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Research Article

A New Oscillation Criterion for Forced Second-Order Quasilinear Differential Equations

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By using the generalized variational principle and Riccati technique, a new oscillation criterion is established for second-order quasilinear differential equation with an oscillatory forcing term, which improves and generalizes some of new results in the literature.

1. Introduction

In this paper, we are concerned with oscillation for the second order forced quasilinear differential equation

$$\left(p(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + q(t)\left|y(t)\right|^{\beta-1}y(t) = e(t), \quad t \geq t_0, \tag{1.1}$$

where $p, q, e \in C([t_0, \infty), \mathbb{R})$ with $p(t) > 0$ and $0 < \alpha \leq \beta$ being constants.

By a solution of (1.1), we mean a function $y(t) \in C^1([T_y, \infty), \mathbb{R})$, where $T_y \geq t_0$ depends on the particular solution, which has the property $p(t)|y'(t)|^{\alpha-1}y'(t) \in C^1[T_y, \infty)$ and satisfies (1.1). We restrict our attention to the nontrivial solutions $y(t)$ of (1.1) only, that is, to solutions $y(t)$ such that $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. A nontrivial solution of (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all its nontrivial solutions are oscillatory.

We note that when $0 < \alpha = \beta$, (1.1) turns into the half-linear differential equation

$$\left(p(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + q(t)\left|y(t)\right|^{\alpha-1}y(t) = e(t), \quad t \geq t_0, \tag{1.2}$$

while $0 < \alpha < \beta$, (1.1) is a super-half-linear differential equation.

Oscillation for (1.2) and its special case (without a forcing term)

$$\left(p(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + q(t)\left|y(t)\right|^{\alpha-1}y(t) = 0, \quad t \geq t_0, \tag{1.3}$$

are widely studied by many authors (see [1–6] and references cited therein) since the foundation work of Elbert [7]. Manojlović [8], Ayanlar and Tiryaki [9] obtained some Kamenev type oscillation criteria for (1.3) by using integral averaging techniques; a more detailed description of oscillation criteria can be found in the book in [10].

The half-linear differential equation (1.3) has similar properties to linear differential equation; for example, Sturm's comparison theorem and separation theorem (see Elbert [7], Jaros and Kusano [5], Li and Yeh [11] for details) are still true for (1.3). In particular, the Riccati type equation (related to (1.3) by the substitution $w = p|y'|^{\alpha-1}y'/|y|^{\alpha-1}y$)

$$w' + q(t) + \alpha p^{-1/\alpha}(t)|w|^{(\alpha+1)/\alpha} = 0 \tag{1.4}$$

and the $(\alpha+1)$-degree functional

$$I(y) := \int_a^b \left[q(t)\left|y\right|^{\alpha+1} - p(t)\left|y'\right|^{\alpha+1}\right]dt \tag{1.5}$$

play the same role as the classical Riccati equation and quadratic functional in the linear oscillation theory. (The proof of the relationship between (1.3), (1.4), and (1.5) is based on the recently established Picone's identity, see [5].)

When $\alpha = 1$, (1.2) is reduced to the forced linear differential equation

$$\left(p(t)y'(t)\right)' + q(t)y(t) = e(t), \quad t \geq t_0. \tag{1.6}$$

In paper [12], based on the oscillatory behavior of the forcing term, Wong proved the following result for (1.6).

Theorem 1.1. *Suppose that, for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that $e(t) \leq 0$ for $t \in [s_1, t_1]$ and $e(t) \geq 0$ for $t \in [s_2, t_2]$. Let $D(s_i, t_i) = \{u \in C^1[s_i, t_i] : u(t) \not\equiv 0,\ u(s_i) = u(t_i) = 0\}$ for $i = 1, 2$. If there exists $u \in D(s_i, t_i)$ such that*

$$Q_i(u) := \int_{s_i}^{t_i} \left[q(t)u^2(t) - p(t)\left(u'(t)\right)^2\right]dt > 0, \quad i = 1, 2, \tag{1.7}$$

then (1.6) *is oscillatory.*

This result involves the "oscillatory interval" of $e(t)$ and Leighton's variational principle for (1.6). Recently, using a similar method, Wong's result was extended to (1.2) by Li and Cheng [1] with a positive and nondecreasing function $\phi \in C^1([t_0, \infty), \mathbb{R})$.

