Research Article

Dynamic Behaviors of a Harvesting Leslie-Gower Predator-Prey Model

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A Leslie-Gower predator-prey model incorporating harvesting is studied. By constructing a suitable Lyapunov function, we show that the unique positive equilibrium of the system is globally stable, which means that suitable harvesting has no influence on the persistent property of the harvesting system. After that, detailed analysis about the influence of harvesting is carried out, and an interesting finding is that under some suitable restriction, harvesting has no influence on the final density of the prey species, while the density of predator species is strictly decreasing function of the harvesting efforts. For the practical significance, the economic profit is considered, sufficient conditions for the presence of bionomic equilibrium are given, and the optimal harvesting policy is obtained by using the *Pontryagin's* maximal principle. At last, an example is given to show that the optimal harvesting policy is realizable.

1. Introduction

Leslie [1, 2] introduced the following predator-prey model, where the "carrying capacity" of the predator's environment is proportional to the number of prey:

$$\frac{dH}{dt} = (r_1 - a_1 P - b_1 H)H, \qquad \frac{dP}{dt} = \left(r_2 - a_2 \frac{P}{H}\right)P, \tag{1.1}$$

where H and P are the density of prey species and the predator species at time t, respectively. The above system admits an unique coexisting fixed point

$$H_* = \frac{r_1 a_2}{a_1 r_2 + a_2 b_1}, \qquad P_* = \frac{r_1 r_2}{a_1 r_2 + a_2 b_1}. \tag{1.2}$$

Recently, by constructing a suitable Lyapunov function, Korobeinikov [3] showed the positive equilibrium is globally stable; consequently, the system could not admits limit cycle. Such an finding is very interesting, since for predator-prey system incorporating Holling-type II or III functional response, limit cycle exists [4, 5]. After the work of Korobeinikov, many scholars have done works on Leslie-type predator prey ecosystem. Aguirre et al. [6] showed that Leslie-Gower predator-prey model with additive Allee effect is possible to admit two limit cycles; Aziz-Alaoui and Daher Okiye [7] argued that a suitable predator prey model should incorporate some kind of functional response, they proposed a predator-prey model with modified Leslie-Gower and Holling-type II schemes, they investigated the boundedness and global stability of the system; Nindjin et al. [8] further incorporated the time delay to the system considered in [7], and they showed that time delay plays important role on the dynamic behaviors of the system; Yafia et al. [9] studied the limit cycle bifurcated from time delay; Nindjin and Aziz-Alaoui [10], and Aziz-Alaoui [11] studied the dynamic behaviors of three Leslie-Gower-type species food chain system; Chen et al. [12] incorporated a prey refuge to system (1.1) and showed that the refuge has no influence on the persistent property of the system; Some scholars argued that nonautonomous case are more realistic if one consider the influence of seasonal effect of the environment. Huo and Li [13] studied the periodic solution of the nonautonomous case Leslie-Gower predator-prey system; Gakkhar and Singh [14] studied a Leslie-Gower predator-prey system with seasonally varying parameters; Song and Li [15] further considered the influence of impulsive effect. For more works on predator-prey ecosystem, one could refer to [4–27] and the references cited therein.

As was pointed out by Makinde [28]: "From the point of view of human needs, the exploitation of biological resources and harvesting of populations are commonly practiced in fishery, forestry, and wildlife management. There is a wide range of interest in the use of bioeconomic models to gain insight into the scientific management of renewable resources like fisheries and forestries." Though there are numerous works on predator-prey system incorporating the harvesting, to this day, still no scholar studies the system (1.1) under the assumption of the harvesting on prey and predator species. In this paper, we assume that the predator and prey species in the model is both of commercial importance and they are subjected to constant effort harvesting with c_1 and c_2, two parameters that measures the effort being spent by a harvesting agency. Thus, we formulating the system as follows:

$$\frac{dH}{dt} = (r_1 - a_1 P - b_1 H)H - c_1 H, \qquad \frac{dP}{dt} = \left(r_2 - a_2 \frac{P}{H}\right)P - c_2 P, \qquad (1.3)$$

where H and P are the density of prey species and the predator species at time t, respectively. To ensure the sustainable development, which means that we try to control the prey and predator species densities in a controllable range, but not to perish the species, it's natural to assume that $0 < c_i < r_i$, $i = 1, 2$.

