w^{\Delta}(t) \le -Lq(t)\delta^{\sigma}(t) + \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}(t)}{A^{\gamma}(t)}. \tag{3.12}$$

Integrating the above inequality from t_* to t, we have

$$\int_{t_*}^{t}\left[Lq(s)\delta^{\sigma}(s) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}(s)}{A^{\gamma}(s)}\right]\Delta s \le w(t_*) \tag{3.13}$$

which leads to a contradiction to (3.1). This completes the proof. □

Remark 3.2. From Theorem 3.1, we can obtain different conditions for oscillation of all solutions of (1.1) with different choice of δ.

Theorem 3.3. *Assume that* (2.3) *holds. Further, suppose that* $1-p(t)\mu(t) > 0$, *and there exist positive* Δ*-differentiable functions* δ *and* r, *such that for all sufficiently large* t_*,

$$\limsup_{t\to\infty}\int_{t_*}^{t}\left[Lq(s)\delta(s)r(s)-\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}(s)}{A^{\gamma}(s)}\right]\Delta s=\infty, \tag{3.14}$$

where $A(t) = \gamma r(t)\delta(t)/(\delta^{\sigma}(t))^{(\gamma+1)/\gamma}$, $B(t) = r(t)(\delta^{\Delta}(t) - p(t)\delta(t))/\delta^{\sigma}(t) + r^{\Delta}(t)$. *Then every solution* x *of* (1.1) *oscillates on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we assume $x(t) > 0$, for $t \ge t_* \ge t_0$. Consider the generalized Riccati substitution as in (3.2). Then $w(t) > 0$, and by the product rule and then the quotient

$$\begin{aligned} w^{\Delta}(t)&=\delta^{\Delta}(t)\left(\frac{(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right)^{\sigma}+\delta(t)\left(\frac{(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right)^{\Delta}\\ &=\delta^{\Delta}(t)\left(\frac{(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right)^{\sigma}+\delta(t)\frac{\left((x^{\Delta}(t))^{\gamma}\right)^{\Delta}}{(x^{\sigma}(t))^{\gamma}}-\delta(t)\frac{(x^{\Delta}(t))^{\gamma}((x(t))^{\gamma})^{\Delta}}{(x(t))^{\gamma}(x^{\sigma}(t))^{\gamma}}, \end{aligned} \tag{3.15}$$

it follows from (1.1) and (3.2) that

$$w^{\Delta}(t)\le w^{\sigma}(t)\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}-p(t)\delta(t)\left(\frac{x^{\Delta\sigma}(t)}{x^{\sigma}(t)}\right)^{\gamma}\left(\frac{x^{\Delta}(t)}{x^{\Delta\sigma}(t)}\right)^{\gamma}-Lq(t)\delta(t)-\delta(t)\frac{(x^{\Delta}(t))^{\gamma}((x(t))^{\gamma})^{\Delta}}{(x(t))^{\gamma}(x^{\sigma}(t))^{\gamma}}. \tag{3.16}$$

If $0 < \gamma \le 1$, from Lemma 2.1, we get

$$\left((x(t))^{\gamma}\right)^{\Delta}\ge\gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t), \tag{3.17}$$

hence

$$\begin{aligned} w^{\Delta}(t)\le{}& w^{\sigma}(t)\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}-p(t)\delta(t)\left(\frac{x^{\Delta\sigma}(t)}{x^{\sigma}(t)}\right)^{\gamma}\left(\frac{x^{\Delta}(t)}{x^{\Delta\sigma}(t)}\right)^{\gamma}-Lq(t)\delta(t)\\ &-\gamma\delta(t)\left(\frac{x^{\Delta\sigma}(t)}{x^{\sigma}(t)}\right)^{\gamma+1}\left(\frac{x^{\sigma}(t)}{x(t)}\right)^{\gamma}\left(\frac{x^{\Delta}(t)}{x^{\Delta\sigma}(t)}\right)^{\gamma+1}. \end{aligned} \tag{3.18}$$

In view of Lemma 2.3, we see that

$$w^{\Delta}(t)\le -Lq(t)\delta(t)+\left(\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}-p(t)\frac{\delta(t)}{\delta^{\sigma}(t)}\right)w^{\sigma}(t)-\frac{\gamma\delta(t)}{(\delta^{\sigma}(t))^{(\gamma+1)/\gamma}}(w^{\sigma}(t))^{(\gamma+1)/\gamma}. \tag{3.19}$$

