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Review Article

Synchronization of Coupled Nonidentical Fractional-Order Hyperchaotic Systems

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Synchronization of coupled nonidentical fractional-order hyperchaotic systems is addressed by the active sliding mode method. By designing an active sliding mode controller and choosing proper control parameters, the master and slave systems are synchronized. Furthermore, synchronizing fractional-order hyperchaotic Lorenz system and fractional-order hyperchaotic Chen system is performed to show the effectiveness of the proposed controller.

1. Introduction

Fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz and L'Hôspital in 1695. Although it is a mathematical topic with more than 300 years old history, the applications of fractional calculus to physics and engineering are just a recent focus of interest [1, 2]. Nowadays, by utilizing fractional calculus technique, many investigations were devoted to the chaotic and hyperchaotic behaviors of fractional-order systems, such as fractional-order Chua circuit [3], fractional-order Lorenz system [4], fractional-order Rössler system [5], fractional-order Chen system [6], and fractional-order conjugate Lorenz system [7].

Over the last two decades, synchronization of chaotic systems has become more and more interesting to researchers in different fields. Since the synchronization of fractional-order chaotic systems was firstly investigated in [8], it has recently attracted increasing attention due to its potential applications in secure communication and control processing [9–13]. Moreover, many theoretical analysis and numerical simulation results about the synchronization of fractional-order chaotic systems are obtained [14–19]. Such synchronization may be safer than those of the classical chaotic systems in secure communications. This can be seen from two aspects: (i) the order of fractional derivatives can be regarded as a parameter and (ii) the fractional derivatives are nonlocal thus more complicated than the regular derivatives.

This paper focuses on synchronization of coupled fractional-order hyperchaotic nonidentical systems. The active sliding mode synchronization method is chosen to achieve this goal. The active sliding mode synchronization technique is a discontinuous control strategy, which relies on two stages of design. The first stage is to select an appropriate active controller to facilitate the design of the sequent sliding mode controller. The second stage is to design a sliding mode controller to achieve the synchronization. This process is verified when active sliding mode synchronization method is used to synchronize fractional-order hyperchaotic Lorenz system and fractional-order hyperchaotic Chen system.

This paper is organized as follows. In Section 2, basic definitions in fractional calculus are briefly presented. In Section 3, design of active sliding mode controller is proposed to synchronize coupled nonidentical fractional-order hyperchaotic systems. The application of the proposed method on fractional-order hyperchaotic Lorenz system and fractional-order hyperchaotic Chen system is numerically investigated in Section 4. Finally, conclusions in Section 5 close the paper.

2. Basic Definitions

There are some definitions for fractional derivatives [20]. Three most commonly used definitions are Grünwald-Letnikov, Riemann-Liouville, and Caputo definitions.

Definition 2.1. The fractional derivative of Grünwald-Letnikov definition is given by

$$D_t^{\alpha} f(t) = \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \lim_{N \to \infty} \left[\frac{t}{N}\right]^{-\alpha} \sum_{j=0}^{N-1} (-1)^j \binom{\alpha}{j} f\left(t - j\left[\frac{t}{N}\right]\right). \tag{2.1}$$

Definition 2.2. Let $m-1 < \alpha < m$, $m \in N$, the Riemann-Liouville fractional derivative of order α of any function $f(t)$ is defined as follows:

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau. \tag{2.2}$$

Definition 2.3. Let $f \in C_{-1}^m$, $m \in N$, the Caputo fractional derivative of $f(t)$ is defined by

$$D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m} d\tau, & \text{if } m-1 < \alpha < m, \\ \dfrac{d^m f(t)}{dt^m}, & \text{if } \alpha = m \in N. \end{cases} \tag{2.3}$$

Note that the main advantage of Caputo approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations. Therefore, in the rest of this paper, the notation D_t^{α} indicates the Caputo fractional derivative.

3. Active Sliding Mode Controller Design and Analysis

Consider a fractional-order hyperchaotic system of order $\alpha (0 < \alpha < 1)$, described by

$$D_t^{\alpha} x = Ax + f(x), \tag{3.1}$$

where $x \in R^4$ denotes the system's 4-dimensional state vector, $A \in R^4 \times R^4$ represents the linear part of the system, and $f : R^4 \rightarrow R^4$ is the nonlinear part of the system. The system (3.1) represents the master system. The controller $u(t) \in R^4$ is added into the slave system, given by

$$D_t^{\alpha} y = By + g(y) + u(t), \tag{3.2}$$

where y, B and g imply the same roles as x, A, and f in the master system.