The rest of the paper is arranged as follows: we will study the stability property of positive equilibrium of system (1.3) in Section 2 and discuss the influence of the harvesting in Section 3. Bionomic equilibrium and optimal harvesting policy for system (1.3) are discussed in Section 4 and 5, respectively. An example of system (1.3) is given in Section 6 to show the feasibility of our results. We end this paper by a briefly discussion.

2. Stability Property of Positive Equilibrium

By simple computation, under the assumption $0 < c_i < r_i$, $i = 1, 2$, system (1.3) admits an unique positive equilibrium

$$H_{1*} = \frac{(r_1 - c_1)a_2}{a_1(r_2 - c_2) + a_2 b_1}, \qquad P_{1*} = \frac{(r_1 - c_1)(r_2 - c_2)}{a_1(r_2 - c_2) + a_2 b_1}. \tag{2.1}$$

Obviously, (H_{1*}, P_{1*}) satisfies the equalities

$$r_1 - c_1 - b_1 H_{1*} - a_1 P_{1*} = 0, \qquad r_2 - c_2 = a_2 \frac{P_{1*}}{H_{1*}}. \tag{2.2}$$

Our result about the local stability property of this equilibrium is stated as follows.

Theorem 2.1. *The positive equilibrium* (H_{1*}, P_{1*}) *of system* (1.3) *is locally asymptotically stable.*

Proof. The variational matrix $J^*(H, P)$ of the system (1.3) is given by

$$\begin{pmatrix} r_1 - c_1 - a_1 P - 2b_1 H & -a_1 H \\ a_2 \dfrac{P^2}{H^2} & r_2 - c_2 - 2a_2 \dfrac{P}{H} \end{pmatrix}. \tag{2.3}$$

So, the characteristic equation for $J^*(H^*, P^*)$ is given by $\lambda^2 + a\lambda + b = 0$, where

$$a = \frac{a_2 b_1(r_1 - c_1)}{a_1(r_2 - c_2 + a_2 b_1)} + r_2 - c_2 (> 0), \qquad b = (r_1 - c_1)(r_2 - c_2)(> 0). \tag{2.4}$$

It is clear that the roots of the characteristic equation are negative or have negative real parts. Hence, the unique positive equilibrium of system (H_{1*}, P_{1*}) is stable. This completes the proof of the Theorem 2.1.

Concerned with the global stability property of the positive equilibrium, we have the following. □

Theorem 2.2. *The positive equilibrium* (H_{1*}, P_{1*}) *of system* (1.3) *is globally stable.*

Definition 2.3. System (1.3) is called permanent if for any positive solution $(H(t), P(t))^T$ of system (1.3) there exist positive constants m_i, M_i, $i = 1, 2$, which are independent of the solution of the system, such that

$$m_1 \leq \liminf_{t \to +\infty} H(t) \leq \limsup_{t \to +\infty} H(t) \leq M_1,$$

$$m_2 \leq \liminf_{t \to +\infty} P(t) \leq \limsup_{t \to +\infty} P(t) \leq M_2. \tag{2.5}$$

Theorem 2.2 shows that

$$\lim_{t \to +\infty} H(t) = H_{1*} > 0, \qquad \lim_{t \to +\infty} P(t) = P_{1*} > 0. \tag{2.6}$$

Noticing that H_{1*} and P_{1*} are only dependent on the coefficients of the system (1.3) and independent of the solution of system (1.3). Thus, (2.6) clearly shows that suitable harvesting (more precisely, with restriction $0 < c_i < r_i, i = 1, 2$) has no influence on the persistent property of the system.

Proof of Theorem 2.2. We will adapt the idea of Korobeinikov [3] to prove Theorem 2.2. More precisely, we construct the following Lyapunov function:

$$V(H, P) = \ln \frac{H}{H_{1*}} + \frac{H_{1*}}{H} - 1 + \frac{a_1 H_{1*}}{a_2} \left(\ln \frac{P}{P_{1*}} + \frac{P_{1*}}{P} - 1 \right). \tag{2.7}$$

Obviously, $V(H, P)$ is well defined and continuous for all $H, P > 0$. By simple computation, we have

$$\frac{\partial V}{\partial H} = \frac{1}{H} \left(1 - \frac{H_{1*}}{H} \right), \qquad \frac{\partial V}{\partial P} = \frac{a_1 H_{1*}}{a_2 P} \left(1 - \frac{P_{1*}}{P} \right). \tag{2.8}$$