If $\gamma > 1$, from Lemma 2.1, we get

$$\left((x(t))^{\gamma}\right)^{\Delta} \geq \gamma (x(t))^{\gamma-1} x^{\Delta}(t). \tag{3.20}$$

So,

$$\begin{aligned} w^{\Delta}(t) \leq{}& w^{\sigma}(t)\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} - p(t)\delta(t)\left(\frac{x^{\Delta\sigma}(t)}{x^{\sigma}(t)}\right)^{\gamma}\left(\frac{x^{\Delta}(t)}{x^{\Delta\sigma}(t)}\right)^{\gamma} - Lq(t)\delta(t) \\ &- \gamma\delta(t)\left(\frac{x^{\Delta\sigma}(t)}{x^{\sigma}(t)}\right)^{\gamma+1}\frac{x^{\sigma}(t)}{x(t)}\left(\frac{x^{\Delta}(t)}{x^{\Delta\sigma}(t)}\right)^{\gamma+1}. \end{aligned} \tag{3.21}$$

In view of Lemma 2.3, we find

$$w^{\Delta}(t) \leq -Lq(t)\delta(t) + \left(\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} - p(t)\frac{\delta(t)}{\delta^{\sigma}(t)}\right)w^{\sigma}(t) - \frac{\gamma\delta(t)}{(\delta^{\sigma}(t))^{(\gamma+1)/\gamma}}(w^{\sigma}(t))^{(\gamma+1)/\gamma}. \tag{3.22}$$

Therefore,

$$w^{\Delta}(t) \leq -Lq(t)\delta(t) + \left(\frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)} - p(t)\frac{\delta(t)}{\delta^{\sigma}(t)}\right)w^{\sigma}(t) - \frac{\gamma\delta(t)}{(\delta^{\sigma}(t))^{(\gamma+1)/\gamma}}(w^{\sigma}(t))^{(\gamma+1)/\gamma}. \tag{3.23}$$

From Lemma 2.2, we obtain

$$w^{\Delta}(t) \leq -Lq(t)\delta^{\sigma}(t) + \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}(t)}{A^{\gamma}(t)}. \tag{3.24}$$

Integrating the above inequality from t_* to t, we get

$$\int_{t_*}^{t}\left[Lq(s)\delta(s)r(s) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}(s)}{A^{\gamma}(s)}\right]\Delta s \leq r(t_*)w(t_*) \tag{3.25}$$

which leads to a contradiction to (3.14). This completes the proof. □

Remark 3.4. From Theorem 3.3, we can obtain different conditions for oscillation of all solutions of (1.1) with different choice of δ and r.

In the following, we will establish Kamenev-type oscillation criteria for (1.1).

Theorem 3.5. *Assume that* (2.3) *holds. Further, suppose that* $1 - p(t)\mu(t) > 0$, *and there exists a positive* Δ*-differentiable function* δ, *such that for* $m > 1$ *and all sufficiently large* t_*,

$$\limsup_{t\to\infty}\frac{1}{t^{m}}\int_{t_*}^{t}(t-s)^{m}\left[Lq(s)\delta^{\sigma}(s) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}(s)}{A^{\gamma}(s)}\right]\Delta s = \infty, \tag{3.26}$$

where $A(t) = \gamma\delta^{\sigma}(t)\beta(t)/(\delta(t))^{(\gamma+1)/\gamma}$, $B(t) = (\delta^{\Delta}(t) - p(t)\delta^{\sigma}(t)\alpha^{\gamma}(t))/\delta(t)$. *Then every solution* x *of* (1.1) *is oscillatory on* $[t_0, \infty)_T$.

Proof. We may assume that (1.1) has a nonoscillatory solution $x(t)$ such that $x(t) > 0$. Define w by (3.2) as before, then we get (3.24). From (3.24), we have

$$Lq(t)\delta^{\sigma}(t) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}(t)}{A^{\gamma}(t)} \leq -w^{\Delta}(t). \tag{3.27}$$

Thus

$$\int_{t_*}^{t}(t-s)^m\left[Lq(s)\delta^{\sigma}(s) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}(s)}{A^{\gamma}(s)}\right]\Delta s \leq -\int_{t_*}^{t}(t-s)^m w^{\Delta}(s)\Delta s. \tag{3.28}$$

