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Research Article

Permanence of a Discrete Model of Mutualism with Infinite Deviating Arguments

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We propose a discrete model of mutualism with infinite deviating arguments, that is $x_1(n+1) = x_1(n)\exp\{r_1(n)[(K_1(n)+\alpha_1(n)\sum_{s=0}^{\infty} J_2(s)x_2(n-s))/(1+\sum_{s=0}^{\infty} J_2(s)x_2(n-s)) - x_1(n-\sigma_1(n))]\}$, $x_2(n+1) = x_2(n)\exp\{r_2(n)[(K_2(n)+\alpha_2(n)\sum_{s=0}^{\infty} J_1(s)x_1(n-s))/(1+\sum_{s=0}^{\infty} J_1(s)x_1(n-s)) - x_2(n-\sigma_2(n))]\}$. By some Lemmas, sufficient conditions are obtained for the permanence of the system.

1. Introduction

Chen and You [1] studied the following two species integro-differential model of mutualism:

$$\frac{dN_1(t)}{dt} = r_1(t)N_1(t)\left[\frac{K_1(t)+\alpha_1(t)\int_0^{\infty} J_2(s)N_2(t-s)ds}{1+\int_0^{\infty} J_2(s)N_2(t-s)ds} - N_1(t-\sigma_1(t))\right],$$
$$\frac{dN_2(t)}{dt} = r_2(t)N_2(t)\left[\frac{K_2(t)+\alpha_2(t)\int_0^{\infty} J_1(s)N_1(t-s)ds}{1+\int_0^{\infty} J_1(s)N_1(t-s)ds} - N_2(t-\sigma_2(t))\right], \tag{1.1}$$

where r_i, K_i, α_i, and $\sigma_i, i = 1,2$ are continuous functions bounded above and below by positive constants: $a_i > K_i$, $i = 1,2$; $J_i \in C([0,+\infty),[0,+\infty))$ and $\int_0^{\infty} J_i(s)ds = 1$, $i = 1,2$. Using the differential inequality theory, they obtained a set of sufficient conditions to ensure the permanence of system (1.1). For more background and biological adjustments of system(1.1), one could refer to [1–4] and the references cited therein.

However, many authors [5–12] have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. Also, since discrete time models can also provide efficient computational models of continuous models for numerical simulations, it is reasonable to study discrete time models governed by difference equations. Another permanence is one of the most important topics on the study of population dynamics. One of the most interesting questions in mathematical biology concerns the survival of species in ecological models. It is reasonable to ask for conditions under which the system is permanent.

Motivated by the above question, we consider the permanence of the following discrete model of mutualism with infinite deviating arguments:

$$x_1(n+1) = x_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n)+\alpha_1(n)\sum_{s=0}^{\infty}J_2(s)x_2(n-s)}{1+\sum_{s=0}^{\infty}J_2(s)x_2(n-s)} - x_1(n-\sigma_1(n))\right]\right\},$$
$$x_2(n+1) = x_2(n)\exp\left\{r_2(n)\left[\frac{K_2(n)+\alpha_2(n)\sum_{s=0}^{\infty}J_1(s)x_1(n-s)}{1+\sum_{s=0}^{\infty}J_1(s)x_1(n-s)} - x_2(n-\sigma_2(n))\right]\right\}, \tag{1.2}$$

where $x_i(n)$, $i = 1,2$ is the density of mutualism species i at the nth generation. For $\{r_i(n)\}, \{K_i(n)\}, \{\alpha_i(n)\}, \{J_i(n)\}$, and $\{\sigma_i(n)\}$, $i = 1,2$ are bounded nonnegative sequences such that

$$0 < r_i^l \le r_i^u, \quad 0 < \alpha_i^l \le \alpha_i^u, \quad 0 < K_i^l \le K_i^u, \quad 0 < \sigma_i^l \le \sigma_i^u, \quad \sum_{n=0}^{\infty} J_i(n) = 1. \tag{1.3}$$

Here, for any bounded sequence $\{a(n)\}$, $a^u = \sup_{n\in N} a(n)$, $a^l = \inf_{n\in N} a(n)$.

