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Research Article

Existence of Periodic Solutions for a Class of Discrete Hamiltonian Systems

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By applying minimax methods in critical point theory, we prove the existence of periodic solutions for the following discrete Hamiltonian systems $\Delta^2 u(t-1) + \nabla F(t, u(t)) = 0$, where $t \in \mathbb{Z}$, $u \in \mathbb{R}^N$, $F : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$, $F(t, x)$ is continuously differentiable in x for every $t \in \mathbb{Z}$ and is T-periodic in t; T is a positive integer.

1. Introduction

Consider the following discrete Hamiltonian system:

$$\Delta^2 u(t-1) + \nabla F(t, u(t)) = 0, \quad t \in \mathbb{Z}, \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\Delta^2 u(t) = \Delta(\Delta u(t))$, $t \in \mathbb{Z}$, $u \in \mathbb{R}^N$, $F : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$, and $F(t, x)$ is continuously differentiable in x for every $t \in \mathbb{Z}$ and is T-periodic in t; T is a positive integer.

Difference equations usually describe evolution of certain phenomena over the course of time. For example, if a certain population has discrete generations, the size of the $(t+1)$ th generation $x(t+1)$ is a function of the tth generation $x(t)$. In fact, difference equations provide a natural description of many discrete models in real world. Since discrete models exist in various fields of science and technology such as statistics, computer science, electrical circuit analysis, biology, neural network, optimal control, and so on, it is of practical importance to investigate the solutions of difference equations. For more details about difference equations, we refer the readers to the books [1–3].

In some recent papers [4–15], the authors studied the existence of periodic solutions and subharmonic solutions of difference equations by applying critical point theory. These papers show that the critical point theory is an effective method to the study of periodic solutions for difference equations. In 2007, Xue and Tang [11] investigated the existence of periodic solutions for (1.1) and obtained the main result.

Theorem A (see [11]). *Suppose that F satisfies the following conditions:*

(F1) *there exists a positive constant T such that $F(t+T,x) = F(t,x)$ for all $(t,x) \in \mathbb{Z} \times \mathbb{R}^N$;*

(F2) *there are constants $L_1 > 0$, $L_2 > 0$, and $0 \le \alpha < 1$ such that*

$$|\nabla F(t,x)| \le L_1|x|^\alpha + L_2, \quad \forall (t,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N, \tag{1.2}$$

where $\mathbb{Z}[a,b] := \cap[a,b]$ for every $a,b \in \mathbb{Z}$ with $a \le b$;

(F3) $|x|^{-2\alpha} \sum_{t=1}^{T} F(t,x) \to +\infty$ *as* $|x| \to +\infty$ *for all* $t \in \mathbb{Z}[1,T]$.

Then problem (1.1) *has at least one periodic solution with period T.*

Let

$$F(t,x) = f(t)|x|^{7/4} + \left(\sin\frac{2\pi t}{T} + 1\right)|x|^{3/2} + (h(t),x), \tag{1.3}$$

where $f : \mathbb{Z}[1,T] \to \mathbb{R}$, $f(t+T) = f(t)$, $h : \mathbb{Z}[1,T] \to \mathbb{R}^N$, and $h(t+T) = h(t)$. It is easy to see that

$$\begin{aligned} |\nabla F(t,x)| &\le \frac{7}{4}\left|f(t)\right||x|^{3/4} + \frac{3}{2}\left|\sin\frac{2\pi t}{T} + 1\right||x|^{1/2} + |h(t)| \\ &\le \frac{7}{4}\left(\left|f(t)\right| + \varepsilon\right)|x|^{3/4} + a(\varepsilon) + |h(t)|, \quad \forall (t,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N, \end{aligned} \tag{1.4}$$

where $\varepsilon > 0$ and $a(\varepsilon)$ is a positive constant and is dependent on ε. The above inequality shows that there are functions not satisfying condition (F2). If we let $\sum_{t=1}^{T} f(t) = 0$, $\alpha = 3/4$, $T = 2$, then we have

$$|x|^{-2\alpha}\sum_{t=1}^{T} F(t,x) = 2 + \left(h(1) + h(2),\ |x|^{-3/2}x\right). \tag{1.5}$$

But the above equality does not satisfy (F3). This example shows that it is valuable to further improve conditions (F2) and (F3).

Before stating our main results, we first introduce some preliminaries.