Upon integration, we arrive at

$$-\int_{t_*}^{t}(t-s)^m w^{\Delta}(t)\Delta s = (t-s)^m\, w(t)|_{t_*}^{t} - \int_{t_*}^{t}\left((t-s)^m\right)^{\Delta_s} w(\sigma(t))\Delta s. \tag{3.29}$$

Note that $((t-s)^m)^{\Delta_s} \leq -m(t-\sigma(s))^{m-1} \leq 0$, $t \geq \sigma(s), m \geq 1$ (see Saker [11]); then using (3.28), we have

$$\int_{t_*}^{t}(t-s)^m\left[Lq(s)\delta^{\sigma}(s) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}(s)}{A^{\gamma}(s)}\right]\Delta s \leq (t-t_*)^m w(t_*). \tag{3.30}$$

Therefore,

$$\frac{1}{t^m}\int_{t_*}^{t}(t-s)^m\left[Lq(s)\delta^{\sigma}(s) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}(s)}{A^{\gamma}(s)}\right]\Delta s \leq \frac{(t-t_*)^m}{t^m}w(t_*). \tag{3.31}$$

Hence,

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_*}^{t}(t-s)^m\left[Lq(s)\delta^{\sigma}(s) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}(s)}{A^{\gamma}(s)}\right]\Delta s = \infty, \tag{3.32}$$

which contradicts (3.26). This completes the proof. □

Theorem 3.6. *Assume that* (2.3) *holds. Further, suppose that* $1 - p(t)\mu(t) > 0$, *and there exists a positive* Δ*-differentiable function* δ, *such that for* $m > 1$ *and all sufficiently large* t_*,

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_*}^{t}(t-s)^m\left[Lq(s)\delta(s) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}(s)}{A^{\gamma}(s)}\right]\Delta s = \infty, \tag{3.33}$$

where $A(t) = \gamma\delta(t)/(\delta^{\sigma}(t))^{(\gamma+1)/\gamma}$, $B(t) = (\delta^{\Delta}(t) - p(t)\delta(t))/\delta^{\sigma}(t)$. *Then every solution* x *of* (1.1) *oscillates on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. In view of Theorem 3.3, the proof is similar to that of [18, Theorem 3.2]. □

In the following, we will establish the Philos-type oscillation criteria for (1.1).

Theorem 3.7. *Assume that* (2.3) *holds. Further, suppose that* $1 - p(t)\mu(t) > 0$, *there exists a positive* Δ*-differentiable function* δ, *and* $H, h \in C_{\mathrm{rd}}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} = \{(t,s) : t \geq s \geq t_0\}$ *such that*

$$H(t,t) = 0, \quad t \geq t_0, \qquad H(t,s) > 0, \quad t > s \geq t_0. \tag{3.34}$$

H *has a continuous and nonpositive* Δ*-partial derivative* $H^{\Delta_s}(t,s)$ *with respect to the second variable and satisfies*

$$H^{\Delta_s}(\sigma(t),s) + H(\sigma(t),\sigma(s))\left(\frac{\delta^{\Delta}(s) - p(s)\delta^{\sigma}(s)\alpha^{\gamma}(s)}{\delta(s)}\right) = -\frac{h(t,s)}{\delta(s)}(H(\sigma(t),\sigma(s)))^{\gamma/(\gamma+1)}, \tag{3.35}$$

and for sufficiently large t_*,

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),t_*)}\int_{t_*}^{\sigma(t)} K(t,s)\Delta s = \infty, \tag{3.36}$$

where

$$K(t,s) = LH(\sigma(t),\sigma(s))\delta^{\sigma}(s)q(s) - \frac{(h_{-}(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\delta^{\sigma}(s)\beta(s))^{\gamma}}. \tag{3.37}$$

Then every solution x *of* (1.1) *oscillates on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume $x(t) > 0$, for $t \geq t_* \geq t_0$. Define w by (3.2) as before, then we have (3.11). From (3.11), we have

$$Lq(t)\delta^{\sigma}(t) \leq -w^{\Delta}(t) + \left(\frac{\delta^{\Delta}(t)}{\delta(t)} - p(t)\frac{\delta^{\sigma}(t)}{\delta(t)}\alpha^{\gamma}(t)\right)w(t) - \frac{\gamma\delta^{\sigma}(t)\beta(t)}{(\delta(t))^{(\gamma+1)/\gamma}}(w(t))^{(\gamma+1)/\gamma}. \tag{3.38}$$

