Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 476381, 7 pages
doi:10.1155/2011/476381

Research Article

On the Values of the Weighted q-Zeta and L-Functions

T. Kim,[1] S. H. Lee,[1] Hyeon-Ho Han,[2] and C. S. Ryoo[3]

[1] *Division of General Education, Kwangwoon University, Seoul 139-701, Republic of Korea*
[2] *Department of Information display, Kwangwoon University, Seoul 139-701, Republic of Korea*
[3] *Department of Mathematics, Hannam University, Daejeon 306-791, Republic of Korea*

Correspondence should be addressed to T. Kim, tkkim@kw.ac.kr

Received 17 August 2011; Accepted 3 October 2011

Academic Editor: Binggen Zhang

Recently, the modified q-Bernoulli numbers and polynomials are introduced in (D. V. Dolgy et al., in press). These numbers are valuable to study the weighted q-zeta and L-functions. In this paper, we study the weighted q-zeta functions and weighted L-functions from the modified q-Bernoulli numbers and polynomials with weight α.

1. Introduction

Let $q \in \mathbb{C}$ with $|q| < 1$. The modified q-Bernoulli numbers and polynomials with weight α are defined by

$$\widetilde{B}_{0,q}^{(\alpha)} = \alpha \frac{q-1}{\log q}, \qquad \left(q^{\alpha}\widetilde{B}_{q}^{(\alpha)} + 1\right)^{n} - \widetilde{B}_{n,q}^{(\alpha)} = \begin{cases} \dfrac{\alpha}{[\alpha]_q} & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.1}$$

with the usual convention about replacing $(\widetilde{B}_{q}^{(\alpha)})^{n}$ by $\widetilde{B}_{n,q}^{(\alpha)}$ (see [1, 2]).

Throughout this paper, we use the notation of q-number as

$$[x]_q = \frac{1-q^{x}}{1-q}, \tag{1.2}$$

(see [1–14]).

From (1.1), we note that

$$\begin{aligned}\widetilde{B}_{n,q}^{(\alpha)} &= \frac{1}{(1-q^{\alpha})^{n}}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}\frac{\alpha l}{[\alpha l]_{q}} \\ &= \frac{1}{(1-q)^{n}[\alpha]_{q}^{n}}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}\frac{\alpha l}{[\alpha l]_{q}}.\end{aligned} \tag{1.3}$$

Let $\widetilde{F}_{q}^{(\alpha)}(t) = \sum_{n=0}^{\infty}\widetilde{B}_{n,q}^{(\alpha)}t^{n}/n!$, Then, by (1.3), we get

$$\widetilde{F}_{q}^{(\alpha)}(t) = \alpha\frac{q-1}{\log q}e^{(1/(1-q^{\alpha}))t} - \frac{\alpha t}{[\alpha]_{q}}\sum_{m=0}^{\infty}q^{\alpha m}e^{[m]_{q^{\alpha}}t}. \tag{1.4}$$

Let us define the modified q-Bernoulli polynomials with weight α as follows:

$$\widetilde{B}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n}\binom{n}{l}[x]_{q^{\alpha}}^{n-l}q^{\alpha l x}\widetilde{B}_{l,q}^{(\alpha)} = \left([x]_{q^{\alpha}} + q^{x\alpha}\widetilde{B}_{q}^{(\alpha)}\right)^{n}, \tag{1.5}$$

with the usual convention about replacing $(\widetilde{B}_{q}^{(\alpha)})^{n}$ by $\widetilde{B}_{n,q}^{(\alpha)}$ (see [1–13]).

From (1.5), we can derive the following equation:

$$\begin{aligned}\widetilde{B}_{n,q}^{(\alpha)}(x) &= \frac{1}{(1-q^{\alpha})^{n}}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}q^{\alpha l x}\frac{\alpha l}{[\alpha l]_{q}} \\ &= \frac{1}{(1-q)^{n}[\alpha]_{q}^{n}}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}q^{\alpha l x}\frac{\alpha l}{[\alpha l]_{q}},\end{aligned} \tag{1.6}$$

(see [2]).