Let $\sigma = \sup_n\{\sigma_i(n),\ i = 1,2\}$, we consider (1.2) together with the following initial condition:

$$x_i(\theta) = \varphi_i(\theta) \ge 0, \quad \theta \in N[-\tau,0] = \{-\tau,-\tau+1,\ldots,0\}, \quad \varphi_i(0) > 0. \tag{1.4}$$

It is not difficult to see that solutions of (1.2) and (1.4) are well defined for all $n \ge 0$ and satisfy

$$x_i(n) > 0, \quad \text{for} \quad n \in Z, i = 1,2. \tag{1.5}$$

The aim of this paper is, by applying the comparison theorem of difference equation and some lemmas, to obtain a set of sufficient conditions which guarantee the permanence of system (1.2).

2. Permanence

In this section, we establish permanence results for system (1.2).

Following Comparison Theorem of difference equation is Theorem 2.6 of [13, page 241].

Lemma 2.1. *Let $k \in N_{k_0}^{+} = \{k_0, k_0+1, \ldots, k_0+l, \ldots\}, r \ge 0$. For any fixed $k, g(k,r)$ is a non-decreasing function with respect to r, and for $k \ge k_0$, following inequalities hold: $y(k+1) \le g(k,y(k)), u(k+1) \ge g(k,u(k))$. If $y(k_0) \le u(k_0)$, then $y(k) \le u(k)$ for all $k \ge k_0$.*

Now let us consider the following single species discrete model:

$$N(k+1) = N(k)\exp\{a(k) - b(k)N(k)\}, \tag{2.1}$$

where $\{a(k)\}$ and $\{b(k)\}$ are strictly positive sequences of real numbers defined for $k \in N = \{0, 1, 2, \ldots\}$ and $0 < a^l \le a^u, 0 < b^l \le b^u$. Similar to the proof of Propositions 1 and 3 in [6], we can obtain the following.

Lemma 2.2. *Any solution of system* (2.1) *with initial condition* $N(0) > 0$ *satisfies*

$$m \le \lim_{k\to+\infty}\inf N(k) \le \lim_{k\to+\infty}\sup N(k) \le M, \tag{2.2}$$

where

$$M = \frac{1}{b^l}\exp\{a^u - 1\}, \quad m = \frac{a^l}{b^u}\exp\left\{a^l - b^u M\right\}. \tag{2.3}$$

Lemma 2.3 (see [14]). *Let* $x(n)$ *and* $b(n)$ *be nonnegative sequences defined on* N, *and* $c \ge 0$ *is a constant. If*

$$x(n) \le c + \sum_{s=0}^{n-1} b(s)x(s), \quad \textit{for } n \in N, \tag{2.4}$$

then

$$x(n) \le c\prod_{s=0}^{n-1}[1 + b(s)], \quad \textit{for } n \in N. \tag{2.5}$$

Lemma 2.4 (see [2]). *Let* $x : Z \to R$ *be a nonnegative bounded sequences, and let* $H : N \to R$ *be a nonnegative sequence such that* $\sum_{n=0}^{\infty} J_i(n) = 1$. *Then*

$$\begin{aligned}\lim_{n\to+\infty}\inf x(n) &\le \lim_{n\to+\infty}\inf \sum_{s=-\infty}^{n} H(n-s)x(s)\\ &\le \lim_{n\to+\infty}\sup \sum_{s=-\infty}^{n} H(n-s)x(s) \le \lim_{n\to+\infty}\sup x(n).\end{aligned} \tag{2.6}$$

Proposition 2.5. *Let* $(x_1(n), x_2(n))$ *be any positive solution of system* (1.2), *then*

$$\lim_{n\to+\infty}\sup x_i(n) \le M_i, \quad i = 1, 2, \tag{2.7}$$

where

$$M_i = \exp\left\{2r_i^u\left[K_i^u + \alpha_i^u\right]\right\}, \quad i = 1, 2. \tag{2.8}$$