Equation (2.8) shows that the positive equilibrium (H_{1*}, P_{1*}) is the only extremum of the function $V(H, P)$ in the positive quadrant. Noting that

$$\frac{\partial^2 V}{\partial H^2} = \frac{1}{H^2} \left(-1 + \frac{2H_{1*}}{H} \right), \qquad \frac{\partial^2 V}{\partial P \partial H} = 0, \qquad \frac{\partial^2 V}{\partial P^2} = \frac{a_1 H_{1*}}{a_2 P^2} \left(-1 + \frac{2P_{1*}}{P} \right). \tag{2.9}$$

Therefore,

$$d^2 V|_{(H_{1*}, P_{1*})} = \frac{1}{H_{1*}} dH^2 + \frac{a_1 H_{1*}}{a_2 P_{1*}} dP^2 > 0. \tag{2.10}$$

Above analysis shows that (H_{1*}, P_{1*}) is the only minimum extremum of the function $V(H, P)$ in the positive quadrant. One could easily verify that

$$\lim_{H \to 0} V(H, P) = \lim_{P \to 0} V(H, P) = \lim_{H \to +\infty} V(H, P) = \lim_{P \to +\infty} V(H, P) = +\infty. \tag{2.11}$$

From(2.8) and (2.11), we can see that the positive equilibrium (H_{1*}, P_{1*}) is the global minimum, that is,

$$V(H, P) > V(H_{1*}, P_{1*}) = 0, \tag{2.12}$$

for all $H, P > 0$.

Calculating the derivative of V along the solution of the system (1.3), by using equalities (2.2), we have

$$
\begin{aligned}
\frac{dV}{dt} &= \frac{1}{H}\left(1 - \frac{H_{1*}}{H}\right)(r_1 - c_1 - b_1 H - a_1 P)H + \frac{a_1 H_{1*}}{a_2 P}\left(1 - \frac{P_{1*}}{P}\right) \cdot \left(r_2 - c_2 - a_2 \frac{P}{H}\right)P \\
&= \frac{H - H_{1*}}{H}(b_1 H_{1*} + a_1 P_{1*} - b_1 H - a_1 P) + \frac{a_1 H_{1*}}{a_2} \cdot \left(1 - \frac{P_{1*}}{P}\right) \cdot \left(a_2 \frac{P_{1*}}{H_{1*}} - a_2 \frac{P}{H}\right) \\
&= -\frac{b_1}{H}(H - H_{1*})^2 - \frac{a_1}{P}(P - P_{1*})^2.
\end{aligned}
\tag{2.13}
$$

Obviously, $(dV/dt) < 0$ strictly for all $H, P > 0$ except the positive equilibrium (H_{1*}, P_{1*}), where $(dV/dt) = 0$. Thus, $V(H, P)$ satisfies Lyapunov's asymptotic stability theorem, and the positive equilibrium (H_{1*}, P_{1*}) of system (1.3) is globally stable. This ends the proof of Theorem 2.2. □

Remark 2.4. With the restriction $0 < c_i < r_i$, $i = 1, 2$, system (1.3) always admits an unique positive equilibrium and from Theorems 2.1 and 2.2 we can see that this equilibrium is globally attractive, since it's stability property is not changed with the variation of parameter c_i, the system could not undergoes Hopf's bifurcation and there is no limit cycle of system (1.3) in $\mathbf{R}_+^2$. In fact, we can also prove this declare by using *Bendixson-Dulac* theorem.

Let

$$
\begin{aligned}
F(H, P) &\doteq (r_1 - a_1 P - b_1 H)H - c_1 H, \\
G(H, P) &\doteq \left(r_2 - a_2 \frac{P}{H}\right)P - c_2 P, \\
B(H, P) &\doteq \frac{1}{HP}.
\end{aligned}
\tag{2.14}
$$

Obviously, $F(H, P)$, $G(H, P)$, and $B(H, P) \in \mathbf{C}^1(\mathbf{R}_+^2)$.

Calculating from above equations we get

$$\frac{\partial BF}{\partial H} + \frac{\partial BG}{\partial P} = -\frac{b_1}{P} - \frac{a_2}{H^2} < 0, \quad (H, P) \in \mathbf{R}_+^2. \tag{2.15}$$

According to *Bendixson-Dulac* theorem, we know that there is no limit cycle of system (1.3) in $\mathbf{R}_+^2$.

3. The Influence of Harvesting

We will discuss this topic on three aspects.

