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Research Article

Extended Mixed Function Method and Its Application for Solving Two Classic Toda Lattice Equations

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The mixed function method is extended from the (1 + 1)-dimensional space to the (2 + 1)-dimensional one, even those forms of exact solution do not exist in (1 + 1)-dimensional NDDEs. By using this extended method, the Toda lattice and (2 + 1)-dimensional Toda lattice equations are studied. Some new solutions such as discrete solitary wave solutions, discrete kink and antikink wave solutions, and discrete breather soliton solutions are obtained, and their dynamic properties are discussed.

1. Introduction

In [1], motivated by the structures of the exact solutions obtained from Darboux transformation, the authors constructed the mixed function method in (1 + 1)-dimensional space. Further, they studied the generalized Hybrid lattice equation and the two-component Volterra lattice equation by using this method, then they obtained some new exact solutions such as discrete solitary wave solutions and kink wave solutions. By this token, the mixed function method is as powerful as the exp method [2–7] and tanh method [8–10]. In this paper, we will extend this method from the (1 + 1)-dimensional space to the (2 + 1)-dimensional one and present some new forms on establishing different kinds of exact solutions, then this extended method will be applied to many higher-order nonlinear differential-difference equations (NDDEs). Moreover, we provide a simplified way to solve large numbers of high-power algebraic equations using a computer. Of course, these high-power algebraic equations mentioned above are all derived from high-order or high-power NDDEs. Therefore, the procedure of computation can be greatly simplified, and time is saved. We notice that this extended method is as powerful as the other symbolic computation methods such as tanh-function method, sine-cosine function method, exp-function method, Jacobian elliptic function method, (G'/G)-expansion method, and Adomian decomposition method, and these methods (please refer to [2–18] and references cited therein) are popular tools in the field of the nonlinear differential-difference equations.

We consider the following $(2 + 1)$-dimensional Toda lattice equation [19]

$$\frac{\partial^2 u_n}{\partial x \partial t} = \left(\frac{\partial u_n}{\partial t} + 1 \right) (u_{n-1} - 2u_n + u_{n+1}), \tag{1.1}$$

and the Toda lattice equation [20]

$$\frac{d^2 u_n}{dt^2} = \left(\frac{du_n}{dt} + 1 \right) (u_{n-1} - 2u_n + u_{n+1}). \tag{1.2}$$

It is well known that there is another form for the $(2 + 1)$-dimensional Toda lattice equation

$$\frac{\partial^2 y_n}{\partial x \partial t} = \exp(y_{n-1} - y_n) - \exp(y_n - y_{n+1}), \tag{1.3}$$

and there are another two forms for the Toda lattice equation

$$\frac{d^2 y_n}{dt^2} = \exp(y_{n-1} - y_n) - \exp(y_n - y_{n+1}), \tag{1.4}$$

$$\dot{u}_n = u_n(v_{n-1} - v_n), \qquad \dot{v}_n = u_n - u_{n+1}. \tag{1.5}$$

Under the transformation $\partial u_n/\partial t = \exp(y_{n-1} - y_n) - 1$, (1.3) and (1.4) can be reduced to (1.1) and (1.2), respectively. For (1.1) and (1.2), there are important physical models and biological models which have been studied by many authors, see [7, 10, 21–39] and references cited therein. For example, the Toda lattice equation was used as DNA models [36–38], this equation can also be used as a model to describe the pressure pulse wave in aorta.

Recently, by using the exp-function method, Zhu [7] studied $(2 + 1)$-dimensional Toda lattice equation and obtained some exact solutions of exp-function type. By using tanh method, S. Zhang and H.-Q. Zhang [10] studied the same equation, they obtained some exact solutions of hyperbolic function type. By using the symbolic computation of hyperbolic tangent solutions, Baldwin et al. [11] studied both (1.1) and (1.2), and they obtained some exact solutions of tanh-function type. By using the modified hyperbolic function method, Zhi et al. [34] studied (1.2), they obtained some discrete soliton solutions of hyperbolic function type. In this paper, using the extended mixed function method, we discuss (1.1) and (1.2), and new solutions which are different from the results in [7, 10, 11, 34] are obtained.

The rest of this paper is organized as follows. In Section 2, we introduce the extended mixed function method. In Section 3, by using the extended method, we obtain new exact solutions of (1.1), and discuss their dynamic properties. In Section 4, we will obtain new exact solutions of (1.2), and also discuss their dynamic properties.

2. The Extended Mixed Function Method

In this section, we begin formally introducing the extended mixed function method motivated by [1].

First, we extend the mixed function method from (1 + 1)-dimensional space to (2 + 1)-dimensional one, and later, we present some new forms of solutions on establishing different kinds of exact solutions of NDDEs. Without loss of generality, we suppose that a polynomial (2 + 1)-dimensional NDDEs and (1 + 1)-dimensional NDDEs have the following form:

$$\Psi\left(\ldots,\ u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}, \ldots, \frac{\partial u_n}{\partial t}, \frac{\partial u_n}{\partial x}, \frac{\partial^2 u_n}{\partial x \partial t}, \ldots\right) = 0, \tag{2.1}$$

$$\Psi\left(\ldots, u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}, \ldots, \frac{\partial u_n}{\partial t}, \frac{\partial^2 u_n}{\partial t^2}, \ldots\right) = 0, \tag{2.2}$$

respectively, where $u_n = u(t, n, x)$ in (2.1) and $u_n = u(t, n)$ in (2.2), n is a discrete variable, and x and t are two continuous variable. The general framework of the extended method for NDDEs is shown in the following.

Step 1. Corresponding to (2.1), as in [1], we suppose that its exact solutions has the following three expansions:

$$u_n = \sum_{p=0}^{m} a_p \left[\frac{w^n e^{\alpha x - \beta t + \gamma_0} - B e^{-(\alpha x - \beta t + \gamma_0)}}{w^n e^{\alpha x - \beta t + \gamma_0} + C e^{-(\alpha x - \beta t + \gamma_0)}}\right]^p, \tag{2.3}$$

$$u_n = \sum_{p=0}^{m} a_p \left[\frac{w^n}{w^{2n} e^{\alpha x - \beta t + \gamma_0} + C e^{-(\alpha x - \beta t + \gamma_0)}}\right]^p, \tag{2.4}$$

$$u_n = \frac{\sum_{p=0}^{m} a_p \left[w^n \exp(\alpha x - \beta t + \gamma_0)\right]^p}{\sum_{p=0}^{m} b_p \left[w^n \exp(\alpha x - \beta t + \gamma_0)\right]^p}, \tag{2.5}$$

where m is a positive integer which can be given by the homogeneous balance principle or the character of the idiographic NDDEs, the parameters B, C, w, α, β, γ_0, a_p, b_p $(p = 0, 1, \ldots, m)$ are constants which need to be determined later, and $k \in N$, $B \neq -C$, $a_0/b_0 \neq a_2/b_2 \neq \cdots \neq a_m/b_m$.

Besides the above forms of solutions, we will add three new expansions. If (2.1) has no term $\partial u_n/\partial x$, then we suppose their expansions as follows:

$$u_n = An\, f(x) + \sum_{p=0}^{m} b_p \left[\frac{w^n e^{\beta t + \gamma_0} - B e^{-(\beta t + \gamma_0)}}{w^n e^{\beta t + \gamma_0} + C e^{-(\beta t + \gamma_0)}}\right]^p, \tag{2.6}$$

$$u_n = An\, f(x) + \sum_{p=0}^{m} b_p \left[\frac{w^n}{w^{2n} e^{\beta t + \gamma_0} + C e^{-(\beta t + \gamma_0)}}\right]^p, \qquad b_{2k+1} = 0, \tag{2.7}$$

$$u_n = An\, f(x) + \sum_{p=0}^{m} \frac{c_p (w^n)^p}{\left(e^{\beta t + \gamma_0}\right)^p + B (w^n)^p}, \qquad c_{2k+1} = 0, \tag{2.8}$$

where $f(x)$ is an arbitrary function which can be chosen as hyperbolic functions and trigonometric functions. In [1], we supposed that (2.2)'s exact solution is one of the following two kinds of expansion forms:

$$u_n = \sum_{i=0}^{m} a_i \left[\frac{w^n e^{\Omega t} - B e^{-\Omega t}}{w^n e^{\Omega t} + C e^{-\Omega t}} \right]^i, \tag{2.9}$$

$$u_n = \sum_{i=0}^{m} a_i \left[\frac{w^n}{w^{2n} e^{\Omega t} + C e^{-\Omega t}} \right]^i. \tag{2.10}$$

We add two new forms of solutions for (2.2) here

$$u_n = \frac{\sum_{p=0}^{m} a_p \left(w^n e^{\Omega t}\right)^p}{\sum_{p=0}^{m} b_p \left(w^n e^{\Omega t}\right)^p}, \quad a_{2k+1} = 0, \ b_{2k+1} = 0, \tag{2.11}$$

$$u_n = \sum_{p=0}^{m} \frac{a_p (w^n)^p}{\left(e^{\Omega t}\right)^p + B (w^n)^p}, \quad a_{2k+1} = 0, \tag{2.12}$$

where m is a positive integer which can be given by the homogeneous balance principle or the character of the idiographic NDDEs, the parameters B, C, w, Ω, a_p, b_p $(p = 0, 1, \ldots, m)$ are constants which need to be determined later, and $k \in N$, $B \neq -C$, $a_0/b_0 \neq a_2/b_2 \neq \cdots \neq a_m/b_m$.

Step 2. By using the homogeneous balance principle or according to character of the idiographic NDDEs, we determine the value of m then substitute it in the expressions (2.3)–(2.12) of Step 1. Sometimes, we can directly assume that $m = 1$ or $m = 2$. In [8], by using the homogeneous balance method, the balance number of the generalized hybrid equation has been obtained successfully.

Step 3. Substituting the presupposed solutions determined by Step 2 in the original equation (2.1) or (2.2), then setting the coefficients of all independent terms in $e^{k\xi}, e^{-k\xi}$ $(k = 0, 1, 2, \ldots, N,\ \xi = \alpha x - \beta t + \gamma_0$ or $\xi = \Omega t)$ to zero, and we get a series of algebraic equations from which the corresponding undetermined constants are explicitly solved by the use of mathematical software *Maple* or *Mathematica*.

Step 4. Substituting the values of these constants B, C, w, Ω, α, β, γ_0, a_p, b_p $(p = 0, 1, \ldots, m)$ given by Step 3 in the solutions presupposed by Step 2, thus, the exact solutions of the original equation (2.1) or (2.2) are obtained finally.

3. New Exact Solutions of (2 + 1)-Dimensional Toda Lattice Equation and Their Dynamic Properties

In this section, by using the extended method shown in the Section 2, we discuss exact solutions of (2 + 1)-dimensional Toda lattice equation (1.1) and their dynamic properties.