2. Preliminaries

Let

$$H_T = \left\{u : \mathbb{Z} \longrightarrow \mathbb{R}^N \mid u(t+T) = u(t),\ t \in \mathbb{Z}\right\}. \tag{2.1}$$

H_T can be equipped with the inner product

$$\langle u, v\rangle = \sum_{t=1}^{T}(u(t), v(t)), \quad \forall u, v \in H_T, \tag{2.2}$$

by which the norm on H_T can be reduced by

$$\|u\| = \left(\sum_{t=1}^{T}|u(t)|^2\right)^{1/2}, \quad \forall u \in H_T, \tag{2.3}$$

where $(\cdot,\cdot)$ and $|\cdot|$ denote the usual inner product and the usual norm in $\mathbb{R}^N$. It is easy to see that $(H_T, \langle\cdot,\cdot\rangle)$ is a finite-dimensional Hilbert space and linear homeomorphic to $\mathbb{R}^{NT}$. For any $r > 1$, define

$$\|u\|_r = \left(\sum_{t=1}^{T}|u(t)|^r\right)^{1/r}, \quad \forall u \in H_T. \tag{2.4}$$

Obviously, $\|u\| = \|u\|_2$ and $\|u\|$ is equivalent to $\|u\|_r$. Hence, there exist two positive constants C_1, C_2, which are independent on r, such that

$$C_1\|u\|_r \le \|u\| \le C_2\|u\|_r, \quad \forall u \in H_T. \tag{2.5}$$

If we define $\|u\|_\infty = \sup_{t\in\mathbb{Z}[1,T]}|u(t)|$, it is easy to see that for any $r > 1$,

$$\|u\|_\infty \le \|u\|_r, \quad \forall u \in H_T. \tag{2.6}$$

For any $u \in H_T$, let

$$\varphi(u) = -\frac{1}{2}\sum_{t=1}^{T}|\Delta u(t)|^2 + \sum_{t=1}^{T}[F(t, u(t)) - F(t, 0)]. \tag{2.7}$$

We can compute the Fréchet derivative of (2.7) as

$$\frac{\partial\varphi(u)}{\partial u(t)} = \Delta^2 u(t-1) + \nabla F(t, u(t)), \quad t \in \mathbb{Z}[1, T]. \tag{2.8}$$

Hence, u is a critical point of φ on H_T if and only if

$$\Delta^2 u(t-1) + \nabla F(t, u(t)) = 0, \quad t \in \mathbb{Z}[1, T], \ u \in \mathbb{R}^N. \tag{2.9}$$

So, the critical points of φ are classical solutions of (1.1). The following lemmas are useful in our proof.

Lemma 2.1 (see [11]). *As a subspace of H_T, N_k is defined by*

$$N_k := \left\{u \in H_T \mid -\Delta^2 u(t-1) = \lambda_k u(t)\right\}, \tag{2.10}$$

where $\lambda_k = 2 - 2\cos k\omega$, $\omega = 2\pi/T$, $k \in \mathbb{Z}[0, [T/2]]$, $[\cdot]$ *and denote the Gauss Function. Then there hold*

(i) $N_k \perp N_j$, $k \neq j$, $j \in \mathbb{Z}[0, [T/2]]$;

(ii) $H_T = \oplus_{k=0}^{[T/2]} N_k$.

Lemma 2.2 (see [11]). *Define* $H_k := \oplus_{j=0}^{k} N_j$, $H_k^{\perp} := \oplus_{j=k+1}^{[T/2]} N_j$, $k \in \mathbb{Z}[0, [T/2]-1]$; *then one has*

$$\sum_{t=1}^{T} |\Delta u(t)|^2 \leq \lambda_k \|u\|_2, \quad \forall u \in H_k; \tag{2.11}$$

$$\sum_{t=1}^{T} |\Delta u(t)|^2 \geq \lambda_{k+1} \|u\|_2, \quad \forall u \in H_k^{\perp}. \tag{2.12}$$

3. Main Results and Proofs

Theorem 3.1. *Suppose that F satisfies (F1) and the following conditions*

(F2)′ *there are* $p, q : \mathbb{Z}[1, T] \to \mathbb{R}^+$, $\alpha \in [0, 1)$ *such that*

$$|\nabla F(t, x)| \leq p(t)|x|^{\alpha} + q(t), \quad \forall (t, x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N, \tag{3.1}$$

where $\mathbb{Z}[a, b] := \mathbb{Z} \cap [a, b]$ *for every* $a, b \in \mathbb{Z}$ *with* $a \leq b$;

(F3)′ $\liminf_{|x| \to \infty} |x|^{-2\alpha} \sum_{t=1}^{T} F(t, x) > ((\lambda_{[T/2]} + 2\lambda_1)/2\lambda_1^2) \sum_{t=1}^{T} p^2(t)$ *for all* $t \in \mathbb{Z}[1, T]$.

Then problem (1.1) *has at least one periodic solution with period T.*

Theorem 3.2. *Suppose that F satisfies (F1) and (F2) with $\alpha = 1$. Moreover, assume the following conditions hold:*

$$\sum_{t=1}^{T} p(t) < \lambda_1, \tag{3.2}$$

and

(F4) $\liminf_{|x| \to \infty} |x|^{-2} \sum_{t=1}^{T} F(t, x) > ((\lambda_{[T/2]} + \lambda_1^{1/2} + \sum_{t=1}^{T} p(t) + \lambda_1^{1/2}(\lambda_1 - \sum_{t=1}^{T} p(t)))/2\lambda_1(\lambda_1 - \sum_{t=1}^{T} p(t))) \sum_{t=1}^{T} p^2(t)$ *for all* $t \in \mathbb{Z}[1, T]$.