(1) The case of only harvesting prey species

In this case,

$$H_{1*} = \frac{(r_1 - c_1)a_2}{a_1 r_2 + a_2 b_1}, \qquad P_{1*} = \frac{(r_1 - c_1)r_2}{a_1 r_2 + a_2 b_1}. \tag{3.1}$$

Obviously, H_{1*}, P_{1*} are all continuous differentiable function of parameter c_1 and

$$\frac{dH_{1*}}{dc_1} = \frac{-a_2}{a_1 r_2 + a_2 b_1} < 0, \qquad \frac{dP_{1*}}{dc_1} = \frac{-r_2}{a_1 r_2 + a_2 b_1} < 0. \tag{3.2}$$

The above inequalities show that H_{1*} and P_{1*} are both the strictly decreasing function of c_1, that is, increasing the capture rate of prey species leads to the decreasing of the density of both prey and predator species.

(2) The case of only harvesting predator species

In this case,

$$H_{1*} = \frac{r_1 a_2}{a_1(r_2 - c_2) + a_2 b_1}, \qquad P_{1*} = \frac{r_1(r_2 - c_2)}{a_1(r_2 - c_2) + a_2 b_1}. \tag{3.3}$$

that is, H_{1*}, P_{1*} are all continuous differentiable function of parameter c_2. Noticing that

$$\frac{dH_{1*}}{dc_2} = \frac{r_1 a_2 a_1}{\left(a_1(r_2 - c_2) + a_2 b_1\right)^2} > 0, \qquad \frac{dP_{1*}}{dc_2} = \frac{-r_1 a_2 b_1}{\left(a_1(r_2 - c_2) + a_2 b_1\right)^2} < 0. \tag{3.4}$$

It is easy to see that H_{1*} is the strictly increasing function of parameter c_2, while P_{1*} is the strictly decreasing function of c_2, that is, increasing the capture rate of predator species leads to the increasing the density of prey species and the decreasing of predator species.

(3) The case of harvesting predator and prey species together

In this case, it follows from (2.1) that H_{1*} and P_{1*} are all continuous differential functions of parameters c_i, $i = 1, 2$. Though we had made the assumption $0 < c_i < r_i$, it still not an easy thing to give an detailed analysis of all of the cases. Here, we only investigate the following problem, which seems very interesting.

Problem 3.1. *Is it possible to choose some suitable parameters c_i such that after the harvesting of predator and prey, the densities of prey species as $t \to +\infty$ still has no change? That is, $H_{1*} = H_*$. If this is possible, what about the dynamic behaviors of predator species in this case?*

The first part of the question is equivalent to say that in what case $H_{1*} = H_*$, that is,

$$\frac{(r_1 - c_1)a_2}{a_1(r_2 - c_2) + a_2 b_1} = \frac{r_1 a_2}{a_1 r_2 + a_2 b_1}. \tag{3.5}$$

Solving the above equality, we obtain

$$c_2 = \frac{a_1 r_2 + a_2 b_1}{r_1 a_1} \cdot c_1. \tag{3.6}$$

It means that with the suitable capture efforts (c_i which satisfy the equality (3.6)), prey species will converge to H_* as $t \to +\infty$.

Now substituting (3.6) into the second equality of (2.1), we have

$$P_{1*} = \frac{r_1^2 a_1 r_2 - r_1 c_1(a_1 r_2 + a_2 b_1)}{a_1(r_1 - c_1)(a_1 r_2 + a_2 b_1)}. \tag{3.7}$$

Obviously, P_{1*} is the continuous differential function of parameter c_1 and

$$\frac{dP_{1*}}{dc_1} = -\frac{b_1 a_2 r_1^2}{a_1(r_1 - c_1)^2(a_1 r_2 + a_2 b_1)} < 0, \tag{3.8}$$

that is, if the capture rate of predator and prey species satisfies (3.6), then increasing the harvesting of prey (and of predator) will lead to the finally decreasing of predator densities (P_{1*}).

4. Bionomic Equilibrium

This section is devoted to study the bionomic equilibrium of system (1.3) since it has the practical significance.

The term bionomic equilibrium is an amalgamation concepts of biological equilibrium and economic equilibrium. As we know, a biological equilibrium is given by $(dH/dt) = (dP/dt) = 0$. And the economic equilibrium is said to be achieved when the TR (total revenue obtained by selling the harvested predators H and P) equals TC (the total cost for the effort devoted to harvesting).