3.1. Exact Solutions of the Form (2.3)

Suppose that (1.1) has exact solutions of the form (2.3). By using the balance procedure, we easily obtain $m = 1$. Thus, we let

$$u_n = a_0 + a_1 \left[\frac{w^n \exp(\alpha x - \beta t + \gamma_0) - B \exp(-(\alpha x - \beta t + \gamma_0))}{w^n \exp(\alpha x - \beta t + \gamma_0) + C \exp(-(\alpha x - \beta t + \gamma_0))} \right], \tag{3.1}$$

where $a_0, a_1, w, \alpha, \beta, \gamma_0, B, C$ are constants to be determined later, and $B \neq -C$. After substituting (3.1) into (1.1), multiplying both sides by the common denominator $(\omega^n e^{\alpha x - \beta t + \gamma_0} + Ce^{-(\alpha x - \beta t + \gamma_0)})^3 (\omega^{n-1} e^{\alpha x - \beta t + \gamma_0} + Ce^{-(\alpha x - \beta t + \gamma_0)})(\omega^{n+1} e^{\alpha x - \beta t + \gamma_0} + Ce^{-(\alpha x - \beta t + \gamma_0)})$ and dividing both sides by the common factor $a_1(B + C)$, it follows:

$$A_{-3} e^{-3(\alpha x - \beta t + \gamma_0)} - A_{-1} e^{-(\alpha x - \beta t + \gamma_0)} + A_1 e^{\alpha x - \beta t + \gamma_0} + A_3 e^{3(\alpha x - \beta t + \gamma_0)} = 0, \tag{3.2}$$

where

$$\begin{aligned}
A_{-3} &= -C^3 \left[w^{n+1} + (4\alpha\beta - 2) w^n + w^{n-1} \right], \\
A_{-1} &= C \Big[(-4\alpha\beta C + 2a_1\beta B - C + 2a_1\beta C) w^{2n+1} + (-4a_1\beta B + 2C - 4a_1\beta C + 4\alpha\beta C) w^{2n} \\
&\quad + (-4\alpha\beta C + 2a_1\beta B - C + 2a_1\beta C) w^{2n-1} \Big], \\
A_1 &= (-4\alpha\beta C + 2a_1\beta B - C + 2a_1\beta C) w^{3n+1} - (-4a_1\beta B + 2C - 4a_1\beta C + 4\alpha\beta C) w^{3n} \\
&\quad - (-4\alpha\beta C + 2a_1\beta B - C + 2a_1\beta C) w^{3n-1}, \\
A_3 &= w^{4n+1} + (4\alpha\beta - 2) w^{4n} + w^{4n-1}.
\end{aligned} \tag{3.3}$$

In (3.2), setting the coefficients of all independent terms $e^{-3(\alpha x - \beta t + \gamma_0)}$, $e^{-(\alpha x - \beta t + \gamma_0)}$, $e^{\alpha x - \beta t + \gamma_0}$, $e^{3(\alpha x - \beta t + \gamma_0)}$ to zero, we get a series of algebraic equations as follows:

$$A_{-3} = 0, \qquad A_{-1} = 0, \qquad A_1 = 0, \qquad A_3 = 0. \tag{3.4}$$

Equation (3.4) is a group of high-power algebraic equations, so the computational load may be heavy when we use computer to solve it. Notice that $A_{-3} = -C^3 w^{n-1} [w^2 + (4\alpha\beta - 2) w + 1]$, we let

$$M_{-3} = \frac{A_{-3}}{(-C^3 w^{n-1})} = w^2 + (4\alpha\beta - 2) w + 1. \tag{3.5}$$

As a result, the equation $A_{-3} = 0$ becomes $M_{-3} = 0$, that is, $w^2 + (4\alpha\beta - 2)w + 1 = 0$, which is a low-power equation. Similarly, we obtain

$$\begin{aligned}
M_{-1} &= \frac{A_{-1}}{(Cw^{2n-1})} = (-4\alpha\beta C + 2a_1\beta B - C + 2a_1\beta C)w^2 + (-4a_1\beta B + 2C - 4a_1\beta C + 4\alpha\beta C)w \\
&\quad + (-4\alpha\beta C + 2a_1\beta B - C + 2a_1\beta C), \\
M_1 &= \frac{A_1}{(-w^{3n-1})} = (-4\alpha\beta C + 2a_1\beta B - C + 2a_1\beta C)w^2 + (-4a_1\beta B + 2C - 4a_1\beta C + 4\alpha\beta C)w \\
&\quad + (-4\alpha\beta C + 2a_1\beta B - C + 2a_1\beta C), \\
M_3 &= \frac{A_3}{w^{4n-1}} = w^2 + (4\alpha\beta - 2)w + 1.
\end{aligned} \tag{3.6}$$

Notice that $M_3 = M_{-3}$, $M_1 = M_{-1}$, thus, (3.4) can be reduced to a group of low-power algebraic equations as follows:

$$M_1 = 0, \qquad M_3 = 0. \tag{3.7}$$

Solving (3.7) yields

$$B = \frac{C(2\alpha - a_1)}{a_1}, \tag{3.8}$$

$$w_{1,2} = 1 - 2\alpha\beta \pm 2\sqrt{\alpha\beta(\alpha\beta - 1)}, \tag{3.9}$$

and C, α, β, γ_0, a_0, a_1 are arbitrary nonzero constants. The process of computation can be greatly simplified by the above settings, and time is saved.

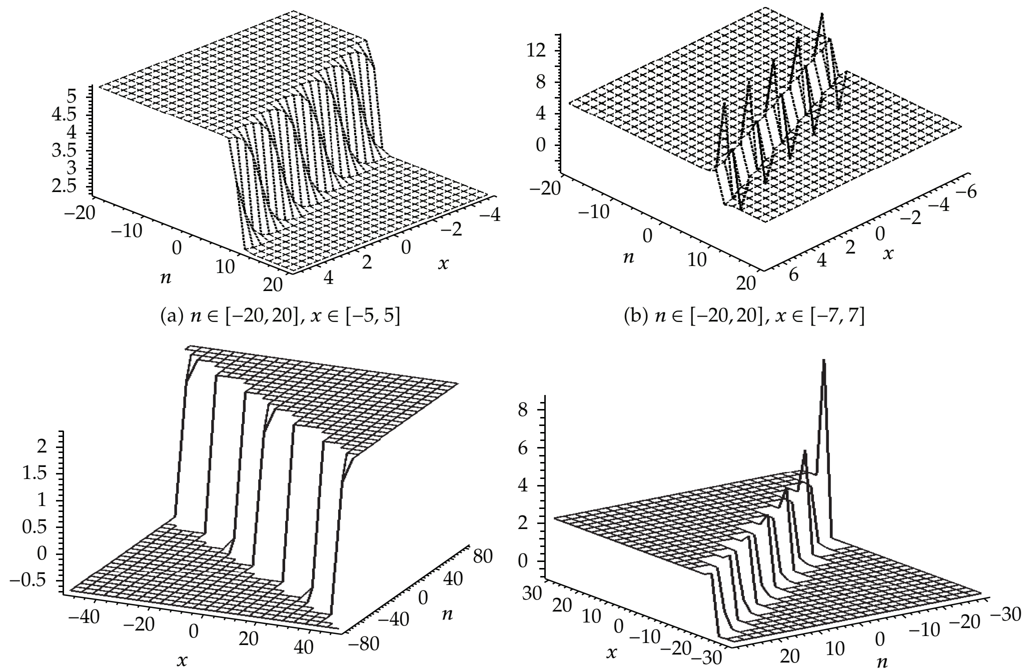
From (3.1), (3.8), and (3.9), we obtain two exact solutions of (1.1) as follows:

$$u_n = a_0 + a_1\left[\frac{w_1^n \exp(\alpha x - \beta t + \gamma_0) - (C(2\alpha - a_1)/a_1)\exp(-\alpha x + \beta t - \gamma_0)}{w_1^n \exp(\alpha x - \beta t + \gamma_0) + C\exp(-\alpha x + \beta t - \gamma_0)}\right], \tag{3.10}$$

$$u_n = a_0 + a_1\left[\frac{w_2^n \exp(\alpha x - \beta t + \gamma_0) - (C(2\alpha - a_1)/a_1)\exp(-\alpha x + \beta t - \gamma_0)}{w_2^n \exp(\alpha x - \beta t + \gamma_0) + C\exp(-\alpha x + \beta t - \gamma_0)}\right], \tag{3.11}$$

where $w_{1,2}$ are given by (3.9).

In fact, (3.10) and (3.11) are solutions of local discretization because the n is a discrete variable while the x, t are continuous variables. Here we still call them discrete exact solutions. In order to describe the dynamic properties of these two discrete soliton solutions intuitively, we plot their profile figures for some fixed parametric values, as shown in Figure 1. Setting $a_0 = 2$, $a_1 = 0.3$, $\alpha = -1.5$, $\beta = 2$, $\gamma_0 = -1$, $t = 1$, when $C = 4 > 0$, the solution (3.10) shows a shape of discrete kink soliton, see Figure 1(a); when $C = -4 < 0$, the solution u_n (3.10) shows another shape of discrete kink soliton, see Figure 1(b). The shape



(a) $n \in [-20, 20]$, $x \in [-5, 5]$

(b) $n \in [-20, 20]$, $x \in [-7, 7]$

(c) $n \in [-30, 30]$, $x \in [-30, 30]$

(d) $n \in [-30, 30]$, $x \in [-30, 30]$

Figure 1: The solutions u_n in (3.10) and (3.11) show four shapes of discrete kink and antikink solitons for fixed parametric values of $a_0, a_1, \alpha, \beta, \gamma_0,\ t$ and different kinds of C values: (a) $C = 4 > 0$; (b) $C = -4 < 0$; (c) $C = 1.4 > 0$; (d) $C = -1.4 < 0$.

shown by Figure 1(b) has jumping phenomenon, but this is not the case shown by Figure 1(a). Setting $a_0 = 2,\ a_1 = 0.3,\ \alpha = 0.8,\ \beta = 1.5,\ \gamma_0 = 3,\ \ t = 1$, when $C = 1.4 > 0$, the solution (3.11) shows a shape of discrete antikink soliton, see Figure 1(c); when $C = -1.4 < 0$, the solution u_n (3.11) shows another shape of discrete antikink soliton, see Figure 1(d). From Figures 1(c) to 1(d), the waveforms have both discrete and continuous character, and the soliton is discrete along the n-axes and is continuous along the x-axes.