Theorem 1.2. *Suppose that, for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that $e(t) \leq 0$ for $t \in [s_1, t_1]$ and $e(t) \geq 0$ for $t \in [s_2, t_2]$. Let $D(s_i, t_i) = \{u \in C^1[s_i, t_i] : u(t) \not\equiv 0, u(s_i) = u(t_i) = 0\}$*

for $i = 1,2$*. If there exist* $H \in D(s_i, t_i)$ *and a positive, nondecreasing function* $\phi \in C^1([t_0, \infty), \mathbb{R}^+)$ *such that*

$$\int_{s_i}^{t_i} H^2(t)\phi(t)q(t)dt > K \int_{s_i}^{t_i} \frac{p(t)\phi(t)}{|H(t)|^{\alpha-1}} \left(2\left|H'(t)\right| + |H(t)|\frac{\phi'(t)}{\phi(t)}\right)^{\alpha+1} dt \tag{1.8}$$

for $i = 1,2$*, where* $K = (1/(\alpha+1))^{\alpha+1}$*, then* (1.2) *is oscillatory.*

However, inequality (1.8) has no relation to the $(\alpha + 1)$-degree functional (1.5), and Theorem 1.2 cannot be applied to oscillation when $\alpha > 1$, since $|H(t)|^{\alpha-1}$ is the denominator of the fraction in the right-side integral of (1.8), and $H(s_i) = H(t_i) = 0$.

Zheng and Meng [4] improved the paper [1] and gave an oscillation criterion without any restriction on the signs of q, e, and ϕ', which is closely related to the $(\alpha + 1)$-degree functional (1.5). Here we list the main result of Zheng and Meng (Jaros et al. [6] obtained a similar result using the generalized Pocone's formulae with $\phi(t) \equiv 1$).

Theorem 1.3. *Suppose that, for any* $T \geq t_0$*, there exist* $T \leq s_1 < t_1 \leq s_2 < t_2$ *such that* $e(t) \leq 0$ *for* $t \in [s_1, t_1]$ *and* $e(t) \geq 0$ *for* $t \in [s_2, t_2]$*. Let* $D(s_i, t_i) = \{u \in C^1[s_i, t_i] : u(t) \not\equiv 0,\ u(s_i) = u(t_i) = 0\}$ *for* $i = 1,2$*. If there exist* $H \in D(s_i, t_i)$ *and a positive function* $\phi \in C^1([t_0, \infty), \mathbb{R}^+)$ *such that*

$$Q_i^{\phi}(H) := \int_{s_i}^{t_i} \phi \left[Q_e H^{\alpha+1} - p\left(\left|H'\right| + \frac{H\left|\phi'\right|}{(\alpha+1)\phi}\right)^{\alpha+1}\right](t)dt > 0 \tag{1.9}$$

for $i = 1,2$*, then* (1.1) *is oscillatory, where* $Q_e(t)$ *is the same as* (2.4)*.*

Meanwhile, among the oscillation criteria, Komkov [13] gave a generalized Leighton's variational principle, which also can be applied to oscillation for linear equation.

Theorem 1.4. *Suppose that there exist a* C^1 *function* $u(t)$ *defined on* $[s_1, t_1]$ *and a function* $G(u)$ *such that* $G(u(t))$ *is not constant on* $[s_1, t_1]$*,* $G(u(s_1)) = G(u(t_1)) = 0$*,* $g(u) = G'(u)$ *is continuous,*

$$\int_{s_1}^{t_1} \left[q(t)G(u(t)) - p(t)\left(u'(t)\right)^2\right]dt > 0, \tag{1.10}$$

and $(g(u(t)))^2 \leq 4G(u(t))$ *for* $t \in [s_1, t_1]$*. Then every solution of the equation* $(p(t)y'(t))' + q(t)y(t) = 0$ *must vanish on* $[s_1, t_1]$*.*

As far as we know, there is no oscillation criterion which relates to the generalized Leighton's variational principle for (1.1). The question then arises as to whether a more general result can be established to this equation. In this paper, we give an affirmative answer for this question and give a new oscillation criterion for the quasilinear differential equation (1.1); this oscillation criterion is closely related to the generalized Leighton's variational principle, which generalizes and improves the results mention above.