Then problem (1.1) *has at least one periodic solution with period T.*

Remark 3.3. It is easy to see that (F2)′ is more general than (F2) and (F3)′ is weaker than (F3). Theorem 3.2 is a new result, which completes Theorem A when $\alpha = 1$.

For the sake of convenience, we denote

$$M_1 = \left(\sum_{t=1}^{T} p^2(t)\right)^{1/2}, \qquad M_2 = \sum_{t=1}^{T} p(t), \qquad M_3 = \sum_{t=1}^{T} q(t). \tag{3.3}$$

Proof of Theorem 3.1. First we prove that φ satisfies the (PS) condition. Suppose that a consequence $\{u_n\} \subset H_T$ is such that $-C_3 \leq \varphi(u_n) \leq C_3$, where $C_3 > 0$ and $\varphi'(u_n) \to 0$ as $n \to \infty$. Then for sufficiently large n,

$$-\|u\| \leq \langle \varphi'(u_n), u \rangle \leq \|u\|. \tag{3.4}$$

From Lemma 2.1, we can write u as $u = \overline{u} + \widetilde{u} \in H_0 \bigoplus H_0^{\perp}$, where $H_0 = N_0$, and $H_0^{\perp} = \bigoplus_{k=1}^{[T/2]} N_k$. From (F3)$'$, we can choose $a_1 > 1/\lambda_1^2 > 0$ such that

$$\liminf_{|x| \to \infty} |x|^{-2\alpha} \sum_{t=1}^{T} F(t,x) > \left(\frac{\lambda_{[T/2]} a_1}{2} + \sqrt{a_1}\right) M_1^2. \tag{3.5}$$

From (F2)$'$, (2.6), Hölder inequality, and Young inequality, we have

$$\begin{aligned}
\left|\sum_{t=1}^{T} (F(t,u(t)) - F(t,\overline{u}))\right| &= \left|\sum_{t=1}^{T} \int_0^1 (\nabla F(t, \overline{u} + s\widetilde{u}(t)), \widetilde{u}(t))\, ds\right| \\
&\leq \sum_{t=1}^{T} \int_0^1 |(\nabla F(t, \overline{u} + s\widetilde{u}(t)), \widetilde{u}(t))|\, ds \\
&\leq \sum_{t=1}^{T} \int_0^1 \left(p(t)|\overline{u} + s\widetilde{u}(t)|^{\alpha} + q(t)\right)|\widetilde{u}(t)|\, ds \\
&\leq \sum_{t=1}^{T} p(t)\left(|\overline{u}|^{\alpha} + |\widetilde{u}(t)|^{\alpha}\right)|\widetilde{u}(t)| \quad + \sum_{t=1}^{T} q(t)|\widetilde{u}(t)| \\
&\leq |\overline{u}|^{\alpha} \left(\sum_{t=1}^{T} p^2(t)\right)^{1/2} \left(\sum_{t=1}^{T} |\widetilde{u}(t)|^2\right)^{1/2} + M_2\|\widetilde{u}\|_{\infty}^{\alpha+1} + M_3\|\widetilde{u}\|_{\infty} \\
&= M_1|\overline{u}|^{\alpha}\|\widetilde{u}\| + M_2\|\widetilde{u}\|_{\infty}^{\alpha+1} + M_3\|\widetilde{u}\|_{\infty} \\
&\leq \frac{1}{2\sqrt{a_1}}\|\widetilde{u}\|^2 + \frac{\sqrt{a_1}}{2} M_1^2 |\overline{u}|^{2\alpha} + M_2\|\widetilde{u}\|^{\alpha+1} + M_3\|\widetilde{u}\|.
\end{aligned} \tag{3.6}$$

In a similar way, we have

$$\sum_{t=1}^{T} (\nabla F(t, u_n(t)), \widetilde{u}_n(t)) \leq \frac{1}{2a_1\lambda_1}\|\widetilde{u}_n\|^2 + \frac{a_1\lambda_1}{2} M_1^2 |\overline{u}_n|^{2\alpha} + M_2\|\widetilde{u}_n\|^{\alpha+1} + M_3\|\widetilde{u}_n\|. \tag{3.7}$$