3.2. Exact Solutions of the Form (2.5)

Suppose that (1.1) has exact solution of the form (2.5). By using the balance procedure, we obtain $m = 2$. For simplicity, here we only consider the case of $a_1 = b_1 = 0$. Suppose that

$$u_n = \frac{a_0 + a_2 w^{2n} \exp\left[2(\alpha x - \beta t + \gamma_0)\right]}{b_0 + b_2 w^{2n} \exp\left[2(\alpha x - \beta t + \gamma_0)\right]}, \tag{3.12}$$

where $a_0,\ a_2,\ b_0,\ b_2,\ w,\ \alpha,\ \beta$ are constants to be determined later, and $a_0/b_0 \neq a_2/b_2$, the γ_0 is an arbitrary constant. As in Section 3.1, after substituting (3.12) into (1.1), multiplying both sides of the result by the common denominator $(b_0 + b_2 w^{2n} e^{2(\alpha x - \beta t + \gamma_0)})^3 (b_0 + b_2 w^{2(n-1)} e^{2(\alpha x - \beta t + \gamma_0)})(b_0 + b_2 w^{2(n+1)} e^{2(\alpha x - \beta t + \gamma_0)})$, it follows:

$$A_2 e^{2(\alpha x - \beta t + \gamma_0)} + A_4 e^{4(\alpha x - \beta t + \gamma_0)} + A_6 e^{6(\alpha x - \beta t + \gamma_0)} + A_8 e^{8(\alpha x - \beta t + \gamma_0)} = 0, \tag{3.13}$$

where

$$\begin{aligned}
A_2 &= \left(b_0^3 w^{2n-2}\right)\left[(b_2 a_0 - a_2 b_0) w^4 + 2(b_2 a_0 - a_2 b_0)\ (2\alpha\beta - 1) w^2 + b_2 a_0 - a_2 b_0\right],\\
A_4 &= \left(b_0 w^{4n-2}\right)\Big[(b_2 a_0 - a_2 b_0)(2b_2 a_0 \beta + 4 b_2 b_0 \alpha\beta + b_0 b_2 - 2a_2 b_0 \beta) w^4\\
&\qquad - 2(b_2 a_0 - a_2 b_0)(2b_2 a_0 \beta + 2b_2 b_0 \alpha\beta + b_0 b_2 - 2a_2 b_0 \beta) w^2 + (b_2 a_0 - a_2 b_0)\\
&\qquad \times (2b_2 a_0 \beta + 4 b_2 b_0 \alpha\beta + b_0 b_2 - 2a_2 b_0 \beta)\Big],\\
A_6 &= \left(-b_2 w^{6n-2}\right)\Big[(b_2 a_0 - a_2 b_0)(2b_2 a_0 \beta + 4 b_2 b_0 \alpha\beta + b_0 b_2 - 2a_2 b_0 \beta) w^4\\
&\qquad - 2(b_2 a_0 - a_2 b_0)(2b_2 a_0 \beta + 2b_2 b_0 \alpha\beta + b_0 b_2 - 2a_2 b_0 \beta) w^2 + (b_2 a_0 - a_2 b_0)\\
&\qquad \times (2b_2 a_0 \beta + 4 b_2 b_0 \alpha\beta + b_0 b_2 - 2a_2 b_0 \beta)\Big],\\
A_8 &= \left(-b_2^3 w^{8n-2}\right)\left[(b_2 a_0 - a_2 b_0) w^4 + 2(b_2 a_0 - a_2 b_0)(2\alpha\beta - 1) w^2 + b_2 a_0 - a_2 b_0\right].
\end{aligned} \tag{3.14}$$

In (3.13), setting the coefficients of all independent exp-function terms to zero, it follows

$$A_2 = 0, \qquad A_4 = 0, \qquad A_6 = 0, \qquad A_8 = 0. \tag{3.15}$$

Let $M_2 = A_2/(b_0^3 w^{2n-2})$, $M_4 = A_4/(b_0 w^{4n-2})$, $M_6 = A_6/(-b_2 w^{6n-2})$, $M_8 = A_8/(-b_2^3 w^{8n-2})$. Then

$$\begin{aligned}
M_2 = M_8 &= (b_2 a_0 - a_2 b_0) w^4 + 2(b_2 a_0 - a_2 b_0)(2\alpha\beta - 1) w^2 + b_2 a_0 - a_2 b_0,\\
M_4 = M_6 &= (b_2 a_0 - a_2 b_0)(2b_2 a_0 \beta + 4 b_2 b_0 \alpha\beta + b_0 b_2 - 2a_2 b_0 \beta) w^4\\
&\qquad - 2(b_2 a_0 - a_2 b_0)(2b_2 a_0 \beta + 2b_2 b_0 \alpha\beta + b_0 b_2 - 2a_2 b_0 \beta) w^2 + (b_2 a_0 - a_2 b_0)\\
&\qquad \times (2b_2 a_0 \beta + 4 b_2 b_0 \alpha\beta + b_0 b_2 - 2a_2 b_0 \beta)].
\end{aligned} \tag{3.16}$$

Thus, (3.15) can be reduced to the following equations:

$$M_2 = 0, \qquad M_4 = 0. \tag{3.17}$$

Solving (3.17), we obtain

$$\alpha = \frac{b_0 a_2 - a_0 b_2}{2 b_0 b_2}, \tag{3.18}$$

$$w_{1,2} = \pm \frac{\sqrt{b_0 b_2 \left(b_0 b_2 - b_0 a_2 \beta + a_0 b_2 \beta + \sqrt{\beta (b_2 a_0 - a_2 b_0)(a_0 b_2 \beta + 2 b_0 b_2 - a_2 b_0 \beta)} \right)}}{b_0 b_2}, \tag{3.19}$$

$$w_{3,4} = \pm \frac{\sqrt{b_0 b_2 \left(b_0 b_2 - b_0 a_2 \beta + a_0 b_2 \beta - \sqrt{\beta (b_2 a_0 - a_2 b_0)(a_0 b_2 \beta + 2 b_0 b_2 - a_2 b_0 \beta)} \right)}}{b_0 b_2}, \tag{3.20}$$

and $a_0, a_2, b_0, b_2, \beta, \gamma_0$ are arbitrary nonzero constants with $a_0/b_0 \neq a_2/b_2$.

Based on (3.12), (3.18), (3.19), and (3.20), we obtain four exact solutions of (1.1) as follows:

$$u_n = \frac{a_0 + a_2 (w_1)^{2n} \exp\left(((b_0 a_2 - a_0 b_2)/b_0 b_2) x - 2\beta t + 2\gamma_0 \right)}{b_0 + b_2 (w_1)^{2n} \exp\left(((b_0 a_2 - a_0 b_2)/b_0 b_2) x - 2\beta t + 2\gamma_0 \right)}, \tag{3.21}$$

$$u_n = \frac{a_0 + a_2 (w_2)^{2n} \exp\left(((b_0 a_2 - a_0 b_2)/b_0 b_2) x - 2\beta t + 2\gamma_0 \right)}{b_0 + b_2 (w_2)^{2n} \exp\left(((b_0 a_2 - a_0 b_2)/b_0 b_2) x - 2\beta t + 2\gamma_0 \right)}, \tag{3.22}$$

$$u_n = \frac{a_0 + a_2 (w_3)^{2n} \exp\left(((b_0 a_2 - a_0 b_2)/b_0 b_2) x - 2\beta t + 2\gamma_0 \right)}{b_0 + b_2 (w_3)^{2n} \exp\left(((b_0 a_2 - a_0 b_2)/b_0 b_2) x - 2\beta t + 2\gamma_0 \right)}, \tag{3.23}$$

$$u_n = \frac{a_0 + a_2 (w_4)^{2n} \exp\left(((b_0 a_2 - a_0 b_2)/b_0 b_2) x - 2\beta t + 2\gamma_0 \right)}{b_0 + b_2 (w_4)^{2n} \exp\left(((b_0 a_2 - a_0 b_2)/b_0 b_2) x - 2\beta t + 2\gamma_0 \right)}, \tag{3.24}$$

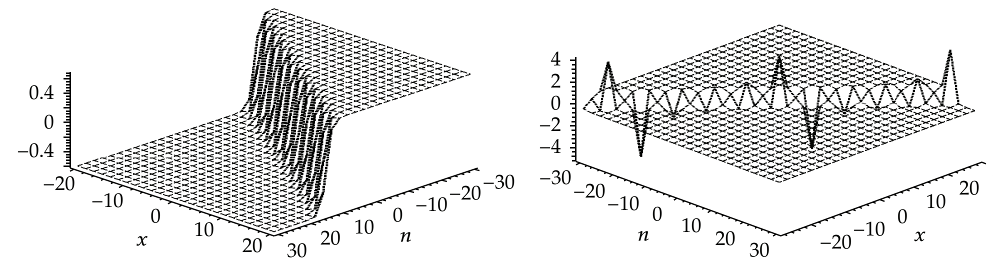
where w_1, w_2, w_3, w_4 are given by (3.19), (3.20), and $w_{1,2,3,4} \neq 1$, $b_0 a_2 \neq a_0 b_2$.

The dynamic properties of solutions (3.21), (3.22), (3.23), and (3.24) are similar to those of (3.10) and (3.11). As an example, we plot two profile figures of the solutions (3.21) and (3.22), see Figure 2.

3.3. Exact Solutions of the Form (2.6)

Suppose that (1.1) has exact solution of the form (2.6). By using the balance procedure, we obtain $m = 1$. So, we suppose that (1.1)'s solution has the following form:

$$u_n = Anf(x) + b_0 + b_1 \left(\frac{w^n e^{\beta t + \gamma_0} - B e^{-\beta t - \gamma_0}}{w^n e^{\beta t + \gamma_0} + C e^{-\beta t - \gamma_0}} \right), \tag{3.25}$$



(a) $n \in [-30, 30]$, $x \in [-20, 20]$ (b) $n \in [-30, 30]$, $x \in [-28, 28]$

Figure 2: The solutions u_n in (3.21) and (3.22) show two shapes of discrete antikink and kink solitons for fixed parametric values: $a_0 = 2$, $b_0 = 3$, $a_2 = -1.5$, $b_2 = 2.5$, $\beta = 0.8$, $\gamma_0 = 5$, $t = 2$.

where $A, B, C, b_0, b_1, w, \beta$ are constants to be determined later, γ_0 is an arbitrary constant, and $f(x)$ is an arbitrary function. As in Section 3.1, after substituting (3.25) into (1.1), multiplying both sides of the result by the common denominator $(w^n e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0})^3 (w^{n-1} e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0})(w^{n+1} e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0})$, it follows:

$$A_{-3} e^{-3(\beta t+\gamma_0)} + A_{-1} e^{-(\beta t+\gamma_0)} + A_1 e^{\beta t+\gamma_0} + A_3 e^{3(\beta t+\gamma_0)} = 0, \tag{3.26}$$

where

$$\begin{aligned}
A_{-3} &= -a_1 C^3 (B + C) w^{n-1} (w - 1)^2, \\
A_{-1} &= -a_1 C (B + C)\left(2a_1 \beta B + C + 2a_1 \beta C\right) w^{2n-1} (w - 1)^2, \\
A_1 &= a_1 (B + C)\left(2a_1 \beta B + C + 2a_1 \beta C\right) w^{3n-1} (w - 1)^2, \\
A_3 &= a_1 (B + C) w^{4n-1} (w - 1)^2.
\end{aligned} \tag{3.27}$$

In (3.26), setting the coefficients of all independent exp-function terms to zero, we obtain

$$A_{-3} = 0, \qquad A_{-1} = 0, \qquad A_1 = 0, \qquad A_3 = 0. \tag{3.28}$$

Directly solving (3.28), we obtain $w = 1$ and $A,\ B,\ C,\ \beta,\ b_0,\ b_1$ are arbitrary constants. Thus, we obtain a class of exact solutions of (1.1) as follows:

$$u_n = An\, f(x) + b_0 + b_1 \left(\frac{e^{\beta t+\gamma_0} - Be^{-\beta t-\gamma_0}}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}} \right), \tag{3.29}$$

where $f(x)$ is an arbitrary function, for which let us set $f(x)$ to be $\tanh(x)$, $\operatorname{sech}(x)$, $\sin(x)$, $\cos(x)$, $\tanh(x)+\sin(x)$, $\operatorname{sech}(x)+\cos(x)$; respectively, we obtain a series of soliton solutions and breather solutions as follows:

$$u_n = An\ \tanh(x) + b_0 + b_1\left(\frac{e^{\beta t+\gamma_0} - Be^{-\beta t-\gamma_0}}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right), \tag{3.30}$$

$$u_n = An\operatorname{sech}(x) + b_0 + b_1\left(\frac{e^{\beta t+\gamma_0} - Be^{-\beta t-\gamma_0}}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right), \tag{3.31}$$

$$u_n = An\ \sin(x) + b_0 + b_1\left(\frac{e^{\beta t+\gamma_0} - Be^{-\beta t-\gamma_0}}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right), \tag{3.32}$$

$$u_n = An\ \cos(x) + b_0 + b_1\left(\frac{e^{\beta t+\gamma_0} - Be^{-\beta t-\gamma_0}}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right), \tag{3.33}$$

$$u_n = An[\tanh(x) + \sin(x)] + b_0 + b_1\left(\frac{e^{\beta t+\gamma_0} - Be^{-\beta t-\gamma_0}}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right), \tag{3.34}$$