Thus,

$$
\begin{aligned}
L\int_{t_*}^{\sigma(t)} H(\sigma(t),\sigma(s))q(s)\delta^{\sigma}(s)\Delta s \le & -\int_{t_*}^{\sigma(t)} H(\sigma(t),\sigma(s))w^{\Delta}(s)\Delta s \\
& + \int_{t_*}^{\sigma(t)} H(\sigma(t),\sigma(s))\left(\frac{\delta^{\Delta}(s)}{\delta(s)} - p(s)\frac{\delta^{\sigma}(s)}{\delta(s)}\alpha^{\gamma}(s)\right)w(s)\Delta s \\
& - \int_{t_*}^{\sigma(t)} H(\sigma(t),\sigma(s))\frac{\gamma\delta^{\sigma}(s)\beta(s)}{(\delta(s))^{(\gamma+1)/\gamma}}(w(s))^{(\gamma+1)/\gamma}\Delta s.
\end{aligned}
\tag{3.39}
$$

Integrating the right side by parts, we have

$$
-\int_{t_*}^{\sigma(t)} H(\sigma(t),\sigma(s))w^{\Delta}(s)\Delta s \le H(\sigma(t),t^*)w(t^*) + \int_{t_*}^{\sigma(t)} H^{\Delta_s}(\sigma(t),s)w(s)\Delta s, \tag{3.40}
$$

and then by using (3.34) and (3.35), we arrive at

$$
\begin{aligned}
& L\int_{t_*}^{\sigma(t)} H(\sigma(t),\sigma(s))q(s)\delta^{\sigma}(s)\Delta s \\
& \quad \le \int_{t_*}^{\sigma(t)} \left[\frac{h_-(t,s)}{\delta(s)(H(\sigma(t),\sigma(s)))^{\gamma/(\gamma+1)}}w(s) - H(\sigma(t),\sigma(s))\frac{\gamma\delta^{\sigma}(s)\beta(s)}{(\delta(s))^{(\gamma+1)/\gamma}}(w(s))^{(\gamma+1)/\gamma}\right]\Delta s \\
& \qquad + H(\sigma(t),t^*)w(t^*).
\end{aligned}
\tag{3.41}
$$

Define

$$
\lambda = \frac{\gamma+1}{\gamma}, \quad A^{\lambda} = H(\sigma(t),\sigma(s))\frac{\gamma\delta^{\sigma}(s)\beta(s)}{\delta^{\lambda}(s)}, \qquad B^{\lambda-1} = \frac{h_-(t,s)}{\lambda\left(\gamma\delta^{\sigma}(s)\beta(s)\right)^{1/\lambda}}. \tag{3.42}
$$

By employing the inequality

$$
\lambda AB^{\lambda-1} - A^{\lambda} \le (\lambda-1)B^{\lambda}, \quad \lambda \ge 1, \tag{3.43}
$$

we obtain

$$
\begin{aligned}
& \frac{h_-(t,s)}{\delta(s)(H(\sigma(t),\sigma(s)))^{\gamma/(\gamma+1)}}w(s) \\
& \quad - H(\sigma(t),\sigma(s))\frac{\gamma\delta^{\sigma}(s)\beta(s)}{(\delta(s))^{(\gamma+1)/\gamma}}(w(s))^{(\gamma+1)/\gamma} \le \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\delta^{\sigma}(s)\beta(s)\right)^{\lambda}}.
\end{aligned}
\tag{3.44}
$$

Therefore,

$$\frac{1}{H(\sigma(t),t^*)}\int_{t_*}^{\sigma(t)} K(t,s)\Delta s \le w(t^*), \tag{3.45}$$

which contradicts (3.36). The proof is complete. □

Theorem 3.8. *Assume that* (2.3) *holds. Further, suppose that* $1-p(t)\mu(t)>0$*, there exists a positive* Δ*-differentiable function* δ*, and* $H,h\in C_{\mathrm{rd}}(\mathbb{D},\mathbb{R})$*, where* $\mathbb{D}=\{(t,s):t\ge s\ge t_0\}$ *such that* (3.28) *holds, and* H *has a continuous and nonpositive* Δ*-partial derivative* $H^{\Delta_s}(t,s)$ *with respect to the second variable and satisfies*

$$H^{\Delta_s}(t,s)+H(t,s)\left(\frac{\delta^{\Delta}(s)}{\delta(s)}-p(s)\frac{\delta^{\sigma}(s)}{\delta(s)}\right)=-\frac{h(t,s)}{\delta^{\sigma}(s)}(H(t,s))^{\gamma/(\gamma+1)}. \tag{3.46}$$