Let $u_n = \overline{u}_n + \widetilde{u}_n \in H_0 \bigoplus H_0^{\perp}$. From (2.12) and (3.7), we have

$$\sum_{t=1}^{T}(\Delta u_n(t), \Delta \widetilde{u}_n(t)) = \sum_{t=1}^{T}(\Delta \widetilde{u}_n(t), \Delta \widetilde{u}_n(t)) \geq \lambda_1 \|\widetilde{u}_n\|^2, \tag{3.8}$$

$$\begin{aligned}\sum_{t=1}^{T}(\Delta u_n(t), \Delta \widetilde{u}_n(t)) &= -\langle \varphi'(u_n), \widetilde{u}_n \rangle + \sum_{t=1}^{T}(\nabla F(t, u_n(t)), \widetilde{u}_n(t)) \\ &\leq \|\widetilde{u}_n\| + \frac{1}{2a_1\lambda_1}\|\widetilde{u}_n\|^2 + \frac{a_1\lambda_1}{2}M_1^2|\overline{u}_n|^{2\alpha} + M_2\|\widetilde{u}_n\|^{\alpha+1} + M_3\|\widetilde{u}_n\|.\end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\frac{\lambda_1}{2}\|\widetilde{u}_n\|^2 + C_4 \leq \frac{a_1\lambda_1}{2}M_1^2|\overline{u}_n|^{2\alpha}, \tag{3.10}$$

where $C_4 = \min_{s\in[0,+\infty)}\{((a_1\lambda_1^2-1)/2a_1\lambda_1)s^2 - M_2 s^{\alpha+1} - (1+M_3)s\}$. The fact that $a_1 > 1/\lambda_1^2 > 0$ implies that $-\infty < C_4 < 0$. So it follows from (3.10) that

$$\|\widetilde{u}_n\|^2 \leq a_1 M_1^2 |\overline{u}_n|^{2\alpha} - 2C_4\lambda_1, \tag{3.11}$$

and so

$$\|\widetilde{u}_n\| \leq \sqrt{a_1}M_1|\overline{u}_n|^{\alpha} + C_5, \tag{3.12}$$

where $C_5 > 0$. It follows from the boundedness of $\varphi(u_n)$, (2.11), (3.6), (3.11), and (3.12) that

$$\begin{aligned}C_3 \geq \varphi(u) &= -\frac{1}{2}\sum_{t=1}^{T}|\Delta u_n(t)|^2 + \sum_{t=1}^{T}[F(t, u_n(t)) - F(t,0)] \\ &= -\frac{1}{2}\sum_{t=1}^{T}|\Delta \widetilde{u}_n(t)|^2 + \sum_{t=1}^{T}[F(t, u_n(t)) - F(t, \overline{u}_n)] + \sum_{t=1}^{T}[F(t, \overline{u}_n) - F(t,0)] \\ &\geq -\frac{1}{2}\lambda_{[T/2]}\|\widetilde{u}_n\| - \frac{1}{2\sqrt{a_1}}\|\widetilde{u}_n\|^2 - \frac{\sqrt{a_1}}{2}M_1^2|\overline{u}_n|^{2\alpha} - M_2\|\widetilde{u}_n\|^{\alpha+1} - M_3\|\widetilde{u}_n\| \\ &\quad + \sum_{t=1}^{T}[F(t, \overline{u}_n) - F(t,0)]\end{aligned}$$

$$
\begin{aligned}
&\geq -\left(\frac{1}{2}\lambda_{[T/2]} + \frac{1}{2\sqrt{a_1}}\right)\left(a_1 M_1^2 |\overline{u}_n|^{2\alpha} - 2C_4\lambda_1\right) + \sum_{t=1}^{T}[F(t,\overline{u}_n) - F(t,0)] \\
&\quad - \frac{\sqrt{a_1}}{2} M_1^2 |\overline{u}_n|^{2\alpha} - M_2\left(\sqrt{a_1}M_1|\overline{u}_n|^{\alpha} + C_5\right)^{\alpha+1} - M_3\left(\sqrt{a_1}M_1|\overline{u}_n|^{\alpha} + C_5\right) \\
&\geq \left(-\frac{1}{2}\lambda_{[T/2]} a_1 M_1^2 - \sqrt{a_1}M_1^2\right)|\overline{u}_n|^{2\alpha} + \lambda_{[T/2]}C_4\lambda_1 + \frac{C_4\lambda_1}{\sqrt{a_1}} \\
&\quad - M_2\left(2^{\alpha}\left(\sqrt{a_1}M_1\right)^{\alpha+1}|\overline{u}_n|^{\alpha(\alpha+1)} + 2^{\alpha}C_5^{\alpha+1}\right) - M_1M_3\sqrt{a_1}|\overline{u}_n|^{\alpha} - M_3C_5 \\
&\quad + \sum_{t=1}^{T}[F(t,\overline{u}_n) - F(t,0)] \\
&= |\overline{u}_n|^{2\alpha}\left[|\overline{u}_n|^{-2\alpha}\sum_{t=1}^{T}F(t,\overline{u}_n) - \left(\frac{1}{2}\lambda_{[T/2]}a_1 + \sqrt{a_1}\right)M_1^2 - M_1M_3\sqrt{a_1}|\overline{u}_n|^{-\alpha}\right. \\
&\qquad\qquad \left. - M_2 2^{\alpha}\left(\sqrt{a_1}M_1\right)^{\alpha+1}|\overline{u}_n|^{\alpha(\alpha-1)}\right] + \lambda_{[T/2]}C_4\lambda_1 + \frac{C_4\lambda_1}{\sqrt{a_1}} - M_3C_5 \\
&\quad - M_2 2^{\alpha}C_5^{\alpha+1} - \sum_{t=1}^{T}F(t,0).
\end{aligned}
\tag{3.13}
$$

Inequalities (3.5) and (3.13) imply that $\{\overline{u}_n\}$ is bounded. Hence, $\{\tilde{u}_n\}$ is bounded by (3.12), and then $\{u_n\}$ is bounded. Since H_T is finite dimensional, there exists a subsequence of $\{u_n\}$ convergent in H_T. Thus, we conclude that (PS) condition is satisfied.