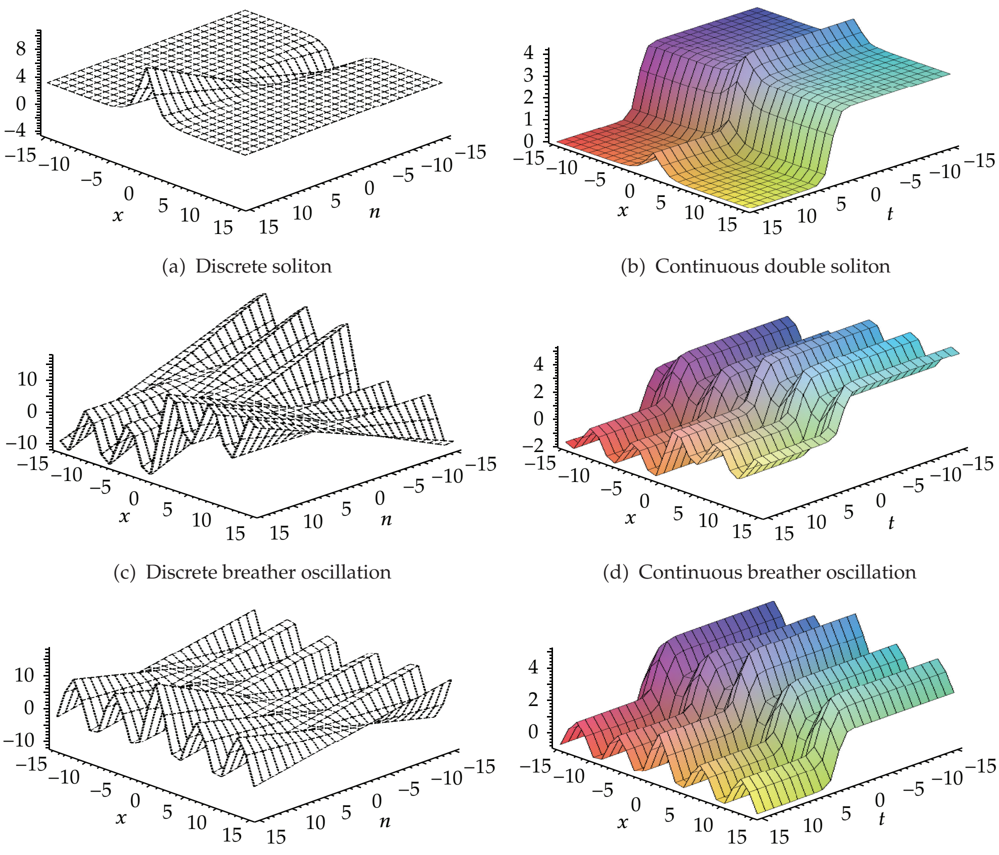
$$u_n = An[\operatorname{sech}(x) + \cos(x)] + b_0 + b_1\left(\frac{e^{\beta t+\gamma_0} - Be^{-\beta t-\gamma_0}}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right). \tag{3.35}$$

In order to describe the dynamic properties of the above solutions intuitively, as examples, we draw the profile figures of the solutions (3.31), (3.34), and (3.35) in Figure 3. Setting $A = 0.5$, $B = 3$, $C = 5$, $b_0 = 2$, $b_1 = -2$, $\beta = 1$, $\gamma_0 = -2$, $t = 1$, $n \in [-15, 15]$, $x \in [-15, 15]$, the solution (3.31) shows a shape of discrete soliton, see Figure 3(a); the solutions (3.34) and (3.35) show two shapes of discrete breather oscillations, see Figures 3(c) and 3(e). Setting $A = 0.5$, $B = 3$, $C = 5$, $b_0 = 2$, $b_1 = -2$, $\beta = 1$, $\gamma_0 = -2$, $n = 2$, $t \in [-15, 15]$, $x \in [-15, 15]$, the solution (3.31) shows a shape of continuous double soliton, see Figure 3(b); the solutions (3.34) and (3.35) show two shapes of continuous breather oscillations, see Figures 3(d) and 3(f). From Figure 3, the solutions (3.30)–(3.35) have diplex dynamic characters, that is, the discrete character and the continuous character coexist. If the parameter t is fixed, the profiles show discrete dynamic character, see Figures 3(a), 3(c), and 3(e); while the parameter n is fixed, the profiles show continuous dynamic character, see Figures 3(b), 3(d), and 3(f).

3.4. Exact Solutions of the Form (2.7)

Suppose that (1.1) has exact solution of the form (2.7). By using the balance procedure, we obtain $m = 2$. So, we suppose that (1.1) has exact solution as the following form:

$$u_n = An\,f(x) + b_0 + b_2\left(\frac{w^n}{w^{2n}e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right)^2, \tag{3.36}$$



(a) Discrete soliton

(b) Continuous double soliton

(c) Discrete breather oscillation

(d) Continuous breather oscillation

(e) Discrete breather oscillation

(f) Continuous breather oscillation

Figure 3: The solutions u_n in (3.31), (3.34), and (3.35) show six shapes of discrete and continuous solitons and discrete and continuous breather oscillations.

where A, C, b_0, b_2, w, β are constants to be determined later, γ_0 is an arbitrary constant, and $f(x)$ is an arbitrary function. As in Section 3.3, by using (3.36) and (1.1), we can obtain the exact solution of (1.1); here, we omit those processes and directly give the result as follows:

$$u_n = An\, f(x) + b_0 + b_2\left(\frac{1}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right)^2. \tag{3.37}$$

Similarly, setting $f(x)$ to be $\tanh^2(x)$, $\operatorname{sech}^2(x)$, $\sin^2(x)$, $\cos^2(x)$, $\tanh^2(x)+\sin^2(x)$, $\operatorname{sech}^2(x)+\cos^2(x)$, respectively, we obtain a series of soliton solutions and breather solutions as follows:

$$u_n = An\tanh^2(x) + b_0 + b_2\left(\frac{1}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right)^2, \tag{3.38}$$

$$u_n = An\operatorname{sech}^2(x) + b_0 + b_2\left(\frac{1}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right)^2, \tag{3.39}$$

$$u_n = An\sin^2(x) + b_0 + b_2\left(\frac{1}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right)^2, \quad (3.40)$$

$$u_n = An\cos^2(x) + b_0 + b_2\left(\frac{1}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right)^2, \quad (3.41)$$

$$u_n = An\left[\tanh^2(x) + \sin^2(x)\right] + b_0 + b_2\left(\frac{1}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right)^2, \quad (3.42)$$

$$u_n = An\left[\operatorname{sech}^2(x) + \cos^2(x)\right] + b_0 + b_2\left(\frac{1}{e^{\beta t+\gamma_0} + Ce^{-\beta t-\gamma_0}}\right)^2. \quad (3.43)$$

The dynamic properties of solutions (3.38)–(3.43) are similar to those of solutions (3.30)–(3.35). As examples, we plot six profile figures of the solutions (3.39), (3.42), and (3.43), see Figure 4. Setting $A = 0.2,\ C = 3,\ b_0 = 4,\ b_2 = 200,\ \beta = 0.8,\ \gamma_0 = -2,\ t = 1,\ n \in [-20, 20],\ x \in [-20, 20]$, the solution (3.39) shows a shape of discrete soliton, see Figure 4(a); the solutions (3.42) and (3.43) show two shapes of discrete breather oscillations, see Figures 4(c) and 4(e). Setting $A = 0.2,\ C = 3,\ b_0 = 4,\ b_2 = 2,\ \beta = 0.8,\ \gamma_0 = -2,\ n = 1,\ t \in [-5, 8],\ x \in [-6, 6]$, (3.39) shows a shape of continuous double soliton, see Figure 4(b); the solutions (3.42) and (3.43) show two shapes of continuous breather oscillations, see Figures 4(d) and 4(f).

3.5. Exact Solutions of the Form (2.8)

Suppose that (1.1) has exact solution of the form (2.8). By using the balance procedure, we obtain $m = 2$. So, we suppose that (1.1) has exact solution as the following form:

$$u_n = An\,f(x) + \frac{c_0}{1+B} + \frac{c_2 w^{2n}}{e^{2(\beta t+\gamma_0)} + Bw^{2n}}, \quad (3.44)$$

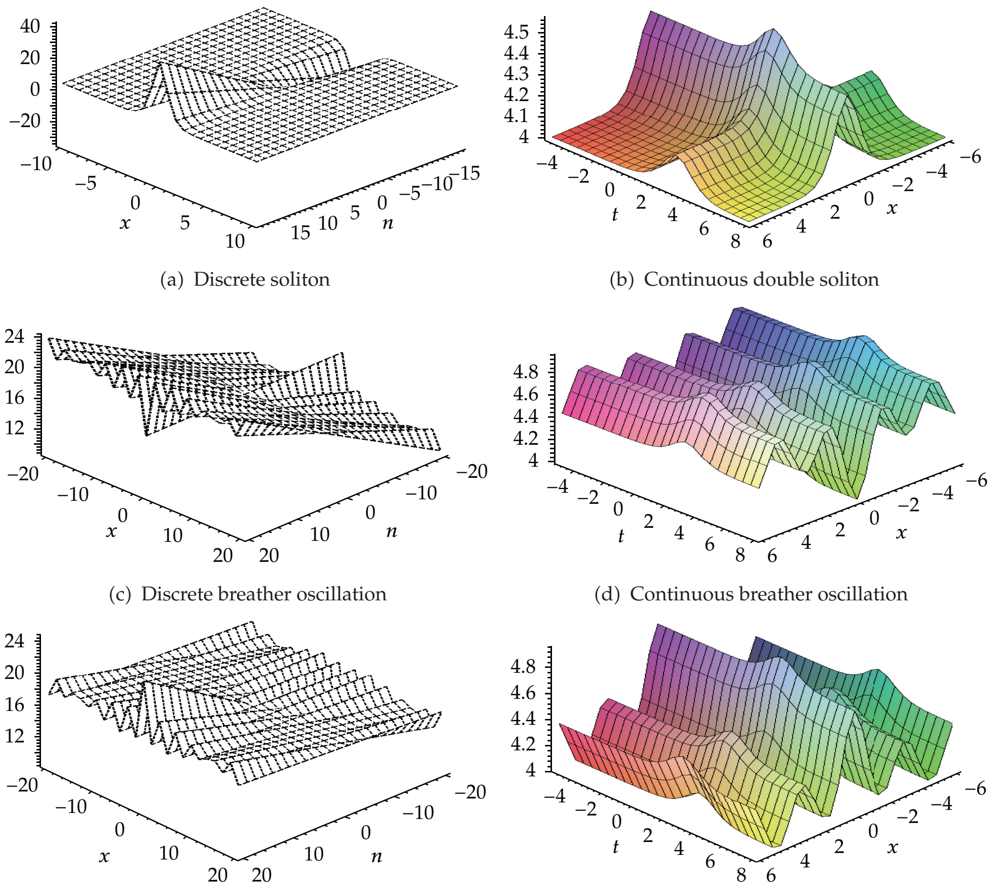
where A, B,c_0,c_2, w,β are constants to be determined later, γ_0 is an arbitrary constant, and $f(x)$ is an arbitrary function. As in Section 3.3, by using (3.44) and (1.1) and the expatiatory computation, we obtain $w = \pm 1$. So we obtain a family of exact solutions with arbitrary function as follows:

$$u_n = An\,f(x) + \frac{c_0}{1+B} + \frac{c_2}{e^{2(\beta t+\gamma_0)} + B}. \quad (3.45)$$

Similarly, assume $f(x)$ the same functions as in Section 3.3, we obtain a series of soliton solutions and breather solutions as follows:

$$u_n = An\,\tanh(x) + \frac{c_0}{1+B} + \frac{c_2}{e^{2(\beta t+\gamma_0)} + B}, \quad (3.46)$$

$$u_n = An\operatorname{sech}(x) + \frac{c_0}{1+B} + \frac{c_2}{e^{2(\beta t+\gamma_0)} + B}, \quad (3.47)$$



(a) Discrete soliton (b) Continuous double soliton

(c) Discrete breather oscillation (d) Continuous breather oscillation

(e) Discrete breather oscillation (f) Continuous breather oscillation

Figure 4: The solutions u_n in (3.39), (3.42), and (3.43) show six shapes of discrete and continuous solitons and discrete and continuous breather oscillations.