To synchronize the master system (3.1) and the slave system (3.2), the synchronization errors dynamics is designed as follows:

$$D_t^{\alpha} e = By + g(y) - Ax - f(x) + u(t) = Be + G(x, y) + u(t), \tag{3.3}$$

where $e = y - x$ and $G(x, y) = (B - A)x + g(y) - f(x)$. The aim is to design the controller $u(t) \in R^4$ such that

$$\lim_{t \to \infty} \|e(t)\| = 0. \tag{3.4}$$

3.1. *Designing the Active Controller*

According to the active control design procedure [21–23], the nonlinear part of the error system is eliminated by the following choice of the input vector:

$$u(t) = H(t) - G(x, y). \tag{3.5}$$

Then the error system (3.3) is rewritten as

$$D_t^{\alpha} e = Be + H(t). \tag{3.6}$$

On the other hand, based on a sliding mode control law, $H(t)$ is designed as

$$H(t) = Kw(t), \tag{3.7}$$

where $K = [k_1, k_2, k_3, k_4]^T$ denotes a constant vector and $w(t) \in R$ is the control input that satisfies

$$w(t) = \begin{cases} w^+(t), & s(e) \geq 0, \\ w^-(t), & s(e) < 0, \end{cases} \tag{3.8}$$

where $s = s(e)$ is a switching surface which prescribes the desired dynamics. Therefore, the error system becomes

$$D_t^{\alpha} e = Be + Kw(t). \tag{3.9}$$

3.2. Designing the Sliding Surface

The sliding surface can be defined by

$$s(e) = Pe, \tag{3.10}$$

in which $P = [p_1, p_2, p_3, p_4]$ is a constant vector. The equivalent control is found by the fact that $\dot{s}(e) = 0$ is a necessary condition for the state trajectory to stay on the switching surface $s(e) = 0$. Hence, when in sliding mode, the controlled system satisfies the following conditions:

$$s(e) = 0, \qquad \dot{s}(e) = 0. \tag{3.11}$$

From (3.9)–(3.11), it follows

$$\dot{s}(e) = \frac{\partial s(e)}{\partial e}\dot{e} = \frac{\partial s(e)}{\partial e} D_t^{1-\alpha}\left(D_t^{\alpha} e\right) = PD_t^{1-\alpha}(Be + Kw(t)). \tag{3.12}$$

Hence,

$$D_t^{1-\alpha} w(t) = -(PK)^{-1} PBD_t^{1-\alpha} e(t). \tag{3.13}$$

The equivalent control $w_{\mathrm{eq}}(t)$ is a solution of (3.13):

$$w_{\mathrm{eq}}(t) = -(PK)^{-1} PBe(t), \tag{3.14}$$

which is realizable whenever PK assumes nonzero value.

Replacing $w(t)$ in (3.9) by $w_{\mathrm{eq}}(t)$ in (3.14), the error dynamics on the sliding surface are determined by the following relation:

$$D_t^{\alpha} e = \left(I - K(PK)^{-1} P\right) Be(t). \tag{3.15}$$

3.3. Designing the Sliding Mode Controller

Based on the constant plus proportional rate reaching law [24–27], the reaching law is chosen as

$$D_t^{\alpha} s = -p\,\mathrm{sgn}(s) - rs, \tag{3.16}$$

where $\mathrm{sgn}(\cdot)$ denotes the sign function. $p > 0$ and $r > 0$ are determined such that the sliding condition is satisfied and the sliding mode motion occurs.

Based on(3.9) and (3.10), it follows

$$D_t^\alpha s = PD_t^\alpha e = PBe + PKw(t). \tag{3.17}$$

Therefore, the control input is determined as

$$w(t) = -(PK)^{-1}\left[P(rI + B)e(t) + p\,\mathrm{sgn}(s)\right]. \tag{3.18}$$

3.4. Stability Analysis

First, two stability theorems on fractional order systems are introduced.