In order to use the saddle point theorem [16, Theorem 4.6], we only need to verify the following conditions:

(I1) $\varphi(x) \to +\infty$ as $|x| \to \infty$ in H_0;

(I2) $\varphi(u) \to -\infty$ as $\|u\| \to \infty$ in $H_0^{\perp}$.

In fact, from (F3)′, we have

$$\sum_{t=1}^{T}F(t,x) \longrightarrow +\infty \quad \text{as } |x| \longrightarrow \infty \text{ in } H_0. \tag{3.14}$$

For any $x \in H_0$, since $\sum_{t=1}^{T}|\Delta x|^2 = 0$, we have

$$\varphi(x) = \sum_{t=1}^{T}[F(t,x) - F(t,0)]. \tag{3.15}$$

It follows from (3.14) and the above inequality that

$$\varphi(x) \longrightarrow +\infty \quad \text{as } |x| \longrightarrow \infty \text{ in } H_0. \tag{3.16}$$

Thus (I1) is easy to verify.

Next, for all $u \in H_0^{\perp}$, from (F2)$'$ and (2.6), we have

$$\begin{aligned}\left|\sum_{t=1}^{T}(F(t,u(t)) - F(t,0))\right| &= \left|\sum_{t=1}^{T}\int_0^1 (\nabla F(t,su(t)),u(t))ds\right| \\ &\leq \sum_{t=1}^{T}p(t)|u(t)|^{\alpha+1} + \sum_{t=1}^{T}q(t)|u(t)| \\ &\leq M_2\|u\|_\infty^{\alpha+1} + M_2\|u\|_\infty \\ &\leq M_2\|u\|^{\alpha+1} + M_2\|u\|.\end{aligned} \tag{3.17}$$

By (2.7), (2.12), and (3.17), we obtain

$$\begin{aligned}\varphi(u) &= -\frac{1}{2}\sum_{t=1}^{T}|\Delta u(t)|^2 + \sum_{t=1}^{T}[F(t,u(t)) - F(t,0)] \\ &\leq -\frac{1}{2}\lambda_1\|u\|^2 + M_2\|u\|^{\alpha+1} + M_2\|u\|.\end{aligned} \tag{3.18}$$

Since $\lambda_1 > 0$ and $\alpha \in [0,1)$, we have $\varphi(u) \to -\infty$ as $\|u\| \to \infty$ in $H_0^{\perp}$. The proof of Theorem 3.1 is complete. □

Proof of Theorem 3.2. By (3.2) and (F4), we can choose an $a_2 \in \mathbb{R}$ such that

$$a_2 > \frac{1}{\lambda_1} > 0, \tag{3.19}$$

$$\liminf_{|x|\to\infty}|x|^{-2}\sum_{t=1}^{T}F(t,x) > \left[\left(\frac{1}{2}\lambda_{[T/2]} + \frac{1}{2\sqrt{a_2}} + \frac{1}{2}M_2\right)\frac{a_2}{\lambda_1 - M_2} + \frac{\sqrt{a_2}}{2}\right]M_1^2. \tag{3.20}$$

It follows from (F2)$'$ with $\alpha = 1$, (2.6), Hölder inequality, and Young inequality that

$$\begin{aligned}\left|\sum_{t=1}^{T}(F(t,u(t)) - F(t,\overline{u}))\right| &= \left|\sum_{t=1}^{T}\int_0^1 (\nabla F(t,\overline{u} + s\widetilde{u}(t)),\widetilde{u}(t))ds\right| \\ &\leq \sum_{t=1}^{T}\int_0^1 |(\nabla F(t,\overline{u} + s\widetilde{u}(t)),\widetilde{u}(t))|ds \\ &\leq \sum_{t=1}^{T}\int_0^1 p(t)(|\overline{u}| + s|\widetilde{u}(t)|)|\widetilde{u}(t)|ds + \sum_{t=1}^{T}\int_0^1 q(t)|\widetilde{u}(t)|ds \\ &= \sum_{t=1}^{T}p(t)\left(|\overline{u}| + \frac{1}{2}|\widetilde{u}(t)|\right)|\widetilde{u}(t)| + \sum_{t=1}^{T}q(t)|\widetilde{u}(t)|\end{aligned}$$