2. Main Results

Firstly, we give an inequality, which is a transformation of Young's inequality.

Lemma 2.1 (see [14]). *Supposing that X and Y are nonnegative, then*

$$\gamma X Y^{\gamma-1} - X^{\gamma} \le (\gamma - 1) Y^{\gamma}, \quad \gamma > 1, \tag{2.1}$$

where equality holds if and only if $X = Y$.

Now, we will give our main results.

Theorem 2.2. *Assume that, for any $T \ge t_0$, there exist $T \le s_1 < t_1 \le s_2 < t_2$ such that*

$$e(t) \begin{cases} \le 0, & t \in [s_1, t_1], \\ \ge 0, & t \in [s_2, t_2]. \end{cases} \tag{2.2}$$

Let $u \in C^1[s_i, t_i]$, and nonnegative functions G_1, G_2 satisfying $G_i(u(s_i)) = G_i(u(t_i)) = 0$, $g_i(u) = G_i'(u)$ are continuous and $(g_i(u(t)))^{\alpha+1} \le (\alpha+1)^{\alpha+1} G_i^{\alpha}(u(t))$ for $t \in [s_i, t_i]$, $i = 1, 2$. If there exists a positive function $\phi \in C^1([t_0, \infty), \mathbb{R})$ such that

$$Q_i^{\phi}(u) := \int_{s_i}^{t_i} \phi \left[Q_e G_i(u) - p\left(|u'| + \frac{G_i^{1/(\alpha+1)}(u)|\phi'|}{(\alpha+1)\phi} \right)^{\alpha+1} \right](t)dt > 0 \tag{2.3}$$

for $i = 1, 2$, then (1.1) *is oscillatory, where*

$$Q_e(t) = \alpha^{-\alpha/\beta} \beta (\beta - \alpha)^{(\alpha-\beta)/\beta} [q(t)]^{\alpha/\beta} |e(t)|^{(\beta-\alpha)/\beta} \tag{2.4}$$

with the convention that $0^0 = 1$.

Proof. Suppose on the contrary that there is a nontrivial nonoscillatory solution. We assume that $y(t) > 0$ on $[T_0, \infty)$ for some $T_0 \ge t_0$. Set

$$w(t) = \phi(t) \frac{p(t)|y'(t)|^{\alpha-1} y'(t)}{|y(t)|^{\alpha-1} y(t)}, \quad t \ge T_0. \tag{2.5}$$

Then for all $t \ge T_0$, we obtain the Riccati type equation

$$w'(t) = \frac{\phi'(t)}{\phi(t)} w(t) - \phi(t) \left[q(t)|y|^{\beta-\alpha} - \frac{e(t)}{|y(t)|^{\alpha-1} y(t)} \right] - \alpha \frac{|w(t)|^{(\alpha+1)/\alpha}}{(p(t)\phi(t))^{1/\alpha}}. \tag{2.6}$$

By the assumption, we can choose $t_1 > s_1 \geq T_0$ so that $e(t) \leq 0$ on the interval $I_1 = [s_1, t_1]$. For given $t \in I_1$, setting $F(x) = q(t)x^{\beta-\alpha} - e(t)/x^{\alpha}$, we have $F'(x^*) = 0$, $F''(x^*) > 0$, where $x^* = [-\alpha e(t)/(\beta-\alpha)q(t)]^{1/\beta}$. So $F(x)$ obtains its minimum on x^*, and

$$F(x) \geq F(x^*) = Q_e(t). \tag{2.7}$$

So (2.6) and (2.7) imply that $w(t)$ satisfies

$$\phi(t)Q_e(t) \leq -w'(t) + \frac{\phi'(t)}{\phi(t)}w(t) - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(p(t)\phi(t))^{1/\alpha}}. \tag{2.8}$$