Let $\widetilde{F}_{q}^{(\alpha)}(t,x) = \sum_{n=0}^{\infty}\widetilde{B}_{n,q}^{(\alpha)}(x)t^{n}/n!$, then, by (1.6), we get

$$\widetilde{F}_{q}^{(\alpha)}(t,x) = \alpha\frac{q-1}{\log q}e^{(1/(1-q^{\alpha}))t} - t\frac{\alpha}{[\alpha]_{q}}\sum_{m=0}^{\infty}q^{\alpha(m+x)}e^{[m+x]_{q^{\alpha}}t}. \tag{1.7}$$

In this paper, we consider the generalized q-Bernoulli numbers with weight α, and we study the weighted q-zeta function and q-analogue of L-function with weight α from the modified q-Bernoulli numbers and polynomials with weight α.

2. Weighted q-Zeta Function and Weighted q-L-Function

From (1.7), we note that

$$\widetilde{B}_{n,q}^{(\alpha)}(x) = \frac{\alpha}{(1-q)^{n}[\alpha]_{q}^{n}}\left(\frac{q-1}{\log q}\right) - \frac{n\alpha}{[\alpha]_{q}}\sum_{m=0}^{\infty}q^{\alpha(m+x)}[m+x]_{q^{\alpha}}^{n-1}. \tag{2.1}$$

For $n \in \mathbb{N}$, we have

$$-\frac{\widetilde{B}_{n,q}^{(\alpha)}(x)}{n} = \left(\frac{\alpha}{[\alpha]_q}\right)\left(\frac{1}{1-q^{\alpha}}\right)^{n-1}\left(\frac{1}{\log q}\right) + \frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty} q^{\alpha(m+x)}[m+x]_{q^{\alpha}}^{n-1}. \tag{2.2}$$

Let $\Gamma(s)$ be the gamma function, then we consider the following complex integral. For $s \in \mathbb{C}$,

$$\frac{1}{\Gamma(s)}\int_0^{\infty} \widetilde{F}_q^{(\alpha)}(-t,x)t^{s-2}dt = \frac{\alpha}{s-1}\frac{q-1}{\log q}\left(1-q^{\alpha}\right)^{s-1} + \frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty}\frac{q^{\alpha(m+x)}}{[m+x]_{q^{\alpha}}^{s}}, \tag{2.3}$$

where $x \neq 0, -1, -2, -3, \ldots$.

Now, we define the twisted Hurwitz's type q-zeta function as follows.

For $s \in \mathbb{C}$, define

$$\widetilde{\zeta}_q^{(\alpha)}(s,x) = \frac{\alpha}{[\alpha]_q}\frac{1}{1-s}\frac{\left(1-q^{\alpha}\right)^{s}}{\log q} + \frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty}\frac{q^{\alpha(m+x)}}{[m+x]_{q^{\alpha}}^{s}}, \tag{2.4}$$

where $x \neq 0, -1, -2, -3, \ldots$.

Note that $\widetilde{\zeta}_q^{(\alpha)}(s,x)$ is meromorphic function whole in complex s-plane except for $s = 1$.

From (2.3) and (2.4), we can derive the following equation:

$$\widetilde{\zeta}_q^{(\alpha)}(s,x) = \frac{1}{\Gamma(s)}\int_0^{\infty} \widetilde{F}_q^{(\alpha)}(-t,x)t^{s-2}dt. \tag{2.5}$$

By (1.7), (2.3), (2.4), (2.5), and Laurent series, we get

$$\widetilde{\zeta}_q^{(\alpha)}(1-k,x) = -\frac{\widetilde{B}_{k,q}^{(\alpha)}(x)}{k}, \tag{2.6}$$

where $k \in \mathbb{N}$.