Proof. Let $(x_1(n), x_2(n))$ be any positive solution of system (1.2), then from the first equation of system (1.2) we have

$$\begin{aligned} x_1(n+1) &\le x_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n)+\alpha_1(n)\sum_{s=0}^{\infty}J_2(s)x_2(n-s)}{1+\sum_{s=0}^{\infty}J_2(s)x_2(n-s)}\right]\right\} \\ &= x_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n)}{1+\sum_{s=0}^{\infty}J_2(s)x_2(n-s)}+\frac{\alpha_1(n)\sum_{s=0}^{\infty}J_2(s)x_2(n-s)}{1+\sum_{s=0}^{\infty}J_2(s)x_2(n-s)}\right]\right\} \\ &\le x_1(n)\exp\left\{r_1(n)\left[\frac{K_1(n)}{1}+\frac{\alpha_1(n)\sum_{s=0}^{\infty}J_2(s)x_2(n-s)}{\sum_{s=0}^{\infty}J_2(s)x_2(n-s)}\right]\right\} \\ &= x_1(n)\exp\{r_1(n)[K_1(n)+\alpha_1(n)]\} \\ &\le x_1(n)\exp\{r_1^u[K_1^u+\alpha_1^u]\}. \end{aligned} \tag{2.9}$$

Let $x_1(n)=\exp\{u_1(n)\}$, then

$$u_1(n+1) \le u_1(n) + r_1^u[K_1^u+\alpha_1^u] = r_1^u[K_1^u+\alpha_1^u] + \sum_{s=0}^{n} b(s)x(s), \tag{2.10}$$

where

$$b(s) = \begin{cases} 0, & 0 \le s \le n-1, \\ 1, & s = n. \end{cases} \tag{2.11}$$

When $u_1(n)$ is nonnegative sequence, by applying Lemma 2.3, it immediately follows that

$$u_1(n+1) \le 2r_1^u[K_1^u+\alpha_1^u]. \tag{2.12}$$

When $u_1(n)$ is negative sequence, (2.12) also holds. From (2.12), we have

$$\lim_{n\to+\infty}\sup x_1(n) \le \exp\{2r_1^u[K_1^u+\alpha_1^u]\} := M_1. \tag{2.13}$$

By using the second equation of system (1.2), similar to the above analysis, we can obtain

$$\lim_{n\to+\infty}\sup x_2(n) \le \exp\{2r_2^u[K_2^u+\alpha_2^u]\} := M_2. \tag{2.14}$$

This completes the proof of Proposition 2.5. □

Now we are in the position of stating the permanence of system (1.2).

Theorem 2.6. *Under the assumption*(1.3), *system* (1.2) *is permanent, that is, there exist positive constants* $m_i, M_i, i = 1, 2$ *which are independent of the solutions of system* (1.2) *such that, for any positive solution* $(x_1(n), x_2(n))$ *of system*(1.2) *with initial condition* (1.4), *one has*

$$m_i \le \lim_{n\to+\infty} \inf x_i(n) \le \lim_{n\to+\infty} \sup x_i(n) \le M_i, \quad i = 1, 2. \tag{2.15}$$

Proof. By applying Proposition 2.5, we see that to end the proof of Theorem 2.6 it is enough to show that under the conditions of Theorem 2.6

$$\lim_{n\to+\infty} \inf x_i(n) \ge m_i. \tag{2.16}$$

From Proposition 2.5, For all $\varepsilon > 0$, there exists a $N_1 > 0$, $N_1 \in N$, For all $n > N_1$,

$$x_i(n) \le M_i + \varepsilon. \tag{2.17}$$

According to Lemma 2.4, from (2.13) and (2.14) we have

$$\lim_{n\to+\infty} \sup \sum_{s=0}^{\infty} J_i(s) x_i(n-s) = \lim_{n\to+\infty} \sup \sum_{k=-\infty}^{n} J_i(n-k) x_i(k) \le M_i, \quad i = 1, 2. \tag{2.18}$$

For above $\varepsilon > 0$, according to (2.18), there exists a positive integer N_2, such that, for all $n > N_2$,

$$\sum_{s=0}^{\infty} J_i(s) x_i(n-s) \le M_i + \varepsilon, \quad i = 1, 2. \tag{2.19}$$