$$\begin{aligned}
&\le |\overline{u}|\left(\sum_{t=1}^{T} p^2(t)\right)^{1/2}\left(\sum_{t=1}^{T}|\widetilde{u}(t)|^2\right)^{1/2} + \frac{1}{2}\|\widetilde{u}\|_\infty^2 \sum_{t=1}^{T} p(t) + \|\widetilde{u}\|_\infty \sum_{t=1}^{T} q(t) \\
&= M_1|\overline{u}|\|\widetilde{u}\| + \frac{M_2}{2}\|\widetilde{u}\|_\infty^2 + M_3\|\widetilde{u}\|_\infty \\
&\le \left(\frac{1}{2\sqrt{a_2}} + \frac{M_2}{2}\right)\|\widetilde{u}\|^2 + \frac{\sqrt{a_2}}{2}M_1^2|\overline{u}|^2 + M_3\|\widetilde{u}\|.
\end{aligned} \tag{3.21}$$

In a similar way, we have

$$\sum_{t=1}^{T}(\nabla F(t,u_n(t)),\widetilde{u}_n(t)) \le \left(\frac{1}{2a_2} + \frac{M_2}{2}\right)\|\widetilde{u}_n\|^2 + \frac{a_2}{2}M_1^2|\overline{u}_n|^2 + M_3\|\widetilde{u}_n\|. \tag{3.22}$$

From (3.8) and (3.22), we have

$$\begin{aligned}
\lambda_1\|\widetilde{u}_n\|^2 &\le \sum_{t=1}^{T}(\Delta u_n(t),\Delta\widetilde{u}_n(t)) \\
&= -\langle\varphi'(u_n),\widetilde{u}_n\rangle + \sum_{t=1}^{T}(\nabla F(t,u_n(t)),\widetilde{u}_n(t)) \\
&\le \|\widetilde{u}_n\| + \left(\frac{1}{2a_2} + \frac{M_2}{2}\right)\|\widetilde{u}_n\|^2 + \frac{a_2}{2}M_1^2|\overline{u}_n|^2 + M_3\|\widetilde{u}_n\|.
\end{aligned} \tag{3.23}$$

It follow from (3.23) that

$$\frac{1}{2}(\lambda_1 - M_2)\|\widetilde{u}_n\|^2 + C_6 \le \frac{a_2}{2}M_1^2|\overline{u}_n|^2, \tag{3.24}$$

where $C_6 = \min_{s\in[0,+\infty)}\{((\lambda_1 a_2 - 1)/2a_2)s^2 - (1 + M_3)s\}$. The fact that $a_2 > 1/\lambda_1 > 0$ implies that $-\infty < C_6 < 0$. So it follows from (3.24) that

$$\|\widetilde{u}_n\|^2 \le \frac{a_2}{\lambda_1 - M_2}M_1^2|\overline{u}_n|^2 - \frac{2C_6}{\lambda_1 - M_2}, \tag{3.25}$$

and so

$$\|\widetilde{u}_n\| \le \sqrt{\frac{a_2}{\lambda_1 - M_2}}M_1|\overline{u}_n| + C_7, \tag{3.26}$$

where $C_7 > 0$. It follows from the boundedness of $\varphi(u_n)$, (2.11), (3.21), (3.25), and (3.26) that

$$
\begin{aligned}
C_3 \geq \varphi(u) &= -\frac{1}{2}\sum_{t=1}^{T}|\Delta u_n(t)|^2 + \sum_{t=1}^{T}[F(t,u_n(t)) - F(t,0)] \\
&= -\frac{1}{2}\sum_{t=1}^{T}|\Delta \widetilde{u}_n(t)|^2 + \sum_{t=1}^{T}[F(t,u_n(t)) - F(t,\overline{u}_n)] + \sum_{t=1}^{T}[F(t,\overline{u}_n) - F(t,0)] \\
&\geq -\frac{1}{2}\lambda_{[T/2]}\|\widetilde{u}_n\|^2 - \left(\frac{1}{2\sqrt{a_2}} + \frac{M_2}{2}\right)\|\widetilde{u}_n\|^2 - \frac{\sqrt{a_2}}{2}M_1^2|\overline{u}_n|^2 - M_3\|\widetilde{u}_n\| \\
&\quad + \sum_{t=1}^{T}[F(t,\overline{u}_n) - F(t,0)] \\
&\geq -\left(\frac{1}{2}\lambda_{[T/2]} + \frac{1}{2\sqrt{a_2}} + \frac{M_2}{2}\right)\left(\frac{a_2}{\lambda_1 - M_2}M_1^2|\overline{u}_n|^2 - \frac{2C_6}{\lambda_1 - M_2}\right) \\
&\quad - \frac{\sqrt{a_2}}{2}M_1^2|\overline{u}_n|^2 - M_3\left(\sqrt{\frac{a_2}{\lambda_1 - M_2}}M_1|\overline{u}_n| + C_7\right) + \sum_{t=1}^{T}[F(t,\overline{u}_n) - F(t,0)] \\
&= |\overline{u}_n|^2\left[|\overline{u}_n|^{-2\alpha}\sum_{t=1}^{T}F(t,\overline{u}_n) - \left(\frac{1}{2}\lambda_{[T/2]} + \frac{1}{2\sqrt{a_2}} + \frac{1}{2}M_2\right)\frac{a_2}{\lambda_1 - M_2}M_1^2\right. \\
&\qquad\quad \left. - \frac{\sqrt{a_2}}{2}M_1^2 - \sqrt{\frac{a_2}{\lambda_1 - M_2}}M_1M_3|\overline{u}_n|^{-1}\right] + \left(\lambda_{[T/2]} + \frac{1}{\sqrt{a_2}} + M_2\right)\frac{C_6}{\lambda_1 - M_2} \\
&\quad - M_3C_7 - \sum_{t=1}^{T}F(t,0).
\end{aligned} \tag{3.27}
$$