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.1. *For $k \in \mathbb{N}$, one has*

$$\widetilde{\zeta}_q^{(\alpha)}(1-k,x) = -\frac{\widetilde{B}_{k,q}^{(\alpha)}(x)}{k}. \tag{2.7}$$

From (2.4), *one notes that*

$$\begin{aligned}\widetilde{\zeta}_q^{(\alpha)}(s,1) &= \frac{\alpha}{[\alpha]_q}\frac{1}{1-s}\frac{\left(1-q^{\alpha}\right)^{s}}{\log q} + \frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty}\frac{q^{\alpha(m+1)}}{[m+1]_{q^{\alpha}}^{s}} \\ &= \frac{\alpha}{[\alpha]_q}\frac{1}{1-s}\frac{\left(1-q^{\alpha}\right)^{s}}{\log q} + \frac{\alpha}{[\alpha]_q}\sum_{m=1}^{\infty}\frac{q^{\alpha m}}{[m]_{q^{\alpha}}^{s}}.\end{aligned} \tag{2.8}$$

Now, by (2.8)*, one defines the weighted q-zeta function as follows:*

$$\begin{aligned}\widetilde{\zeta}_q^{(\alpha)}(s) &= \frac{\alpha}{[\alpha]_q}\frac{1}{1-s}\frac{(1-q^{\alpha})^s}{\log q} + \frac{\alpha}{[\alpha]_q}\sum_{m=1}^{\infty}\frac{q^{\alpha m}}{[m]_{q^{\alpha}}^s}. \\ &= \widetilde{\zeta}_q^{(\alpha)}(s,1).\end{aligned} \tag{2.9}$$

For $k \in \mathbb{N}$, by (1.1) *and* (1.5)*, one gets*

$$\begin{aligned}\widetilde{\zeta}_q^{(\alpha)}(1-k) = \widetilde{\zeta}_q^{(\alpha)}(1-k,1) &= -\frac{\widetilde{B}_{k,q}^{(\alpha)}(1)}{k} \\ &= \begin{cases} -\left(\dfrac{\alpha}{[\alpha]_q} + \widetilde{B}_{1,q}^{(\alpha)}\right) & \text{if } k = 1, \\ -\dfrac{\widetilde{B}_{k,q}^{(\alpha)}}{k} & \text{if } k > 1. \end{cases}\end{aligned} \tag{2.10}$$

Therefore, by (2.10)*, one obtains the following corollary.*

Corollary 2.2. *For $k \in \mathbb{N}$, one has*

$$\widetilde{\zeta}_q^{(\alpha)}(1-k) = \begin{cases} -\left(\dfrac{\alpha}{[\alpha]_q} + \widetilde{B}_{1,q}^{(\alpha)}\right) & \text{if } k = 1, \\ -\dfrac{\widetilde{B}_{k,q}^{(\alpha)}}{k} & \text{if } k > 1. \end{cases} \tag{2.11}$$

Let χ be the Dirichlet's character with conductor $d \in \mathbb{N}$. Let us consider the generalized q-Bernoulli polynomials with weight α as follows:

$$\begin{aligned}\widetilde{F}_{q,\chi}^{(\alpha)}(t,x) &= \frac{\alpha}{[\alpha]_q}t\sum_{m=0}^{\infty}\chi(m)q^{\alpha(m+x)}e^{[m+x]_{q^{\alpha}}t} \\ &= \sum_{n=0}^{\infty}\widetilde{B}_{n,\chi,q}^{(\alpha)}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.12}$$

The sequence $\widetilde{B}_{n,\chi,q}^{(\alpha)}(x)$ will be called the nth generalized q-Bernoulli polynomials with weight α attached to χ.

In the special case, $x = 0$, $\widetilde{B}_{n,\chi,q}^{(\alpha)}(0) = \widetilde{B}_{n,\chi,q}^{(\alpha)}$ are called the nth generalized q-Bernoulli numbers with weight α attached to χ.