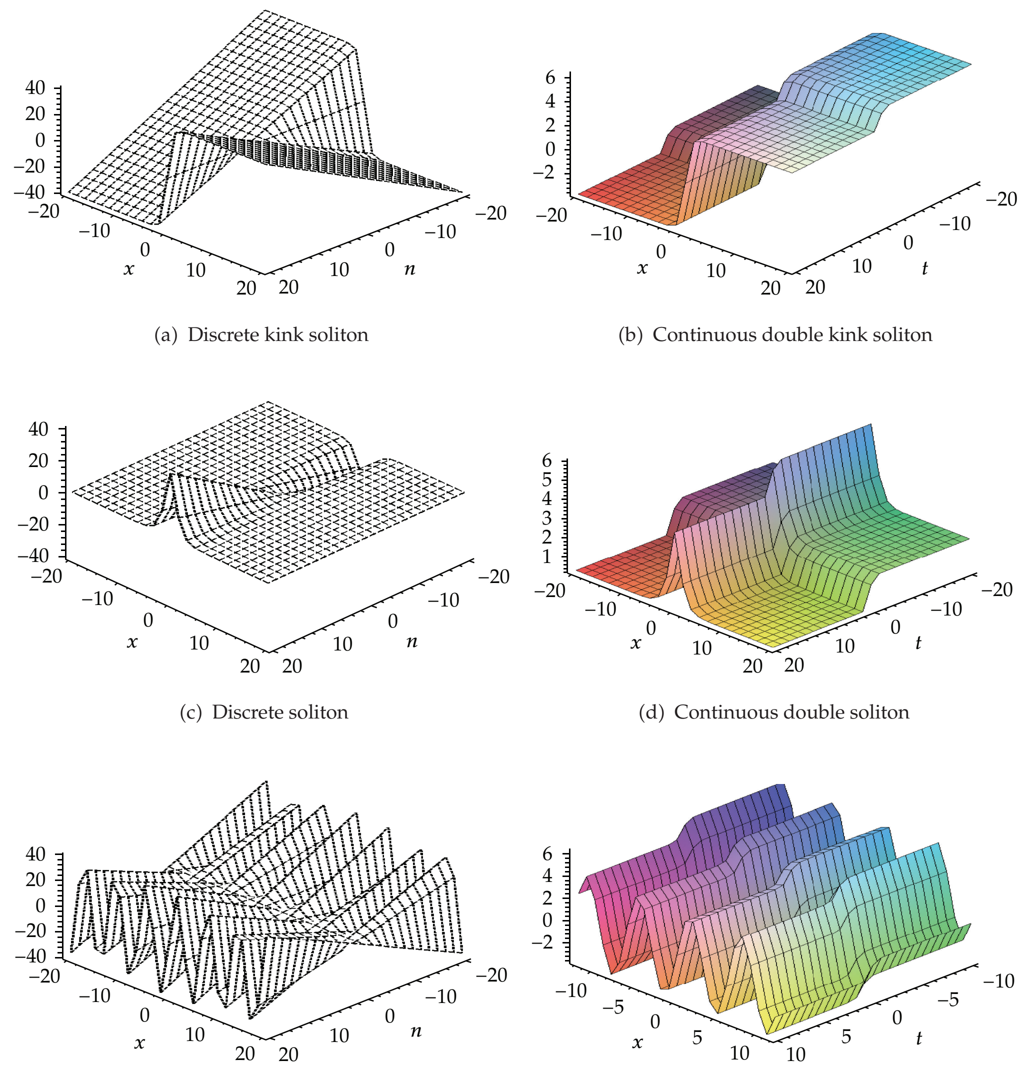
$$u_n = An\sin(x) + \frac{c_0}{1+B} + \frac{c_2}{e^{2(\beta t+\gamma_0)} + B}, \tag{3.48}$$

$$u_n = An\cos(x) + \frac{c_0}{1+B} + \frac{c_2}{e^{2(\beta t+\gamma_0)} + B}, \tag{3.49}$$

$$u_n = An[\tanh(x) + \sin(x)] + \frac{c_0}{1+B} + \frac{c_2}{e^{2(\beta t+\gamma_0)} + B}, \tag{3.50}$$

$$u_n = An[\operatorname{sech}(x) + \cos(x)] + \frac{c_0}{1+B} + \frac{c_2}{e^{2(\beta t+\gamma_0)} + B}. \tag{3.51}$$

The dynamic properties of solutions (3.46)–(3.51) are similar to those of solutions (3.30)–(3.35). As examples, we plot six profile figures of the solutions (3.46), (3.47), and (3.48), see Figure 5. Setting $A = 0.2$, $C = 3$, $b_0 = 4$, $b_2 = 200$, $\beta = 0.8$, $\gamma_0 = -2$, $t = 1$, $n \in [-20, 20]$, $x \in [-20, 20]$, the solution (3.46) shows a shape of discrete kink soliton, see Figure 5(a); the solution (3.47) shows a shape of discrete soliton, see Figure 5(c); the solution (3.48) shows a shape of discrete breather oscillations, see Figure 5(e). Setting $A = 0.2$,



(a) Discrete kink soliton

(b) Continuous double kink soliton

(c) Discrete soliton

(d) Continuous double soliton

(e) Discrete breather oscillation

(f) Continuous breather oscillation

Figure 5: The solutions u_n in (3.46), (3.47) and (3.48) show six shapes of discrete and continuous solitons, and discrete and continuous breather oscillations.

$C = 3$, $b_0 = 4$, $b_2 = 2$, $\beta = 0.8$, $\gamma_0 = -2$, $n = 1$, $t \in [-5, 8]$, $x \in [-6, 6]$, the solution (3.46) shows a shape of continuous double kink soliton, see Figure 5(b); the solution (3.47) shows a shape of continuous double soliton, see Figure 5(c); the solution (3.48) shows a shape of continuous breather oscillations, see Figure 5(e).

4. New Exact Solutions of Toda Lattice Equation and Its Dynamic Properties

In this section, using the extended method offered in Section 2, we discuss the exact solutions of the Toda lattice equation (1.2).

4.1. Exact Solutions of the Form (2.9)

Suppose that (1.2) has exact solution of the form (2.9) as follows:

$$u_n = a_0 + a_1\left(\frac{w^n e^{\Omega t} - Be^{-\Omega t}}{w^n e^{\Omega t} + Ce^{-\Omega t}}\right). \tag{4.1}$$

After substituting (4.1) in (1.2), multiplying both sides by the common denominator $(w^n e^{\Omega t} + Ce^{-\Omega t})^3(w^{n-1}e^{\Omega t} + Ce^{-\Omega t})(w^{n+1}e^{\Omega t} + Ce^{-\Omega t})$, it follows:

$$A_{-3}e^{-3\Omega t} + A_{-1}e^{-\Omega t} + A_1 e^{\Omega t} + A_3 e^{3\Omega t} = 0, \tag{4.2}$$

where

$$\begin{aligned}
A_{-3} &= -a_1 C^3(B+C)w^{n-1}\left[w^2 - (4\Omega+2)w + 1\right],\\
A_{-1} &= -a_1 C(B+C)w^{2n-1}\left[\left(2a_1 B\Omega + 2a_1 C\Omega + C - 4C\Omega^2\right)w^2\right.\\
&\quad + \left(4C\Omega^2 - 4a_1 B\Omega - 4a_1 C\Omega - 2C\right)w\\
&\quad \left.+2a_1 B\Omega + 2a_1 C\Omega + C - 4C\Omega^2\right],\\
A_1 &= a_1(B+C)w^{3n-1}\left[\left(2a_1 B\Omega + 2a_1 C\Omega + C - 4C\Omega^2\right)w^2\right.\\
&\quad + \left(4C\Omega^2 - 4a_1 B\Omega - 4a_1 C\Omega - 2C\right)w\\
&\quad \left.+2a_1 B\Omega + 2a_1 C\Omega + C - 4C\Omega^2\right],\\
A_3 &= a_1(B+C)w^{4n-1}\left[w^2 - (4\Omega+2)w + 1\right].
\end{aligned} \tag{4.3}$$

In (4.2), setting the coefficients of all independent exp-function terms to zero, we obtain

$$A_{-3} = 0, \qquad A_{-1} = 0, \qquad A_1 = 0, \qquad A_3 = 0. \tag{4.4}$$

Let $M_{-3} = A_{-3}/[-a_1 C^3(B+C)w^{n-1}]$, $M_{-1} = A_{-1}/[-a_1 C(B+C)w^{2n-1}]$, $M_1 = A_1/[a_1(B+C)w^{3n-1}]$, $M_3 = A_3/[a_1(B+C)w^{4n-1}]$, then

$$\begin{aligned}
M_3 = M_{-3} &= w^2 - (4\Omega+2)w + 1,\\
M_1 = M_{-1} &= \left(2a_1 B\Omega + 2a_1 C\Omega + C - 4C\Omega^2\right)w^2\\
&\quad +\left(4C\Omega^2 - 4a_1 B\Omega - 4a_1 C\Omega - 2C\right)w + 2a_1 B\Omega + 2a_1 C\Omega + C - 4C\Omega^2\Big].
\end{aligned} \tag{4.5}$$

Thus, (4.4) can be reduced to

$$M_1 = 0, \qquad M_3 = 0. \tag{4.6}$$

Solving (4.6) yields

$$B = \frac{C(2\Omega - a_1)}{a_1}, \qquad w_{1,2} = 2\Omega^2 + 1 \pm 2\Omega\sqrt{\Omega^2 + 1} \tag{4.7}$$

and $C,\ a_0,\ a_1,\ \Omega$ are arbitrary constants. From (4.1), (4.7), we obtain discrete kink or antikink soliton solution of (1.2) as follows:

$$u_n = a_0 + a_1 \left[\frac{\left(2\Omega^2 + 1 + 2\Omega\sqrt{\Omega^2+1}\right)^n e^{\Omega t} - (C(2\Omega - a_1)/a_1)e^{-\Omega t}}{w^n e^{\Omega t} + Ce^{-\Omega t}} \right],$$
$$u_n = a_0 + a_1 \left[\frac{\left(2\Omega^2 + 1 - 2\Omega\sqrt{\Omega^2+1}\right)^n e^{\Omega t} - (C(2\Omega - a_1)/a_1)e^{-\Omega t}}{w^n e^{\Omega t} + Ce^{-\Omega t}} \right]. \tag{4.8}$$

Similarly, by using the same method, we can obtain exact solutions as the form of (2.10); here, we omit them.