If for sufficiently large t_*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t}K(t,s)\Delta s=\infty, \tag{3.47}$$

where

$$K(t,s)=LH(t,s)\delta(s)q(s)-\frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\delta^{\sigma}(s))^{\gamma}}, \tag{3.48}$$

then every solution x *of* (1.1) *oscillates on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. In view of Theorem 3.3, the proof is similar to [16, Theorem 2.2]. □

Theorem 3.9. *Assume that* (2.3) *holds. Further, suppose that* $1-p(t)\mu(t)>0$*, and for all sufficiently large* t_**,*

$$\limsup_{t\to\infty}(t-t_*)^{\gamma}\int_{t}^{\infty}q(s)\Delta s>\frac{1}{L}. \tag{3.49}$$

Then every solution x *of* (1.1) *oscillates on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we assume $x(t)>0$, for $t\ge t_*\ge t_0$. From (1.1) and Lemma 2.3, we get for $T\ge t\ge t_*$,

$$L\int_{t}^{T}q(s)x^{\gamma}(s)\Delta s<L\int_{t}^{T}q(s)x^{\gamma\sigma}(s)\Delta s<\left(x^{\Delta}(t)\right)^{\gamma}-\left(x^{\Delta}(T)\right)^{\gamma}<\left(x^{\Delta}(t)\right)^{\gamma}. \tag{3.50}$$

Letting $T \to \infty$, we obtain

$$L\int_t^{\infty} q(s)x^{\gamma}(s)\Delta s < \left(x^{\Delta}(t)\right)^{\gamma}. \tag{3.51}$$

In view of Lemma 2.3, we obtain

$$L\int_t^{\infty} q(s)\Delta s < \left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma} < \left(\frac{1}{t-t_*}\right)^{\gamma}. \tag{3.52}$$

Thus

$$\limsup_{t\to\infty}\,(t-t_*)^{\gamma}\int_t^{\infty} q(s)\Delta s \le \frac{1}{L}, \tag{3.53}$$

which is a contradiction. This completes the proof. □

4. Example

In this section, we will give an example to illustrate our results.

Example 4.1. Consider the second-order damped dynamic equation on time scales

$$\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \frac{1}{t}\left(x^{\Delta}(t)\right)^{\gamma} + t(x^{\sigma}(t))^{\gamma} = 0, \tag{4.1}$$

where

$$\mu(t) < t, \qquad p(t) = \frac{1}{t}, \qquad q(t) = t, \qquad \delta(t) = 1, \qquad L = 1, \qquad f(x) = x^{\gamma}. \tag{4.2}$$

Obviously, $f(x)/x^{\gamma} = 1 \ge L = 1$.

It is easy to see that (2.3) holds. For $0 < \gamma \le 1$, one has

$$\limsup_{t\to\infty}\int_{t^*}^{t}\left[s-\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{((1/s)\alpha^{\gamma}(s))^{\gamma+1}}{(\gamma\alpha^{\gamma}(S))^{\gamma}}\right]\Delta s = \limsup_{t\to\infty}\int_{t^*}^{t}\left[s-\frac{1}{(\gamma+1)^{\gamma+1}}\frac{1}{s^{\gamma+1}}\alpha^{\gamma^2}(s)\right]\Delta s = \infty, \tag{4.3}$$

and for $\gamma > 1$,

$$\limsup_{t\to\infty}\int_{t^*}^{t}\left[s-\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{((1/s)\alpha^{\gamma}(s))^{\gamma+1}}{(\gamma\alpha(s))^{\gamma}}\right]\Delta s = \limsup_{t\to\infty}\int_{t^*}^{t}\left[s-\frac{1}{(\gamma+1)^{\gamma+1}}\frac{1}{s^{\gamma+1}}\alpha^{\gamma}(s)\right]\Delta s = \infty. \tag{4.4}$$

Hence, by Theorem 3.1, every solution x of (4.1) is oscillatory.

Remark 4.2. It is easy to see that the results in [16–21] cannot be applied in (4.1), and to the best of our knowledge nothing is known regarding the oscillatory behavior of (1.1), so our results are new.

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