From (1.7) *and* (2.12)*, one notes that*

$$\widetilde{F}_{q,\chi}^{(\alpha)}(t,x) = \frac{1}{[d]_q}\sum_{a=0}^{d-1}\chi(a)\widetilde{F}_{q^d}^{(\alpha)}\left([d]_{q^{\alpha}}t, \frac{x+a}{d}\right). \tag{2.13}$$

Thus, by (2.13), *one gets*

$$\widetilde{B}_{n,\chi,q}^{(\alpha)}(x) = \frac{[d]_{q^{\alpha}}^{n}}{[d]_q}\sum_{a=0}^{d-1}\chi(a)\widetilde{B}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right). \tag{2.14}$$

Therefore, by (2.14), *one obtains the following theorem.*

Theorem 2.3. *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{B}_{n,\chi,q}^{(\alpha)}(x) = \frac{[d]_{q^{\alpha}}^{n}}{[d]_q}\sum_{a=0}^{d-1}\chi(a)\widetilde{B}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right). \tag{2.15}$$

In the special case, $x = 0$, one obtains the following corollary.

Corollary 2.4. *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{B}_{n,\chi,q}^{(\alpha)} = \frac{[d]_{q^{\alpha}}^{n}}{[d]_q}\sum_{a=0}^{d-1}\chi(a)\widetilde{B}_{n,q^d}^{(\alpha)}\left(\frac{a}{d}\right). \tag{2.16}$$

Let

$$\begin{aligned}\widetilde{F}_{q,\chi}^{(\alpha)}(t) &= \frac{\alpha}{[\alpha]_q}t\sum_{m=0}^{\infty}\chi(m)q^{\alpha m}e^{[m]_{q^{\alpha}}t}\\ &= \sum_{n=0}^{\infty}\widetilde{B}_{n,\chi,q}^{(\alpha)}\frac{t^n}{n!},\end{aligned} \tag{2.17}$$

then, by (2.12) *and* (2.17), *one easily gets*

$$\frac{\widetilde{B}_{n,\chi,q}^{(\alpha)}(d) - \widetilde{B}_{n,\chi,q}^{(\alpha)}}{n} = \frac{\alpha}{[\alpha]_q}\sum_{l=0}^{d-1}\chi(l)q^{\alpha l}[l]_{q^{\alpha}}^{n-1}. \tag{2.18}$$

For $s \in \mathbb{C}$, consider

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^{\infty}\widetilde{F}_{q,\chi}^{(\alpha)}(-t,x)t^{s-2}dt &= \frac{\alpha}{[\alpha]_q}\frac{1}{\Gamma(s)}\int_0^{\infty}\sum_{m=0}^{\infty}\chi(m)q^{\alpha(m+x)}e^{-[m+x]_{q^{\alpha}}t}t^{s-1}dt\\ &= \frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty}\frac{\chi(m)q^{\alpha(m+x)}}{[m+x]_{q^{\alpha}}^{s}}\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-y}y^{s-1}dy\\ &= \frac{\alpha}{[\alpha]_q}\sum_{m=0}^{\infty}\frac{\chi(m)q^{\alpha(m+x)}}{[m+x]_{q^{\alpha}}^{s}},\end{aligned} \tag{2.19}$$

where $x \neq 0, -1, -2, -3, \ldots$.

Now, one defines Hurwitz's type q-L-function with weight α as follows. For $s \in \mathbb{C}$,

$$\widetilde{L}_q^{(\alpha)}(s,\chi \mid x)(-t,x) = \frac{\alpha}{[\alpha]_q}\sum_{n=0}^{\infty}\frac{\chi(n)q^{(n+x)\alpha}}{[n+x]_{q^\alpha}^s}, \tag{2.20}$$

where $x \neq 0,-1,-2,-3,\ldots$.

From (2.19) *and* (2.20)*, one notes that*

$$\widetilde{L}_q^{(\alpha)}(s,\chi \mid x) = \frac{1}{\Gamma(s)}\int_0^{\infty}\widetilde{F}_{q,\chi}^{(\alpha)}(-t,x)t^{s-2}dt. \tag{2.21}$$

By (1.7) *and* (2.21) *and Laurent series, one obtains the following theorem.*

Theorem 2.5. *For $k \in \mathbb{N}$, one has*

$$\widetilde{L}_q^{(\alpha)}(1-k,\chi \mid x) = -\frac{\widetilde{B}_{k,\chi,q}^{(\alpha)}(x)}{k}. \tag{2.22}$$

In the special case, $x = 0$, $\widetilde{L}_q^{(\alpha)}(1-k,\chi \mid 0) = \widetilde{L}_q^{(\alpha)}(1-k,\chi)$ are called the q-L-function with weight α.