Inequalities (3.20) and (3.27) imply that $\{\overline{u}_n\}$ is bounded. Hence, $\{\widetilde{u}_n\}$ is bounded by (3.26), and then $\{u_n\}$ is bounded. Since H_T is finite dimensional, there exists a subsequence of $\{u_n\}$ convergent in H_T. Thus, we conclude that (PS) condition is satisfied.

In the following, we prove that φ satisfies (I1) and (I2). In fact, from (F4), we have

$$\sum_{t=1}^{T}F(t,x) \longrightarrow +\infty \text{ as } \quad |x| \longrightarrow \infty \text{ in } H_0. \tag{3.28}$$

It follows from (3.27) and $\sum_{t=1}^{T}|\Delta x|^2 = 0$ that

$$\varphi(x) = \sum_{t=1}^{T}[F(t,x) - F(t,0)] \longrightarrow +\infty \quad \text{as } |x| \longrightarrow \infty \text{ in } H_0. \tag{3.29}$$

Thus (I1) is easy to verify.

Next, for all $u \in H_0^{\perp}$, from (F2)$'$ with $\alpha = 1$ and (2.6), we have

$$\begin{aligned}\left|\sum_{t=1}^{T}(F(t,u(t)) - F(t,0))\right| &= \left|\sum_{t=1}^{T}\int_0^1 (\nabla F(t, su(t)), u(t))ds\right| \\ &\leq \frac{1}{2}\sum_{t=1}^{T} p(t)|u(t)|^2 + \sum_{t=1}^{T} q(t)|u(t)| \\ &\leq \frac{1}{2}M_2\|u\|_\infty^2 + M_3\|u\|_\infty \\ &\leq \frac{1}{2}M_2\|u\|^2 + M_3\|u\|.\end{aligned} \tag{3.30}$$

By (2.7), (2.12), and (3.30), we obtain

$$\begin{aligned}\varphi(u) &= -\frac{1}{2}\sum_{t=1}^{T}|\Delta u(t)|^2 + \sum_{t=1}^{T}[F(t,u(t)) - F(t,0)] \\ &\leq -\frac{1}{2}\lambda_1\|u\|^2 + \frac{M_2}{2}\|u\|^2 + M_2\|u\|.\end{aligned} \tag{3.31}$$

Since $\lambda_1 > M_2$, we have $\varphi(u) \to -\infty$ as $\|u\| \to \infty$ in $H_0^{\perp}$. The proof of Theorem 3.2 is complete. □

4. Examples

In this section, we give two examples to illustrate our results.

Example 4.1. Let

$$F(t,x) = \sin\frac{2\pi t}{T}|x|^{7/4} + \left(\sin\frac{2\pi t}{T} + 1\right)|x|^{3/2} + (h(t), x), \tag{4.1}$$

where $h : \mathbb{Z}[1,T] \to \mathbb{R}^N$ and $h(t+T) = h(t)$. It is easy to see that

$$\begin{aligned}|\nabla F(t,x)| &\leq \frac{7}{4}\left|\sin\frac{2\pi t}{T}\right||x|^{3/4} + \frac{3}{2}\left|\sin\frac{2\pi t}{T} + 1\right||x|^{1/2} + |h(t)| \\ &\leq \frac{7}{4}\left(\left|\sin\frac{2\pi t}{T}\right| + \varepsilon\right)|x|^{3/4} + a(\varepsilon) + |h(t)|, \quad \forall (t,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N,\end{aligned} \tag{4.2}$$

where $\varepsilon > 0$ and $a(\varepsilon)$ is a positive constant and is dependent on ε. It is easy to see that $F(t,x)$ satisfies (F1). From (4.2), we can let p, q, and α be

$$p(t) = \frac{7}{4}\left(\left|\sin\frac{2\pi t}{T}\right| + \varepsilon\right), \qquad q(t) = a(\varepsilon) + |h(t)|, \qquad \alpha = \frac{3}{4}, \tag{4.3}$$

which shows that (F2)′ is satisfied. Moreover, if we let $T = 2$, then we have

$$\liminf_{|x|\to+\infty} |x|^{-2\alpha} \sum_{t=1}^{T} F(t,x) = 2,$$
$$\lambda_1 = \lambda_{[T/2]} = 4, \quad \frac{\lambda_{[T/2]} + 2\lambda_1}{2\lambda_1^2} \sum_{t=1}^{T} p^2(t) = \frac{147}{128} \sum_{t=1}^{T} \left(\left| \sin \frac{2\pi t}{T} \right| + \varepsilon \right)^2 = \frac{147}{128}\varepsilon^2. \tag{4.4}$$