4.2. Exact Solutions of the Form (2.11)

Suppose that (1.2) has exact solution of the form (2.11) as follows:

$$u_n = \frac{a_0 + a_2 w^{2n} e^{2\Omega t} + a_4 w^{4n} e^{4\Omega t}}{b_0 + b_2 w^{2n} e^{2\Omega t} + b_4 w^{4n} e^{4\Omega t}}, \tag{4.9}$$

where a_0, a_2,a_4, b_0, b_2, b_4 are constants to be determined later and $a_0/b_0 \neq a_2/b_2 \neq a_4/b_4$. After substituting (4.9) in (1.2), multiplying both sides by the common denominator $(b_0 + b_2 w^{2n} e^{2\Omega t} + b_4 w^{4n} e^{4\Omega t})^3 (b_0 + b_2 w^{2(n-1)} e^{2\Omega t} + b_4 w^{4(n-1)} e^{4\Omega t})(b_0 + b_2 w^{2(n+1)} e^{2\Omega t} + b_4 w^{4(n+1)} e^{4\Omega t})$, it follows:

$$A_{18}e^{18\Omega t} + A_{16}e^{16\Omega t} + A_{14}e^{14\Omega t} + A_{12}e^{12\Omega t} + A_{10}e^{10\Omega t} + A_8 e^{8\Omega t} + A_6 e^{6\Omega t} + A_4 e^{4\Omega t} + A_2 e^{2\Omega t} = 0, \tag{4.10}$$

where

$$A_{18} = -b_4^3(a_2b_4 - a_4b_2)w^{18n-2}[w^4 - (4\Omega^2+2)w^2+1],$$

$$A_{16} = b_4 w^{16n-4}[(a_4b_0b_4^2 - a_0b_4^3)w^8 + (a_2b_4 - a_4b_2)(2\Omega b_4a_2 - b_2b_4 + 4b_4\Omega^2 b_2 - 2\Omega_4 b_2)w^6 + (-2a_4b_0b_4^2 + 16\Omega^2a_0b_4^3 - 4\Omega a_2^2b_4^2 - 2a_4b_2^2b_4 - 4\Omega a_4^2b_2^2 + 8\Omega a_2b_4a_4b_2 + 2a_2b_4^2b_2 + 2a_0b_4^3 - 16\Omega^2a_4b_0b_4^2 - 4\Omega^2a_2b_4^2b_2 + 4\Omega^2a_4b_2^2b_4)w^4 + (a_2b_4 - a_4b_2)(2\Omega b_4a_2 - b_2b_4 + 4b_4\Omega^2b_2 - 2\Omega_4b_2)w^2 + a_4b_0b_4^2 - a_0b_4^3],$$

$$A_{14} = w^{14n-4}[(-3a_0b_2b_4^3 + 2a_2b_0b_4^3 + 4\Omega^2a_2b_0b_4^3 + 2\Omega a_2b_4^3a_0 - 2\Omega a_4b_2a_0b_4^2 + a_4b_0b_2b_4^2 + 2\Omega a_4^2b_0b_2b_4 - 2\Omega a_2b_4^2a_4b_0 - 4\Omega^2a_4b_0b_2b_4^2)w^8 + (4\Omega a_2b_4^3a_0 + a_0b_2b_4^3 + 4\Omega^2a_4b_2^3b_4 - a_4b_2^3b_4 - 2\Omega a_4^2b_2^3 - 2\Omega a_2^2b_4^2b_2 + a_2b_2^2b_4^2 - 4a_2b_0b_4^3 + 4\Omega a_2b_4a_4b_2^2 - 16\Omega^2a_4b_0b_2b_4^2 + 16\Omega^2a_0b_2b_4^3 + 4\Omega a_4^2b_0b_2b_4 - 4\Omega^2a_2b_4^2b_2^2 + 3a_4b_0b_2b_4^2 - 4\Omega a_4b_2a_0b_4^2 - 4\Omega a_2b_4^2a_4b_0)w^6 + (4a_0b_2b_4^3 + 4a_2b_0b_4^3 - 12\Omega a_2b_4^3a_0 - 12\Omega a_4^2b_0b_2b_4 + 12\Omega a_4b_2a_0b_4^2 - 2a_2b_2^2b_4^2 + 4\Omega a_4^2b_2^3 - 8a_4b_0b_2b_4^2 + 2a_4b_2^3b_4 - 24\Omega^2a_2b_0b_4^3 + 12\Omega a_2b_4^2a_4b_0 + 4\Omega^2a_2b_4^2b_2^2 + 12\Omega^2a_4b_0b_2b_4^2 + 12\Omega^2a_0b_2b_4^3 + 4\Omega a_2^2b_4^2b_2 - 4\Omega^2a_4b_2^3b_4 - 8\Omega a_2b_4a_4b_2^2)w^4 + (4\Omega a_2b_4^3a_0 + a_0b_2b_4^3 + 4\Omega^2a_4b_2^3b_4 - a_4b_2^3b_4 - 2\Omega a_4^2b_2^3 - 2\Omega a_2^2b_4^2b_2 + a_2b_2^2b_4^2 - 4a_2b_0b_4^3 + 4\Omega a_2b_4a_4b_2^2 - 16\Omega^2a_4b_0b_2b_4^2 + 16\Omega^2a_0b_2b_4^3 + 4\Omega a_4^2b_0b_2b_4 - 4\Omega^2a_2b_4^2b_2^2 + 3a_4b_0b_2b_4^2 - 4\Omega a_4b_2a_0b_4^2 - 4\Omega a_2b_4^2a_4b_0)w^2 - 3a_0b_2b_4^3 + 2a_2b_0b_4^3 + 4\Omega^2a_2b_0b_4^3 + 2\Omega a_2b_4^3a_0 - 2\Omega a_4b_2a_0b_4^2 + a_4b_0b_2b_4^2 + 2\Omega a_4^2b_0b_2b_4 - 2\Omega a_2b_4^2a_4b_0 - 4\Omega^2a_4b_0b_2b_4^2],$$

$$A_{12} = w^{12n-4}[(-3a_0b_2^2b_4^2 + a_4b_4^2b_0^2 - a_0b_4^3b_0 + 4\Omega a_0^2b_4^3 + 4a_2b_0b_2b_4^2 - a_4b_0b_4b_2^2 - 16\Omega^2a_4b_0^2b_4^2 + 16\Omega^2a_0b_4^3b_0 - 4\Omega a_2^2b_0b_4^2 - 2\Omega a_4^2b_0b_2^2 + 4\Omega a_4^2b_0^2b_4 + 2\Omega a_0b_2a_2b_4^2 + 6\Omega a_2b_0a_4b_2b_4 - 8\Omega a_4b_0a_0b_4^2 - 2\Omega a_0b_2^2a_4b_4 + 4\Omega^2a_4b_0b_4b_2^2 - 4\Omega^2a_2b_0b_2b_4^2)w^8 + (-4\Omega a_0b_2a_2b_4^2 + 12\Omega^2a_0b_2^2b_4^2 + 8\Omega^2a_4b_0b_4b_2^2 + a_0b_2^2b_4^2 + 2a_4b_0b_4b_2^2 + 4\Omega a_0b_2^2a_4b_4 - 3a_2b_0b_2b_4^2 - 20\Omega^2a_2b_0b_2b_4^2 + 2\Omega a_2^2b_0b_4^2 - 2\Omega a_4^2b_0b_2^2 + a_2b_2^3b_4 - a_4b_2^4)w^6 + (4a_0b_2^2b_4^2 + 2a_0b_4^3b_0 - 4\Omega^2a_2b_2^3b_4 + 2a_4b_2^4 + 20\Omega^2a_0b_2^2b_4^2 + 4\Omega^2a_4b_2^4 - 2a_4b_0b_4b_2^2 - 16\Omega^2a_0b_4^3b_0 + 4\Omega a_2^2b_0b_4^2 - 4\Omega^2a_2b_0b_2b_4^2 - 2a_2b_2^3b_4 - 2a_2b_0b_2b_4^2 + 8\Omega a_4^2b_0b_2^2 + 16\Omega^2a_4b_0^2b_4^2 - 2a_4b_4^2b_0^2 - 4\Omega a_0b_2^2a_4b_4 - 8\Omega a_4^2b_0^2b_4 - 16\Omega^2a_4b_0b_4b_2^2 - 12\Omega a_2b_0a_4b_2b_4 + 16\Omega a_4b_0a_0b_4^2 + 4\Omega a_0b_2a_2b_4^2 - 8\Omega a_0^2b_4^3)w^4 - (4\Omega a_0b_2a_2b_4^2 - 12\Omega^2a_0b_2^2b_4^2 - 8\Omega^2a_4b_0b_4b_2^2 - a_0b_2^2b_4^2 - 2a_4b_0b_4b_2^2 - 4\Omega a_0b_2^2a_4b_4 + 3a_2b_0b_2b_4^2 + 20\Omega^2a_2b_0b_2b_4^2 - 2\Omega a_2^2b_0b_4^2 + 2\Omega a_4^2b_0b_2^2 - a_2b_2^3b_4 + a_4b_2^4)w^2 - 3a_0b_2^2b_4^2 + a_4b_4^2b_0^2 - a_0b_4^3b_0 + 4\Omega a_0^2b_4^3 + 4a_2b_0b_2b_4^2 - a_4b_0b_4b_2^2 - 16\Omega^2a_4b_0^2b_4^2 + 16\Omega^2a_0b_4^3b_0 - 4\Omega a_2^2b_0b_4^2 - 2\Omega a_4^2b_0b_2^2 + 4\Omega a_4^2b_0^2b_4 + 2\Omega a_0b_2a_2b_4^2 + 6\Omega a_2b_0a_4b_2b_4 - 8\Omega a_4b_0a_0b_4^2 - 2\Omega a_0b_2^2a_4b_4 + 4\Omega^2a_4b_0b_4b_2^2 - 4\Omega^2a_2b_0b_2b_4^2],$$