Multiplying $G_1(u(t))$ through (2.8) and integrating (2.8) from s_1 to t_1, using the fact that $G_1(u(s_1)) = G_1(u(t_1)) = 0$, we obtain

$$\begin{aligned}
&\int_{s_1}^{t_1}\phi(t)Q_e(t)G_1(u(t))dt \\
&\quad\leq \int_{s_1}^{t_1}G_1(u(t))\left\{-w'(t) + \frac{\phi'(t)}{\phi(t)}w(t) - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(p(t)\phi(t))^{1/\alpha}}\right\}dt \\
&\quad= -G_1\left(u(t)\right)w(t)\big|_{s_1}^{t_1} + \int_{s_1}^{t_1}g_1(u(t))u'(t)w(t)dt \\
&\qquad+ \int_{s_1}^{t_1}G_1(u(t))\left\{\frac{\phi'(t)}{\phi(t)}w(t) - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{(p(t)\phi(t))^{1/\alpha}}\right\}dt \\
&\quad= \int_{s_1}^{t_1}\left[g_1(u(t))u'(t) + G_1(u(t))\frac{\phi'(t)}{\phi(t)}\right]w(t)dt - \alpha\int_{s_1}^{t_1}G_1(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{(p(t)\phi(t))^{1/\alpha}}dt \\
&\quad\leq \int_{s_1}^{t_1}\left[\left|g_1(u(t))\right|\left|u'(t)\right| + G_1(u(t))\frac{\left|\phi'(t)\right|}{\phi(t)}\right]|w(t)|dt - \alpha\int_{s_1}^{t_1}G_1(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{(p(t)\phi(t))^{1/\alpha}}dt \\
&\quad\leq (\alpha+1)\int_{s_1}^{t_1}\left[G_1^{\alpha/(\alpha+1)}(u(t))\left|u'(t)\right| + G_1(u(t))\frac{\left|\phi'(t)\right|}{(\alpha+1)\phi(t)}\right]|w(t)|dt \\
&\qquad- \alpha\int_{s_1}^{t_1}G_1(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{(p(t)\phi(t))^{1/\alpha}}dt.
\end{aligned} \tag{2.9}$$

Let

$$\begin{aligned}
X &= \left[\frac{\alpha}{(p(t)\phi(t))^{1/\alpha}}\right]^{\alpha/(\alpha+1)}G_1^{\alpha/(\alpha+1)}|w(t)|, \qquad \gamma = 1 + \frac{1}{\alpha}, \\
Y &= \left(\alpha\phi(t)p(t)\right)^{\alpha/(\alpha+1)}\left[\left|u'(t)\right| + \frac{G_1^{1/(\alpha+1)}\left|\phi'(t)\right|}{(\alpha+1)\phi(t)}\right]^{\alpha}.
\end{aligned} \tag{2.10}$$

By Lemma 2.1 and (1.2), we have

$$\int_{s_1}^{t_1} \phi(t)Q_e(t)G_1(u(t))dt \le \int_{s_1}^{t_1} \phi(t)p(t)\left[|u'(t)| + \frac{G_1^{1/(\alpha+1)}(u(t))\left|\phi'(t)\right|}{(\alpha+1)\phi(t)}\right]^{\alpha+1} dt, \tag{2.11}$$

which contradicts (2.3) with $i = 1$.

When $y(t)$ is a negative solution for $t \ge T_0 > t_0$, we define the Riccati transformation as (2.5), and we obtain (2.8) also. In this case, we choose $t_2 > s_2 \ge T_0$ so that $e(t) \ge 0$ on the interval $I_2 = [s_2, t_2]$. For given $t \in I_2$, setting $F(x) = q(t)x^{\beta-\alpha} - (e(t)/x^{\alpha})$, we have $F(x) \ge F(x^*) = Q_e(t)$. We use $u \in C^1[s_2, t_2]$ and G_2 on $[s_2, t_2]$ to reach a similar contradiction. The proof is complete. □

Corollary 2.3. *If* $\phi(t) \equiv 1$ *in Theorem 2.2 and* (2.3) *is replaced by*

$$Q_i(u) := \int_{s_i}^{t_i} \left[Q_e(t)G_i(u(t)) - p(t)\left|u'(t)\right|^{\alpha+1}\right] dt > 0 \tag{2.12}$$

for $i = 1, 2$, *then* (1.1) *is oscillatory.*

Remark 2.4. Corollary 2.3 is closely related to the generalized variational formulae similar to (1.10), so Theorem 2.2, Corollary 2.3 are generalizations of Theorems 1.1 and 1.3 (if we choose $G_1(u) = G_2(u) = u^{\alpha+1}$ in Corollary 2.3, then we obtain Corollary 2.3 of paper [4]), and when $0 < \alpha = \beta$, we have $Q_e(t) = q(t)$, so Corollary 2.3 improves the main result of paper [1], since the positive constant α in Theorem 2.2, Corollary 2.3 can be selected as any number lying in $(0, \infty)$.