If we let $\varepsilon^2 < 256/147$, then we obtain

$$\liminf_{|x|\to+\infty} |x|^{-2\alpha} \sum_{t=1}^{T} F(t,x) = 2 > \frac{147}{128}\varepsilon^2 = \frac{\lambda_{[T/2]} + 2\lambda_1}{2\lambda_1^2} \sum_{t=1}^{T} p^2(t), \tag{4.5}$$

which shows that (F3)′ holds. Then from Theorem 3.1, problem (1.1) has at least one periodic solution with period T.

Example 4.2. Let

$$F(t,x) = \frac{1}{4}\left(\sin \frac{2\pi t}{T} + \frac{1}{2} \right)|x|^2 + \sin\left(\frac{2\pi t}{T} \right)|x|^{3/2} + (h(t), x), \tag{4.6}$$

where $h : \mathbb{Z}[1,T] \to \mathbb{R}^N$ and $h(t+T) = h(t)$. It is easy to see that $F(t,x)$ satisfies (F1) and

$$\begin{aligned} |\nabla F(t,x)| &\leq \frac{1}{2}\left| \sin \frac{2\pi t}{T} + \frac{1}{2} \right||x| + \frac{3}{2}\left| \sin \frac{2\pi t}{T} \right||x|^{1/2} + |h(t)| \\ &\leq \frac{1}{2}\left(\left| \sin \frac{2\pi t}{T} + \frac{1}{2} \right| + \varepsilon \right)|x| + b(\varepsilon) + |h(t)|, \quad \forall (t,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N, \end{aligned} \tag{4.7}$$

where $\varepsilon > 0$ and $b(\varepsilon)$ is a positive constant and is dependent on ε. The above shows that (F2)′ holds with $\alpha = 1$ and

$$p(t) = \frac{1}{2}\left(\left| \sin \frac{2\pi t}{T} + \frac{1}{2} \right| + \varepsilon \right), \quad q(t) = b(\varepsilon) + |h(t)|. \tag{4.8}$$

Let $T = 2$, then $\lambda_0 = 0$, $\lambda_1 = \lambda_{[T/2]} = 4$. Observe that

$$\begin{aligned} |x|^{-2} \sum_{t=1}^{T} F(t,x) &= |x|^{-2} \sum_{t=1}^{T} \left(\frac{1}{4}\left(\sin \frac{2\pi t}{T} + \frac{1}{2} \right)|x|^2 + \sin\left(\frac{2\pi t}{T} \right)|x|^{3/2} + (h(t), x) \right) \\ &= \frac{1}{4} + \left(\sum_{t=1}^{T} h(t), |x|^{-2}x \right). \end{aligned} \tag{4.9}$$

On the other hand, we have

$$\sum_{t=1}^{T} p(t) = \sum_{t=1}^{T} \frac{1}{2}\left(\left|\sin\frac{2\pi t}{T} + \frac{1}{2}\right| + \varepsilon\right) = \frac{1}{2} + \varepsilon,$$
$$\sum_{t=1}^{T} p^2(t) = \sum_{t=1}^{T} \frac{1}{4}\left(\left|\sin\frac{2\pi t}{T} + \frac{1}{2}\right| + \varepsilon\right)^2 = \frac{1}{2}\left(\frac{1}{2} + \varepsilon\right)^2. \tag{4.10}$$

We can choose ε sufficiently small such that $\sum_{t=1}^{T} p(t) < 4$ and

$$\begin{aligned}\liminf_{|x|\to+\infty} |x|^{-2}\sum_{t=1}^{T} F(t,x) = \frac{1}{4} &> \frac{27-2\varepsilon}{16(7-\varepsilon)}\left(\frac{1}{2}+\varepsilon\right)^2 \\ &= \frac{\lambda_{[T/2]} + \lambda_1^{1/2} + \sum_{t=1}^{T} p(t) \ + \lambda_1^{1/2}\left(\lambda_1 - \sum_{t=1}^{T} p(t)\right)}{2\lambda_1\left(\lambda_1 - \sum_{t=1}^{T} p(t)\ \right)} \sum_{t=1}^{T} p^2(t),\end{aligned} \tag{4.11}$$

which shows that (F4) holds. Then from Theorem 3.2, problem (1.1) has at least one periodic solution with period T.

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