$$A_{10} = w^{10n-4}[(-a_4b_0b_2^3 - 2a_4b_2b_4b_0^2 - a_0b_2^3b_4 + 4a_2b_4^2b_0^2 + 12\Omega^2a_0b_2b_0b_4^2 - 2a_0b_2b_0b_4^2 - 24\Omega^2a_2b_4^2b_0^2 - 6\Omega a_4^2b_2b_0^2 + 2a_2b_0b_4b_2^2 + 12\Omega^2a_4b_2b_4b_0^2 + 6\Omega a_0^2b_2b_4^2 + 12\Omega a_2b_0^2a_4b_4 - 12\Omega a_2b_0a_0b_4^2)w^8 - (4\Omega a_0^2b_2b_4^2 - 2a_2b_0b_4b_2^2 + 6a_2b_4^2b_0^2 - 4\Omega^2a_0b_2^3b_4 + a_0b_2^3b_4 + 8\Omega^2a_2b_0b_4b_2^2 + a_4b_0b_2^3 - 3a_4b_2b_4b_0^2 - 4\Omega^2a_4b_0b_2^3 - 3a_0b_2b_0b_4^2 - 8\Omega a_2b_0a_0b_4^2 - 4\Omega a_4^2b_2b_0^2 + 8\Omega a_2b_0^2a_4b_4)w^6 + (4a_2b_4^2b_0^2 - 8a_2b_0b_4b_2^2 + 4a_0b_2^3b_4 + 4a_4b_0b_2^3 - 4\Omega^2a_4b_2b_4b_0^2 - 4\Omega^2a_0b_2b_0b_4^2 + 8\Omega^2a_2b_4^2b_0^2 - 2a_4b_2b_4b_0^2 - 8\Omega a_2b_0^2a_4b_4 + 12\Omega^2a_4b_0b_2^3 - 2a_0b_2b_0b_4^2 + 8\Omega a_2b_0a_0b_4^2 + 4\Omega a_4^2b_2b_0^2 - 4\Omega a_0^2b_2b_4^2 - 24\Omega^2a_2b_0b_4b_2^2 + 12\Omega^2a_0b_2^3b_4)w^4 - (4\Omega a_0^2b_2b_4^2 - 2a_2b_0b_4b_2^2 + 6a_2b_4^2b_0^2 - 4\Omega^2a_0b_2^3b_4 + a_0b_2^3b_4 + 8\Omega^2a_2b_0b_4b_2^2 + a_4b_0b_2^3 - 3a_4b_2b_4b_0^2 - 4\Omega^2a_4b_0b_2^3 - 3a_0b_2b_0b_4^2 - 8\Omega a_2b_0a_0b_4^2 - 4\Omega a_4^2b_2b_0^2 + 8\Omega a_2b_0^2a_4b_4)w^2 - a_4b_0b_2^3 - 2a_4b_2b_4b_0^2 - a_0b_2^3b_4 + 4a_2b_4^2b_0^2 + 12\Omega^2a_0b_2b_0b_4^2 - 2a_0b_2b_0b_4^2 - 24\Omega^2a_2b_4^2b_0^2 - 6\Omega a_4^2b_2b_0^2 + 2a_2b_0b_4b_2^2 + 12\Omega^2a_4b_2b_4b_0^2 + 6\Omega a_0^2b_2b_4^2 + 12\Omega a_2b_0^2a_4b_4 - 12\Omega a_2b_0a_0b_4^2],$$

$$A_8 = w^{8n-4}[-(4\Omega^2a_2b_2b_4b_0^2 + 4\Omega a_4^2b_0^3 + 3a_4b_2^2b_0^2 - b_0^2a_0b_4^2 + b_0^3a_4b_4 - 2\Omega a_4b_0a_0b_2^2 + 2\Omega a_4b_0^2a_2b_2 - 4a_2b_2b_4b_0^2 - 4\Omega a_2^2b_4b_0^2 + a_0b_4b_0b_2^2 - 2\Omega a_0^2b_4b_2^2 + 4\Omega a_0^2b_4^2b_0 - 16\Omega^2a_4b_0^3b_4 - 4\Omega^2a_0b_4b_0b_2^2 + 16\Omega^2a_0b_4^2b_0^2 - 8\Omega a_4b_0^2a_0b_4 + 6\Omega a_2b_4a_0b_2b_0)w^8 + (a_2b_0b_2^3 - a_0b_2^4 + 2\Omega a_0^2b_4b_2^2 + 2a_0b_4b_0b_2^2 - 2\Omega a_2^2b_4b_0^2 + a_4b_2^2b_0^2 + 8\Omega^2a_0b_4b_0b_2^2 - 20\Omega^2a_2b_2b_4b_0^2 + 12\Omega^2a_4b_2^2b_0^2 - 3a_2b_2b_4b_0^2 + 4\Omega a_4b_0^2a_2b_2 - 4\Omega a_4b_0a_0b_2^2)w^6 - (8\Omega a_0^2b_4b_2^2 + 2b_0^2a_0b_4^2 + 4\Omega a_2^2b_4b_0^2 - 4a_4b_2^2b_0^2 - 16\Omega^2a_0b_4^2b_0^2 - 2b_0^3a_4b_4 + 4\Omega^2a_2b_2b_4b_0^2 + 16\Omega^2a_0b_4b_0b_2^2 + 2a_2b_2b_4b_0^2 - 8\Omega a_4^2b_0^3 + 4\Omega a_4b_0^2a_2b_2 + 2a_0b_4b_0b_2^2 + 16\Omega a_4b_0^2a_0b_4 - 8\Omega a_0^2b_4^2b_0 - 4\Omega a_4b_0a_0b_2^2 + 16\Omega^2a_4b_0^3b_4 + 2a_2b_0b_2^3 - 12\Omega a_2b_4a_0b_2b_0 - 4\Omega^2a_0b_2^4 - 2a_0b_2^4 + 4\Omega^2a_2b_0b_2^3 - 20\Omega^2a_4b_2^2b_0^2)w^4 + (a_2b_0b_2^3 - a_0b_2^4 + 2\Omega a_0^2b_4b_2^2 + 2a_0b_4b_0b_2^2 - 2\Omega a_2^2b_4b_0^2 + a_4b_2^2b_0^2 + 8\Omega^2a_0b_4b_0b_2^2 - 20\Omega^2a_2b_2b_4b_0^2 + 12\Omega^2a_4b_2^2b_0^2 - 3a_2b_2b_4b_0^2 + 4\Omega a_4b_0^2a_2b_2 - 4\Omega a_4b_0a_0b_2^2)w^2 - 4\Omega^2a_2b_2b_4b_0^2 - 4\Omega a_4^2b_0^3 - 3a_4b_2^2b_0^2 + b_0^2a_0b_4^2 - b_0^3a_4b_4 + 2\Omega a_4b_0a_0b_2^2 - 2\Omega a_4b_0^2a_2b_2 + 4a_2b_2b_4b_0^2 + 4\Omega a_2^2b_4b_0^2 - a_0b_4b_0b_2^2 + 2\Omega a_0^2b_4b_2^2 - 4\Omega a_0^2b_4^2b_0 + 16\Omega^2a_4b_0^3b_4 + 4\Omega^2a_0b_4b_0b_2^2 - 16\Omega^2a_0b_4^2b_0^2 + 8\Omega a_4b_0^2a_0b_4 - 6\Omega a_2b_4a_0b_2b_0],$$

$$A_6 = w^{6n-4}[-(2\Omega a_2b_0^3a_4 - 4\Omega^2a_2b_0^3b_4 - a_0b_2b_4b_0^2 - 2\Omega a_2b_0^2a_0b_4 + 2\Omega a_0^2b_2b_4b_0 - 2\Omega a_4b_0^2a_0b_2 - 2a_2b_0^3b_4 + 3a_4b_0^3b_2 + 4\Omega^2a_0b_2b_4b_0^2)w^8 + (2\Omega a_2^2b_0^2b_2 - 4\Omega^2a_2b_0^2b_2^2 - a_0b_2^3b_0 + 16\Omega^2a_4b_0^3b_2 + 4\Omega^2a_0b_2^3b_0 +$$

$3a_0b_2b_4b_0^2 - 4a_2b_0^3b_4 + a_2b_2^2b_0^2 + a_4b_0^3b_2 + 2\Omega a_0^2b_2^3 - 16\Omega^2a_0b_2b_4b_0^2 - 4\Omega a_2b_0a_0b_2^2 + 4\Omega a_4b_0^2a_0b_2 + 4\Omega a_2b_0^2a_0b_4 - 4\Omega a_0^2b_2b_4b_0 - 4\Omega a_2b_0^3a_4)w^6 - (24\Omega^2a_2b_0^3b_4 - 12\Omega a_2b_0^3a_4 + 8a_0b_2b_4b_0^2 + 12\Omega a_4b_0^2a_0b_2 - 12\Omega a_0^2b_2b_4b_0 - 4a_2b_0^3b_4 - 12\Omega^2a_0b_2b_4b_0^2 - 4a_4b_0^3b_2 + 12\Omega a_2b_0^2a_0b_4 + 2a_2b_2^2b_0^2 - 2a_0b_2^3b_0 + 4\Omega a_0^2b_2^3 + 4\Omega^2a_0b_2^3b_0 + 4\Omega a_2^2b_0^2b_2 - 12\Omega^2a_4b_0^3b_2 - 4\Omega^2a_2b_0^2b_2^2 - 8\Omega a_2b_0a_0b_2^2)w^4 + (2\Omega a_2^2b_0^2b_2 - 4\Omega^2a_2b_0^2b_2^2 - a_0b_2^3b_0 + 16\Omega^2a_4b_0^3b_2 + 4\Omega^2a_0b_2^3b_0 + 3a_0b_2b_4b_0^2 - 4a_2b_0^3b_4 + a_2b_2^2b_0^2 + a_4b_0^3b_2 + 2\Omega a_0^2b_2^3 - 16\Omega^2a_0b_2b_4b_0^2 - 4\Omega a_2b_0a_0b_2^2 + 4\Omega a_4b_0^2a_0b_2 + 4\Omega a_2b_0^2a_0b_4 - 4\Omega a_0^2b_2b_4b_0 - 4\Omega a_2b_0^3a_4)w^2 - 2\Omega a_2b_0^3a_4 + 4\Omega^2a_2b_0^3b_4 + a_0b_2b_4b_0^2 + 2\Omega a_2b_0^2a_0b_4 - 2\Omega a_0^2b_2b_4b_0 + 2\Omega a_4b_0^2a_0b_2 + 2a_2b_0^3b_4 - 3a_4b_0^3b_2 - 4\Omega^2a_0b_2b_4b_0^2]$,

$A_4 = -b_0w^{4n-4}[(a_4b_0^3 - a_0b_4b_0^2)w^8 + (2\Omega a_2^2b_0^2 + a_2b_0^2b_2 - 4\Omega^2a_2b_0^2b_2 + 2\Omega a_0^2b_2^2 - 4\Omega a_2b_0a_0b_2 + 4\Omega^2a_0b_2^2b_0 - a_0b_2^2b_0)w^6 - (4\Omega^2a_0b_2^2b_0 - 2a_0b_4b_0^2 + 2a_2b_0^2b_2 - 4\Omega^2a_2b_0^2b_2 + 16\Omega^2a_4b_0^3 + 4\Omega a_2^2b_0^2 + 4\Omega a_0^2b_2^2 - 8\Omega a_2b_0a_0b_2 + 2a_4b_0^3 - 2a_0b_2^2b_0 - 16\Omega^2a_0b_4b_0^2)w^4 + (2\Omega a_2^2b_0^2a_2b_0^2b_2 - 4\Omega^2a_2b_0^2b_2 + 2\Omega a_0^2b_2^2 - 4\Omega a_2b_0a_0b_2 + 4\Omega^2a_0b_2^2b_0 - a_0b_2^2b_0)w^2 + a_4b_0^3 - a_0b_4b_0^2]$,

$A_2 = b_0^3(a_0b_2 - a_2b_0)w^{2n-2}[w^4 - (4\Omega^2 + 2)w^2 + 1]$.