Some symbols should be given at first.

Let

p_1 is the price per unit biomass of the prey H,

p_2 is the price per unit biomass of the predator P,

q_1 is the fishing cost per unit effort of the prey H,

q_2 is the fishing cost per unit effort of the predator P.

Then, the economic rent (revenue at any time) is given by

$$N = TR - TC = (p_1 H - q_1)c_1 + (p_2 P - q_2)c_2 \stackrel{\text{def}}{=} N_1 + N_2, \tag{4.1}$$

where $N_1 \overset{\text{def}}{=} (p_1 H - q_1)c_1$, $N_2 \overset{\text{def}}{=} (p_2 P - q_2)c_2$, that is, N_1 and N_2 represent the net revenues for the population H and P, respectively.

For convenience, we take the price per unit biomass of the predators and the fishing cost per unit effort of the predators to be constant. So, the bionomic equilibrium is given by the following simultaneous equations

$$\frac{dH}{dt} = (r_1 - a_1 P - b_1 H)H - c_1 H, \tag{4.2}$$

$$\frac{dP}{dt} = \left(r_2 - a_2 \frac{P}{H}\right)P - c_2 P, \tag{4.3}$$

$$N = (p_1 H - q_1)c_1 + (p_2 P - q_2)c_2 = 0. \tag{4.4}$$

Since the price and the cost of the predators are not sure, we will consider the following cases in order to determine the bionomic equilibrium.

Case 4.1. If

$$\frac{q_1}{p_1} > H, \tag{4.5}$$

that is,

$$p_1 H - q_1 < 0 \tag{4.6}$$

holds, that is to say the total cost exceed the total revenue for the harvesting of prey, obviously, the prey harvesting will be stopped (i.e., $c_1 = 0$) and the predator harvesting remains operational if $p_2 P - q_2 > 0$.

Then, from (4.4), we have

$$P_{1\infty} = \frac{q_2}{p_2}. \tag{4.7}$$

Substituting it into (4.3), it follows that

$$H_{1\infty} = \frac{r_1 p_2 - a_1 q_2}{b_1 p_2}. \tag{4.8}$$

Again, substituting (4.7) and (4.8) into (4.2) leads to

$$c_{2\infty} = r_2 - a_2 \frac{p_{1\infty}}{H_{1\infty}} = r_2 - \frac{a_2 b_1 q_2}{r_1 p_2 - a_1 q_2}. \tag{4.9}$$

So, if $r_1 > a_2(q_2/p_2)$ and $r_2 > (a_2 b_1 q_2 / r_1 p_2 - a_1 q_2)$ hold together, we have the bionomic equilibrium $[H_{1\infty}, P_{1\infty}, 0, c_{2\infty}]$.

Case 4.2. If

$$\frac{q_2}{p_2} > P, \tag{4.10}$$

that is,

$$p_2 P - q_2 < 0 \tag{4.11}$$

holds, that is to say the total cost exceeds the total revenue for the harvesting of predator obviously, the prey harvesting will be stopped (i.e., $c_2 = 0$) and the predator harvesting remains operational if $p_1 H - q_1 > 0$.

Then, it follows from (4.4) that

$$H_{1\infty} = \frac{q_1}{p_1}. \tag{4.12}$$

Substituting (4.12) into (4.3), we obtain

$$P_{1\infty} = \frac{r_2 q_1}{a_2 p_1}. \tag{4.13}$$

Again, substituting (4.12) and (4.13) into (4.2) leads to

$$c_{1\infty} = r_1 - a_1 P_{1\infty} - b_1 H_{1\infty} = r_1 - \frac{(a_1 r_2 - a_2 b_1) q_1}{a_2 p_1}. \tag{4.14}$$

So, if $r_1 > ((a_1 r_2 - a_2 b_1) q_1 / a_2 p_1)$ hold, we have the bionomic equilibrium $[H_{1\infty}, P_{1\infty}, c_{1\infty}, 0]$.

Case 4.3. If

$$\frac{q_2}{p_2} > P, \qquad \frac{q_1}{p_1} > H, \tag{4.15}$$

that is,

$$p_2 P - q_2 < 0, \qquad p_1 H - q_1 < 0 \tag{4.16}$$

hold, then it is equivalent to say that the total cost exceeds the total revenue for two populations.

Obviously, the harvesting will be stopped, that is, $c_1 = 0$, $c_2 = 0$. In this case, there is no bionomic equilibrium.