Theorem 3.1 (see [28]). *The following autonomous system:*

$$D_t^\alpha x = Ax, \qquad x(0) = x_0, \tag{3.19}$$

with $0 < \alpha < 1$, $x \in R^n$, and $A \in R^{n\times n}$, is asymptotically stable if and only if $|\arg(\lambda)| > \alpha\pi/2$ is satisfied for all eigenvalues of matrix A. Also, this system is stable if and only if $|\arg(\lambda)| \ge \alpha\pi/2$ is satisfied for all eigenvalues of matrix A with those critical eigenvalues satisfying $|\arg(\lambda)| = \alpha\pi/2$ having geometric multiplicity of one. The geometric multiplicity of an eigenvalue λ of the matrix A is the dimension of the subspace of vectors v for which $Av = \lambda v$.

Theorem 3.2 (see [28]). *Consider a system given by the following linear state space form with inner dimension n*

$$\begin{aligned} D_t^\alpha x &= Ax + Bu \\ y &= Cx, \end{aligned} \qquad x(0) = x_0, \tag{3.20}$$

with $0 < \alpha < 1$, $x \in R^n$, and $A \in R^{n\times n}$. Also, assume that the triplet (A, B, C) is minimal. System (3.20) *is bounded-input bounded-output stable if and only if $|\arg(\lambda)| > \alpha\pi/2$. When system* (3.20) *is externally stable, each component of its impulse response behaves like $t^{-1-\alpha}$ at infinity.*

According to Theorem 3.2, the sliding surface $s = s(e)$ is bounded since the dynamic of the sliding surface $s = s(e)$ is linear with bounded input ($-p$ for $s \ge 0$ and p for $s < 0$).

Substituting (3.18) into (3.9), the error system reads

$$D_t^\alpha e = \left[B - K(PK)^{-1}P(rI + B)\right]e(t) - K(PK)^{-1}p\,\mathrm{sgn}(s). \tag{3.21}$$

As a linear fractional-order system with bounded input ($-K(PK)^{-1}p$ for $s \ge 0$ and $K(PK)^{-1}p$ for $s < 0$), the error system is stable if

$$\left|\arg\left(\mathrm{eig}\left(B - K(PK)^{-1}P(rI + B)\right)\right)\right| > \frac{\alpha\pi}{2}. \tag{3.22}$$

In this case, the error system is asymptotically stable when $p = 0$. The error signals will not converge to zero if $p \ne 0$. Parameter p can be used to enhance the robustness of the controller in the presence of noise and mismatches.

Furthermore, it is easy to check that the eigenvalues of matrix $[K(PK)^{-1}P]$ are $\{1,0,0,0\}$. This means that the eigenvalues of matrix $[I - K(PK)^{-1}P]$ are $\{0,1,1,1\}$. Then the rank of matrix $[I - K(PK)^{-1}P(rI + B)]$ satisfies

$$\operatorname{rank}\Big[I - K(PK)^{-1}P(rI + B)\Big] < 4, \tag{3.23}$$

and one of eigenvalues of matrix $[I - K(PK)^{-1}P(rI + B)]$ is always 0. Thus,

$$\exists \nu \in R^{4\times 1} : \Big[I - K(PK)^{-1}P(rI + B)\Big]\nu = 0 \Longrightarrow \Big[B - K(PK)^{-1}P(rI + B)\Big]\nu = -r\nu. \tag{3.24}$$

Therefore, one of the eigenvalues of matrix $[B - K(PK)^{-1}P(rI + B)]$ is always $-r < 0$. It can be shown that the three other eigenvalues are independent from r and determined by the other control parameters (K and P). These three eigenvalues must satisfy condition (3.22).

Remark 3.3. Since we consider the fractional-order hyperchaotic systems (3.1) and (3.2) in the general form, the approach in this section is generic and can also be used for other fractional-order chaotic or hyperchaotic systems.

Remark 3.4. Since the chain rule is not valid in fractional-order systems, the expression (3.12) is significantly different from the integer-order one. In addition, the classical Lyapunov stability approach is difficult in checking the asymptotical stability of the error system. To overcome this shortcoming, we utilize the obtained stability results on fractional order systems to deduce the condition (3.22). Therefore, our aim is to find the controller such that the condition (3.22) is satisfied.