In (4.10), we let

$$A_{18} = 0, \quad A_{16} = 0, \quad A_{14} = 0, \quad A_{12} = 0, \quad A_{10} = 0,$$
$$A_8 = 0, \quad A_6 = 0, \quad A_4 = 0, \quad A_2 = 0. \tag{4.11}$$

Solving the group of (4.11), we obtain three group of parametric conditions which satisfy (4.10).

Case 1. We have the following:

$$a_0 = \frac{a_2b_0}{b_2} \pm \frac{w^2 - 1}{w}, \qquad a_4 = b_4 = 0, \qquad \Omega = \pm\frac{w^2 - 1}{2w}, \tag{4.12}$$

where a_2, b_0, b_2, w are arbitrary nonzero constants with $a_2 \neq b_0 \neq b_2,\ w \neq \pm 1$.

Case 2. We have the following:

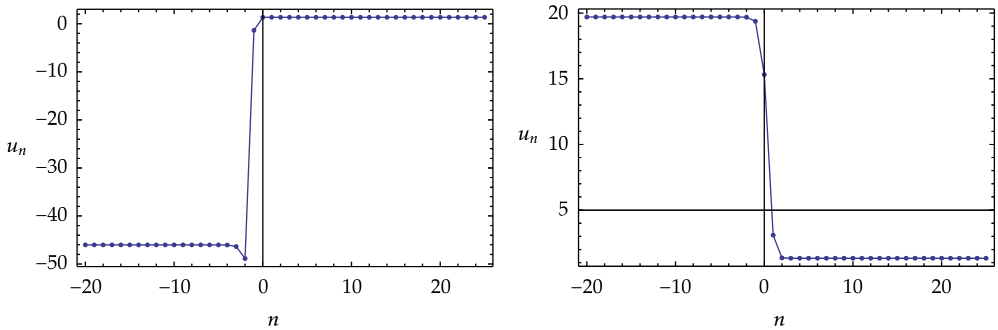
$$a_0 = -\frac{p_4w^4 \pm p_3w^3 + p_2w^2 \pm p_1w + p_0}{b_4^2(w^3 - w^2 - w + 1)(w + 1)}, \qquad b_0 = \frac{\pm q_2w^2 + q_1w \pm q_0}{b_4^3(w - 1)^2(w + 1)^2},$$
$$\Omega = \pm\frac{w^2 - 1}{2w}, \tag{4.13}$$

where $p_4 = -a_2b_4^3b_2 + a_4b_2^2b_4^2$, $p_3 = 2a_4^2b_2^2b_4 - 3a_2b_4^2a_4b_2 + a_2^2b_4^3$, $p_2 = -2a_4^2b_4b_2a_2 - 2a_4b_2^2b_4^2 + 2a_2b_4^3b_2 + a_4b_4^2a_2^2 + a_4^3b_2^2$, $p_1 = 3a_2b_4^2a_4b_2 - 2a_4^2b_2^2b_4 - a_2^2b_4^3$, $p_0 = -a_2b_4^3b_2 + a_4b_2^2b_4^2$, $q_2 = a_4b_2^2b_4 - a_2b_4^2b_2$, $q_1 = -2b_4b_2a_4a_2 + b_4^2a_2^2 + b_2^2a_4^2$, $q_0 = a_2b_4^2b_2 - a_4b_2^2b_4$ and a_2, a_4, b_2, b_4, w are arbitrary nonzero constants with $a_2 \neq a_4 \neq b_2 \neq b_4,\ w \neq \pm 1$.

Under Case 1, substituting (4.12) in (4.9), we obtain discrete kink or antikink soliton solution of (1.2) as follows:

$$u_n = \frac{a_2b_0 \pm (w - (1/w)) + a_2b_2w^{2n}\exp[\pm(w - (1/w))t]}{b_0b_2 + b_2^2w^{2n}\exp[\pm(w - (1/w))t]}, \tag{4.14}$$

where $a_2, \ b_0, \ b_2, \ w$ are arbitrary nonzero constants with $a_2 \neq b_0 \neq b_2,\ w \neq \pm 1$.



(a) Discrete antikink wave (b) Discrete kink wave

Figure 6: The solution u_n in (4.15) shows shapes of discrete antikink and kink solitons.

Under Case 2, substituting (4.13) into (4.9), we obtain discrete kink or antikink soliton solution of (1.2) as follows:

$$u_n = \frac{\mathfrak{L} + a_2 w^{2n}\exp[\pm(w-(1/w))t] + a_4 w^{4n}\exp[\pm(2w-(2/w))t]}{\mathfrak{S} + b_2 w^{2n}\exp[\pm(w-(1/w))t] + b_4 w^{4n}\exp[\pm(2w-(2/w))t]}, \tag{4.15}$$

where $\mathfrak{L}$ denotes $-((p_4w^4 \pm p_3w^3 + p_2w^2 \pm p_1w + p_0)/(w^3 - w^2 - w + 1))$ and $\mathfrak{S}$ denotes $(\pm q_2w^2 + q_1w \pm q_0)/(b_4(w+1)(w-1)^2)$, and $p_4, p_3, p_2, p_1, p_0, q_2, q_1, q_0$ have been given above, and a_2, a_4, b_2, b_4, w are arbitrary nonzero constants with $a_2 \neq a_4 \neq b_2 \neq b_4$, $w \neq \pm 1$.

In order to describe the dynamic properties of the above solutions intuitively, as an example, we draw the profile figures of the solution (4.15), see Figure 6. Setting $a_2 = 0.8$, $a_4 = 2$, $b_2 = -0.5 < 0$, $b_4 = 1.5$, $w = 4$, $t = 1$, $n \in [-20, 25]$, the solution (4.15) shows a shape of discrete antikink wave, see Figure 6(a). Setting $a_2 = 0.8$, $a_4 = 2$, $b_2 = 0.5 > 0$, $b_4 = 1.5$, $w = 4$, $t = 1$, $n \in [-20, 25]$, the solution (4.15) shows a shape of discrete kink wave, see Figure 6(b).

4.3. Exact Solutions of the Form (2.12)

Suppose that (1.2) has exact solution of the form (2.12) as follows:

$$u_n = \tilde{a}_0 + \frac{a_2 w^{2n}}{e^{2\Omega t} + b_2 w^{2n}}, \tag{4.16}$$

where $\tilde{a}_0 = a_0/(1 + b_2)$ and a_0, a_2, b_2 are constants to be determined later. After substituting (4.16) in (1.2), multiplying both sides by the common denominator $(e^{2\Omega t} + b_2w^{2n})^3(e^{2\Omega t} + b_2w^{2(n-1)})(e^{2\Omega t} + b_2w^{2(n+1)})$, it follows

$$a_2w^{2n-2}M_8e^{8\Omega t} + a_2w^{4n-2}M_6e^{6\Omega t} - b_2a_2w^{6n-2}M_4e^{4\Omega t} + b_2^3a_2w^{8n-2}M_2e^{2\Omega t} = 0, \tag{4.17}$$

where $M_8 = -1 - w^4 + (4\Omega^2+2)w^2$, $M_6 = (-b_2+2a_2\Omega+4\Omega^2 b_2)w^4 + (2b_2 - 4\Omega^2 b_2 - 4a_2\Omega)w^2 - b_2 + 2a_2\Omega + 4\Omega^2 b_2$, $M_4 = (-b_2+2a_2\Omega+4\Omega^2 b_2)w^4 + (2b_2 - 4\Omega^2 b_2 - 4a_2\Omega)w^2 - b_2 + 2a_2\Omega + 4\Omega^2 b_2$, $M_2 = 1 + w^4 + (-4\Omega^2 - 2)w^2$.

In (4.17), we let

$$M_8 = 0, \qquad M_6 = 0, \qquad M_4 = 0, \qquad M_2 = 0. \tag{4.18}$$

Solving the group of (4.18), we obtain two groups of parametric conditions which satisfy (4.17).

Case 1. We have the following:

$$a_2 = -2b_2\Omega, \qquad w = -\Omega \pm \sqrt{\Omega^2+1}, \tag{4.19}$$

where $b_2,\ \Omega$ are two arbitrary nonzero constants.

Case 2. We have the following:

$$a_2 = -2b_2\Omega, \qquad w = \Omega \pm \sqrt{\Omega^2+1}, \tag{4.20}$$

where $b_2,\ \Omega$ are also two arbitrary nonzero constants.

Under Case 1, substituting (4.19) in (4.16), we obtain discrete kink or antikink soliton solution of (1.2) as follows:

$$u_n = \frac{a_0}{1+b_2} - \frac{2b_2\Omega\left(-\Omega \pm \sqrt{\Omega^2+1}\right)^{2n}}{e^{2\Omega t} + b_2\left(-\Omega \pm \sqrt{\Omega^2+1}\right)^{2n}}, \tag{4.21}$$

where $a_0,\ b_2,\ \Omega$ are arbitrary nonzero constants.

Under Case 2, substituting (4.20) in (4.16), we obtain discrete kink or antikink soliton solution of (1.2) as follows:

$$u_n = \frac{a_0}{1+b_2} - \frac{2b_2\Omega\left(\Omega \pm \sqrt{\Omega^2+1}\right)^{2n}}{e^{2\Omega t} + b_2\left(\Omega \pm \sqrt{\Omega^2+1}\right)^{2n}}, \tag{4.22}$$

where a_0, b_2, Ω are arbitrary nonzero constants.

The dynamic properties of solutions (4.21) and (4.22) are similar to those of solutions (4.14) and (4.15). So we omit their profile figures here.

5. Conclusion

In this work, we introduced an extended method based on the mixed function method. Using this extended method, we studied the Toda lattice equation and (2 + 1)-dimensional

Toda lattice equation. We obtained some new exact solutions of discrete type for these two classic Toda lattice equations. As for the (2 + 1)-dimensional Toda lattice equation, its exact solutions which we obtained contain the discrete soliton solutions (3.31), (3.38), (3.39), and (3.47), the discrete kink and antikink wave solutions (3.10)-(3.11), (3.21)–(3.23) and (3.46), and the discrete breather soliton solutions (3.32)–(3.35), (3.40)–(3.43) and (3.48)–(3.51). These exact solutions have both discrete and continuous dynamic properties. In other words, their waveforms have discrete character along the n-axes and have continuous character along the x-axes.

Among the above discrete soliton solutions and kink (or antikink) wave solutions, the waveforms of the solutions (3.10), (3.21), (3.31), (3.39), (3.46), and (3.47) are similar to those smooth traveling waves which appear in continuous systems except their discrete characters, see Figures 1(a), 1(c), 2(a), 3(a), 4(a), 5(a), and 5(c); the waveforms of the solutions (3.11) and (3.22) are similar to those nonsmooth traveling waves such as peakon, and cusp wave which appear in continuous systems except their discrete characters, see Figures 1(b), 1(d), and 2(b). It is worthy to regard that Li et al. [40–42] explained the causes of the smooth traveling waves, nonsmooth traveling waves, peakons and cusp waves by the bifurcation theory. In addition, the waveforms with continuous characters of the breather soliton solutions and double kink wave solutions obtained in this paper are very similar to those in [43] though the studied problems (model equations) are different. This shows that the waveforms of these exact solutions obtained by us are partly similar to some of traveling waves appeared in continuous systems though the obtained solutions and the studied problems are different.

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