Case 4.4. If

$$\frac{q_2}{p_2} < P, \qquad \frac{q_1}{p_1} < H, \tag{4.17}$$

that is,

$$p_2 P - q_2 > 0, \qquad p_1 H - q_1 > 0 \tag{4.18}$$

hold. In this case, the total revenue exceeds the total cost for two populations and the harvesting is in operational because it can bring profit for fishery.

From (4.4), we have

$$H_{1\infty} = \frac{q_1}{p_1}, \qquad P_{1\infty} = \frac{q_2}{p_2}. \tag{4.19}$$

Substituting above equalities into (4.2) and (4.3), it is easy to obtain

$$c_{1\infty} = r_1 - \frac{(a_1 p_2 q_1 - b_1 p_1 q_2)}{p_1 p_2}, \qquad c_{2\infty} = r_2 - \frac{a_2 p_1 q_2}{p_2 q_1}. \tag{4.20}$$

So, if

$$r_1 > \frac{(a_1 p_2 q_1 - b_1 p_1 q_2)}{p_1 p_2},$$

$$r_2 > \frac{a_2 p_1 q_2}{p_2 q_1} \tag{4.21}$$

hold together, we have the bionomic equilibrium $[H_{1\infty}, P_{1\infty}, c_{1\infty}, c_{2\infty}]$.

It is obviously that the bionomic equilibrium may exist if the intrinsic growth rates of two species exceed some value.

5. Optimal Harvesting Policy

In order to determine the optimal harvesting policy, we consider the present value J of a continuous time-stream of revenue

$$J = \int_0^\infty e^{-\delta t} \{(p_1 H - q_1) c_1(t) + (p_2 P - q_2) c_2(t)\} dt, \tag{5.1}$$

where δ denotes the instantaneous annual rate of discount and $c_i(t)$ $(i = 1, 2)$ are the control variables, which are subject to the assumption $0 < c_i(t) < r_i$, $i = 1, 2$.

Now, our objective is to maximize J subject to the state equations (1.3) by invoking *Pontryagin's* maximal principle.

The Hamiltonian for the problem is given at first

$$\overline{H} = e^{-\delta t} \{(p_1 H - q_1) c_1(t) + (p_2 P - q_2) c_2(t)\} + \lambda_1 [(r_1 - a_1 P - b_1 H) H - c_1 H]$$

$$+ \lambda_2 \left[\left(r_2 - a_2 \frac{P}{H} \right) P - c_2 P \right], \tag{5.2}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are the adjoint variables.

Obviously, the control variables c_1 and c_2 appear linearly in the Hamiltonian function $\overline{H}$. So, the conditions

$$\frac{\partial \overline{H}}{\partial c_i} = 0, \quad i = 1, 2 \tag{5.3}$$

are necessary for the singular control to be optimal.

Then, we have

$$\lambda_1 = e^{-\delta t}\left(p_1 - \frac{q_1}{H}\right), \quad \lambda_2 = e^{-\delta t}\left(p_2 - \frac{q_2}{P}\right), \tag{5.4}$$

that is

$$\lambda_1 e^{\delta t} = p_1 - \frac{q_1}{H}, \qquad \lambda_2 e^{\delta t} = p_2 - \frac{q_2}{P}. \tag{5.5}$$

Therefore, the shadow prices $e^{\delta t}\lambda_i(t)$ do not vary with time in the optimal equilibrium. Hence they remain bounded as $t \to \infty$.

By the maximal principle, the adjoint variables satisfy $d\lambda_1/dt = -\partial \overline{H}/\partial H$ and $d\lambda_2/dt = -(\partial \overline{H}/\partial P)$, for all $t \geq 0$, that is.

$$\frac{d\lambda_1}{dt} = -\left\{e^{-\delta t}p_1 c_1 + \lambda_1(r_1 - c_1 a_1 p_1 - 2b_1 H) - \lambda_2 a_2 \frac{P}{H^2}\right\},$$

$$\frac{d\lambda_2}{dt} = -\left\{e^{-\delta t}p_2 c_2 + \lambda_2\left(r_2 - c_2 - \frac{2a_2 P}{H}\right) - \lambda_1 a_1 H\right\}. \tag{5.6}$$

Substituting (5.4) and (5.5) into (5.6), we get

$$2a_2 p_2 P^2 + a_1 p_1 H^2 P + (p_2\delta - r_2 p_2 - a_1 q_1)HP - a_2 q_2 P - q_2\delta H = 0,$$

$$2b_1 p_1 H^3 + a_1 p_1 PH^2 + (p_1\delta - r_1 p_1 - b_1 q_1)H^2 - q_1\delta H - a_2 p_2 P + a_2 q_2 = 0. \tag{5.7}$$

From (5.7), we may find the positive values of (H_δ, P_δ).