Thus, for all $n > max\{N_1, N_2\} + \sigma$, from the first equation of system(1.2), it follows that

$$\begin{aligned} x_1(n+1) &\ge x_1(n) \exp\left\{ r_1(n) \left[\frac{K_1^l}{1 + (M_2 + \varepsilon)} - (M_1 + \varepsilon) \right] \right\} \\ &\ge x_1(n) \exp\left\{ \frac{r_1^l K_1^l}{1 + (M_2 + \varepsilon)} - r_1^u (M_1 + \varepsilon) \right\}. \end{aligned} \tag{2.20}$$

It follows that, for $n \ge \sigma_1(n)$,

$$\prod_{i=n-\sigma_1(n)}^{n-1} x_1(i+1) \ge \prod_{i=n-\sigma_1(n)}^{n-1} x_1(i) \exp\left\{ \frac{r_1^l K_1^l}{1 + (M_2 + \varepsilon)} - r_1^u (M_1 + \varepsilon) \right\}. \tag{2.21}$$

Hence

$$x_1(n) \ge x_1(n - \sigma_1(n)) \exp\left\{ \frac{r_1^l K_1^l}{1 + (M_2 + \varepsilon)} \sigma_1^l - r_1^u (M_1 + \varepsilon) \sigma_1^u \right\}. \tag{2.22}$$

In other words,

$$x_1(n-\sigma_1(n)) \le x_1(n)\exp\left\{-\frac{r_1^l K_1^l}{1+(M_2+\varepsilon)}\sigma_1^l + r_1^u(M_1+\varepsilon)\sigma_1^u\right\}. \tag{2.23}$$

From the first equation of system (1.2) and (2.23), for all $n > max\{N_1, N_2\} + \sigma$, it follows that

$$x_1(n+1) \ge x_1(n)\exp\left\{-\frac{r_1^l K_1^l}{1+(M_2+\varepsilon)} - r_1^u \exp\left\{-\frac{r_1^l K_1^l}{1+(M_2+\varepsilon)}\sigma_1^l + r_1^u(M_1+\varepsilon)\sigma_1^u\right\}x_1(n)\right\}. \tag{2.24}$$

By applying Lemmas 2.1 and 2.2 to (2.24), it immediately follows that

$$\begin{aligned}\liminf_{n\to+\infty} x_1(n) \ge{}& \frac{r_1^l K_1^l}{r_1^u(1+(M_2+\varepsilon))}\exp\left\{\frac{r_1^l K_1^l}{1+(M_2+\varepsilon)}\sigma_1^l - r_1^u(M_1+\varepsilon)\sigma_1^u\right\}\\ &\times\exp\left\{\frac{r_1^l K_1^l}{1+(M_2+\varepsilon)} - r_1^u\exp\left\{-\frac{r_1^l K_1^l}{1+(M_2+\varepsilon)}\sigma_1^l + r_1^u(M_1+\varepsilon)\sigma_1^u\right\}M_1\right\}.\end{aligned} \tag{2.25}$$

Setting $\varepsilon \to 0$, it follows that

$$\begin{aligned}\liminf_{n\to+\infty} x_1(n) \ge{}& \frac{r_1^l K_1^l}{r_1^u(1+M_2)}\exp\left\{\frac{r_1^l K_1^l}{1+M_2}\sigma_1^l - r_1^u M_1\sigma_1^u\right\}\\ &\times\exp\left\{\frac{r_1^l K_1^l}{1+M_2} - r_1^u\exp\left\{-\frac{r_1^l K_1^l}{1+M_2}\sigma_1^l + r_1^u M_1\sigma_1^u\right\}M_1\right\}.\end{aligned} \tag{2.26}$$

Similar to the above analysis, from the second equation of system (1.2), we have that

$$\begin{aligned}\liminf_{n\to+\infty} x_2(n) \ge{}& \frac{r_2^l K_2^l}{r_2^u(1+M_1)}\exp\left\{\frac{r_2^l K_2^l}{1+M_1}\sigma_2^l - r_2^u M_2\sigma_2^u\right\}\\ &\times\exp\left\{\frac{r_2^l K_2^l}{1+M_1} - r_2^u\exp\left\{-\frac{r_2^l K_2^l}{1+M_1}\sigma_2^l + r_2^u M_2\sigma_2^u\right\}M_2\right\}.\end{aligned} \tag{2.27}$$

This completes the proof of Theorem 2.6. □

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