Let

$$\begin{aligned}\widetilde{F}_q^{(\alpha)}(s,a \mid F) &= \frac{\alpha}{[F]_q[\alpha]_q}\left(\sum_{\substack{m\equiv a(\mathrm{mod}\ F)\\ m>0}}^{\infty}\frac{q^{\alpha m}}{[m]_{q^\alpha}^s}+\frac{(1-q^\alpha)^s}{F(1-s)\log q}\right)\\ &= \frac{\alpha}{[F]_q[\alpha]_q}\left(\sum_{n=0}^{\infty}\frac{q^{\alpha(a+nF)}}{[a+nF]_{q^\alpha}^s}+\frac{(1-q^\alpha)^s}{F(1-s)\log q}\right)\\ &= \frac{[F]_{q^\alpha}}{[F]_q[F]_{q^\alpha}^s}\widetilde{\zeta}_{q^F}^{(\alpha)}\left(s,\frac{a}{F}\right),\end{aligned} \tag{2.23}$$

where a and F are positive integers with $0 < a < F$.

Then, by (2.23)*, one gets*

$$\widetilde{H}_q^{(\alpha)}(1-n,a \mid F) = -\frac{[F]_{q^\alpha}^n\widetilde{B}_{n,\chi,q}^{(\alpha)}(a/F)}{[F]_q n}, \quad n \geq 1, \tag{2.24}$$

and $\widetilde{H}_q^{(\alpha)}(s,a \mid F)$ has as simple pole as $s = 1$ with residue $(\alpha/[F]_q)((q-1)/\log q^F)$.

Let χ be the Dirichlet character with conductor F, then one easily sees that

$$\widetilde{L}_q^{(\alpha)}(s,\chi) = \sum_{a=1}^{F}\chi(a)\widetilde{H}_q^{(\alpha)}(s,a \mid F). \tag{2.25}$$

References

[1] T. Kim, "On the weighted q-Bernoulli numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 21, pp. 207–215, 2011.

[2] D. V. Dolgy, T. Kim, S. H. Lee, B. Lee, and S.-H. Rim, "A note on the modified q-Bernoulli numbers and polynomials with weight α," communicated.

[3] S. Araci, D. Erdal, and D.-J. Kang, "Some new properties on the q-Genocchi numbers and polynomials associated with q-Bernstein polynomials," *Honam Mathematical Journal*, vol. 33, pp. 261–270, 2011.

[4] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.

[5] T. Kim, "p-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients," *Russian Journal of Mathematical Physics*, vol. 15, no. 1, pp. 51–57, 2008.

[6] T. Kim, "Multiple p-adic L-function," *Russian Journal of Mathematical Physics*, vol. 13, no. 2, pp. 151–157, 2006.

[7] T. Kim, "Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 186–196, 2005.

[8] H. Ozden, I. N. Cangul, and Y. Simsek, "Remarks on q-Bernoulli numbers associated with Daehee numbers," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 1, pp. 41–48, 2009.

[9] L.-C. Jang, "On multiple generalized w-Genocchi polynomials and their applications," *Mathematical Problems in Engineering*, vol. 2010, Article ID 316870, 8 pages, 2010.