Remark 2.5. Hypothesis (2.2) in Theorem 2.2, Corollary 2.3 can be replaced by the following condition

$$e(t) \begin{cases} \ge 0, & t \in [s_1, t_1], \\ \le 0, & t \in [s_2, t_2]. \end{cases} \tag{2.13}$$

The conclusion is still true for this case.

Now we give an example to illustrate the efficiency of our result.

Example 2.6. Consider the following forced quasilinear differential equation:

$$\left(\gamma t^{-\lambda/3} y'(t)\right)' + q(t)\left|y(t)\right|^2 y(t) = -\sin^3 t, \quad t \ge 2\pi, \tag{2.14}$$

where $\gamma, \lambda > 0$ are constants, $q(t) = t^{-\lambda}\exp(3\sin t)$ for $t \in [2n\pi, (2n+1)\pi)$, and $q(t) = t^{-\lambda}\exp(-3\sin t)$ for $t \in [(2n+1)\pi, (2n+2)\pi)$, $n > 0$ is an integer. Here $\alpha = 1$, $\beta = 3$ in Theorem 2.2. Since $\alpha < \beta$, Theorems 1.1 and 1.2 cannot be applied to this case. However, we can obtain oscillation for (2.14) using Theorem 2.2. In fact, we can easily verify that $Q_e(t) = (3/2)\sqrt[3]{2}t^{-\lambda/3}\exp(\sin t)\sin^2 t$ for $t \in [2n\pi, (2n+1)\pi)$ and $Q_e(t) = (3/2)\sqrt[3]{2}t^{-\lambda/3}\exp(-\sin t)\sin^2 t$

for $t \in [(2n+1)\pi, (2n+2)\pi)$. For any $T \geq 1$, we choose n sufficiently large so that $n\pi = 2k\pi \geq T$ and $s_1 = 2k\pi$ and $t_1 = (2k+1)\pi$; we select $u(t) = \sin t \geq 0$, $G_1(u) = u^2 \exp(-u)$ (we note that $(G_1'(u))^2 \leq 4G_1(u)$ for $u \geq 0$), $\phi(t) = t^{\lambda/3}$, then we have

$$\int_{s_1}^{t_1} \phi(t) Q_e(t) G_1(u(t))dt = \frac{3}{2}\sqrt[3]{2}\int_{2k\pi}^{(2k+1)\pi} \sin^4 t \, dt = \frac{9}{8}\sqrt[3]{2},$$

$$\begin{aligned}\int_{s_1}^{t_1} \phi(t)p(t)\left[\left|u'(t)\right| + \frac{G_1^{1/(\alpha+1)}(u(t))\left|\phi'(t)\right|}{(\alpha+1)\phi(t)}\right]^{\alpha+1} dt \\ = \gamma\int_{2k\pi}^{(2k+1)\pi}\left[|\cos t| + \frac{\lambda|\sin t|\exp(3\sin t/2)}{2t}\right]^2 dt \\ < \gamma\int_{2k\pi}^{(2k+1)\pi}\left(1+\frac{\lambda e^{3/2}}{2}\right)^2 dt = \gamma\left(1+\frac{\lambda e^{3/2}}{2}\right)^2\pi.\end{aligned} \tag{2.15}$$

So we have $Q_1^{\phi}(u) > 0$ provided $0 < \gamma < 9\sqrt[3]{2}/2(2+\lambda e^{3/2})^2\pi$. Similarly, for $s_2 = (2k+1)\pi$ and $t_2 = (2k+2)\pi$, we select $u(t) = \sin t \leq 0$, $G_2(u) = u^2\exp(u)$ (we note that $(G_2'(u))^2 \leq 4G_2(u)$ for $u \leq 0$), and we can show that the integral inequality $Q_2^{\phi}(u) > 0$ for $0 < \gamma < 9\sqrt[3]{2}/2(2+\lambda e^{3/2})^2\pi$. So (2.14) is oscillatory for $0 < \gamma < 9\sqrt[3]{2}/2(2+\lambda e^{3/2})^2\pi$ by Theorem 2.2. We note further that Theorem 1.3 cannot be applied to (2.14) with $G(u) = u^2$.

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