4. Numerical Results

The hyperchaotic Lorenz system [29] was found by adding a nonlinear controller to the classical Lorenz system [30]. It has been shown that the fractional-order hyperchaotic Lorenz system can exhibit hyperchaotic behavior [31]. The fractional-order hyperchaotic Lorenz system reads

$$\begin{aligned} D_t^{\alpha} x_1 &= \sigma(x_2 - x_1) + x_4, \\ D_t^{\alpha} x_2 &= \gamma x_1 - x_2 - x_1 x_3, \\ D_t^{\alpha} x_3 &= x_1 x_2 - \beta x_3, \\ D_t^{\alpha} x_4 &= -x_2 x_3 + \delta x_4. \end{aligned} \tag{4.1}$$

By adding a nonlinear controller to the Chen system [32], the authors obtained a hyperchaotic Chen system [33], which can demonstrate hyperchaotic behavior. The fractional-order hyperchaotic Chen system reads

$$\begin{aligned} D_t^{\alpha} y_1 &= a(y_2 - y_1) + y_4, \\ D_t^{\alpha} y_2 &= d y_1 - y_1 y_3 + c y_2, \\ D_t^{\alpha} y_3 &= y_1 y_2 - b y_3, \\ D_t^{\alpha} y_4 &= y_2 y_3 + k y_4. \end{aligned} \tag{4.2}$$

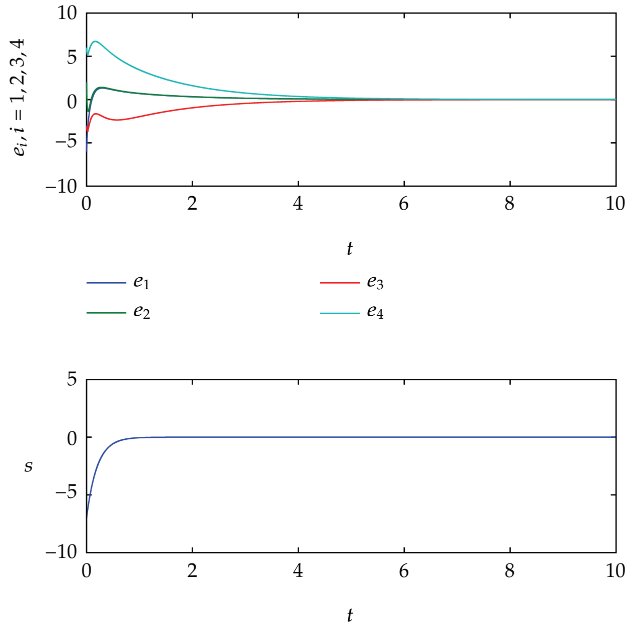

Figure 1: The evolvement of error system and sliding surface.

In what follows, the active sliding mode synchronization method is applied to synchronize fractional-order hyperchaotic Lorenz system and fractional-order hyperchaotic Chen system. In this case, matrix A and B are given as

$$A = \begin{bmatrix} -\sigma & \sigma & 0 & 1 \\ \gamma & -1 & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \delta \end{bmatrix}, \qquad B = \begin{bmatrix} -a & a & 0 & 1 \\ d & c & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & k \end{bmatrix}. \tag{4.3}$$

Assume that order of the master and slave systems is 2.94 ($\alpha = 0.98$) and system parameters are $(\sigma, \gamma, \beta, \delta) = (10, 28, 8/3, -1)$ and $(a, b, c, d, k) = (35, 3, 12, 7, 0.58)$. The controller parameters are chosen as $K = [-2, -2, -6, -2]^T$, $P = [1, 1, -, 1-1]$, $r = 5$ and $p = 0.3$. This selection of parameters results in eigenvalues $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-5, -71.4194, -11.4542, -0.7564)$ that are located in the stable region.

The numerical simulation has been carried out using MATLAB subroutines written based on the predictor-corrector scheme [34]. The time step size employed in the simulation is 0.001 ($h = 0.001$), and the initial conditions of master and slave systems are $(x_{10}, x_{20}, x_{30}, x_{40}) = (4, 1, 2, -4)$ and $(y_{10}, y_{20}, y_{30}, y_{40}) = (-2, 3, 1, 2)$. The simulation results are given in Figure 1. As one can see, the designed controller is effective to synchronize fractional-order hyperchaotic Lorenz system and fractional-order hyperchaotic Chen system.

5. Conclusions

In this paper, the active sliding mode method for synchronization of coupled nonidentical fractional-order hyperchaotic systems is addressed. By designing the active sliding mode controller and choosing proper control parameters (K, P, and r), the master and slave systems are synchronized. Furthermore, the application of the proposed method on fractional-order hyperchaotic Lorenz system and fractional-order hyperchaotic Chen system is investigated. Numerical results show the efficiency of the proposed controller to synchronize coupled nonidentical fractional-order hyperchaotic systems.

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