Substituting (H_δ, P_δ) into (1.3), we get the equations as follows:

$$c_{1\delta} = r_1 - a_1 P_\delta - b_1 H_\delta,$$

$$c_{2\delta} = r_2 - a_2 \frac{P_\delta}{H_\delta}. \tag{5.8}$$

Then, we may have the optimal equilibrium effort levels $c_{1\delta}$ and $c_{2\delta}$, if

$$r_1 > a_1 P_\delta + b_1 H_\delta, \qquad r_2 > a_2 \frac{P_\delta}{H_\delta}. \tag{5.9}$$

It means that, under the harvesting for both prey and predator species, the optimal harvesting policy can be obtained only under the assumption that the intrinsic growth rates of two species exceed some values.

The optimal harvesting policy means the economic rent for harvesting will be the maximal in the future time, with the populations of the system are permanent.

6. Numerical Example

Let

$$r_1 = 1.6, \qquad a_1 = 0.7, \qquad b_1 = 0.3, \qquad p_1 = 0.4, \qquad q_1 = 0.2,$$
$$r_2 = 1, \qquad a_2 = 0.25, \qquad \delta = 1.25, \qquad p_2 = 0.5, \qquad q_2 = 0.2. \tag{6.1}$$

From Section 6, we know that the optimal harvesting policy can be obtained by substituting the result of (5.7) into (1.3). So, the corresponding equations of (5.7) are

$$0.25P^2 + 0.28H^2P - 0.015HP - 0.05P - 0.25H = 0,$$
$$0.24H^3 + 0.28PH^2 - 0.2H^2 - 0.25H - 0.125P + 0.05 = 0. \tag{6.2}$$

Solving (6.2) by using Maple, we get the results as follow:

$$\{H = 0.03875658339, \ P = 0.3212762700\},$$
$$\{H = 0.3442951974, \ P = -0.5444632570\},$$
$$\{H = 1.121042647, \ P = 0.6323992784\},$$
$$\{H = 0.8269704200 + 1.125716729I, \ P = 0.1137766719 - 1.326783285I\},$$
$$\{H = 0.8269704200 - 1.125716729I, \ P = 0.1137766719 + 1.326783285I\},$$
$$\{H = -0.7529121920 + 0.04647885793I, \ P = -0.2550446252 - 0.7947461329I\}. \tag{6.3}$$

From the above results, one could easily see that there is only one result ($H_\delta = 1.121042647$, $P_\delta = 0.6323992784$) meeting the condition

$$H_\delta > \frac{q_1}{p_1}(= 0.5), \qquad P_\delta > \frac{q_2}{p_2}(= 0.4). \tag{6.4}$$

In this case, the corresponding system of (1.3) is as follows:

$$\frac{dH}{dt} = (1.6 - 0.7P - 0.3H)H - c_1H, \qquad \frac{dP}{dt} = \left(1 - 0.25\frac{P}{H}\right)P - c_2P. \tag{6.5}$$

Substituting ($H_\delta = 1.121042647$, $P_\delta = 0.6323992784$) into (6.5), we obtain the optimal harvesting efforts

$$c_{1\delta} = 0.821008 < r_1 (= 1.6), \qquad c_{2\delta} = 0.858971 < r_2 (= 1). \tag{6.6}$$

7. Conclusion

A Leslie-Gower predator-prey model incorporating harvesting is studied in this paper. We first show that suitable harvesting has no influence on the persistent property of the harvesting system. After that, we try to give the detail analysis of harvesting on the dynamic behaviors of the system. Our study shows that for the system having both harvesting on predator and prey species, it admits some interesting phenomenon; maybe such a finding could be applied to help human improving the scientific management of renewable resources such as fisheries and forest trees. Then, for the practical significance, we consider the economic profit of the harvesting. The bionomic equilibrium and optimal harvesting policy are studied. The results show that the optimal harvesting policy may exist. Finally, an example is given to show that the optimal harvesting policy of system (1.3) is realizable.

Acknowledgments

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