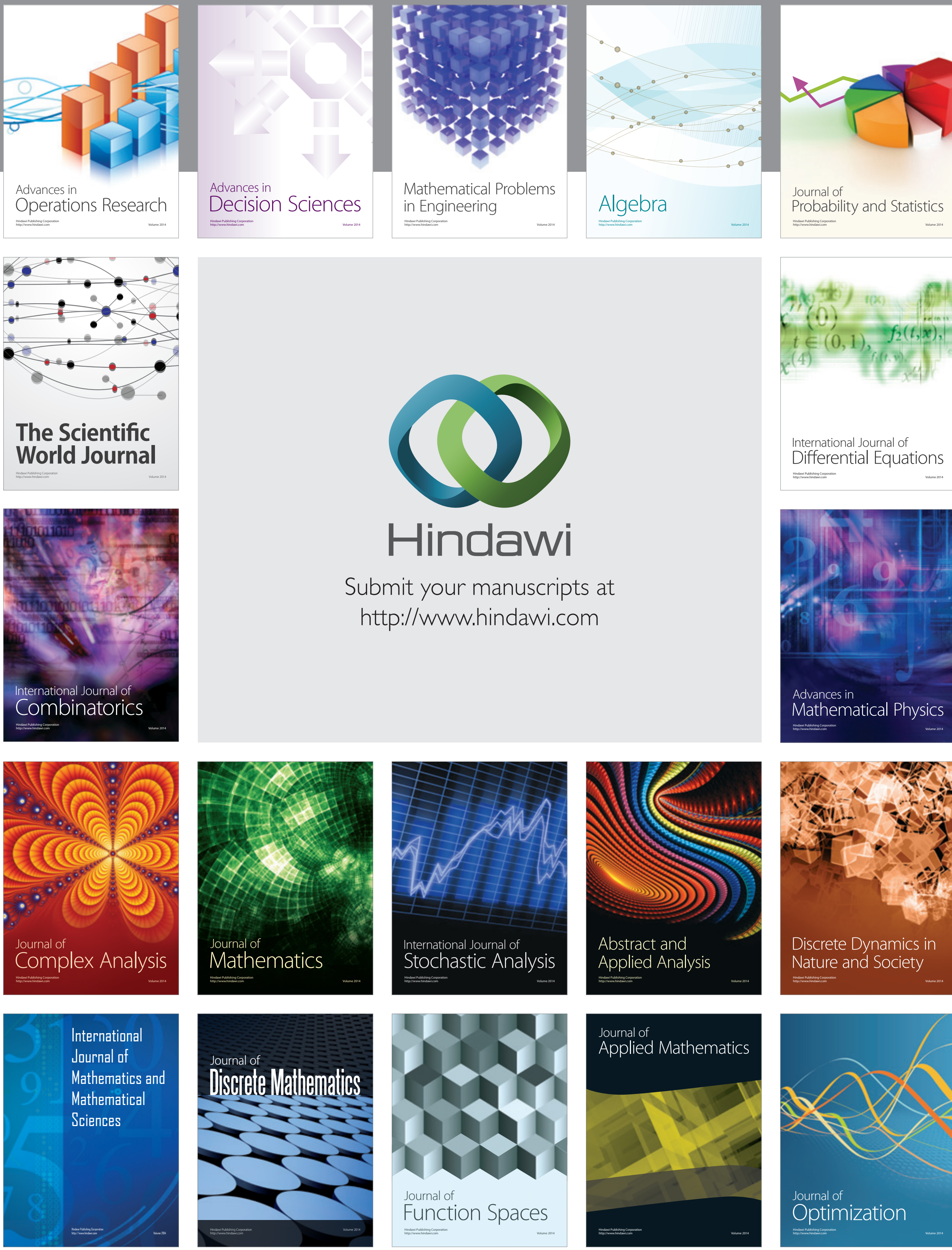
[10] S.-H. Rim, S. J. Lee, E. J. Moon, and J. H. Jin, "On the q-Genocchi numbers and polynomials associated with q-zeta function," *Proceedings of the Jangjeon Mathematical Society*, vol. 12, no. 3, pp. 261–267, 2009.

[11] Y. Simsek, "Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions," *Advanced Studies in Contemporary Mathematics*, vol. 16, no. 2, pp. 251–278, 2008.

[12] Y. Simsek, "Theorems on twisted L-function and twisted Bernoulli numbers," *Advanced Studies in Contemporary Mathematics*, vol. 11, no. 2, pp. 205–218, 2005.

[13] M. Cenkci, Y. Simsek, and V. Kurt, "Multiple two-variable p-adic q-L-function and its behavior at $s = 0$," *Russian Journal of Mathematical Physics*, vol. 15, no. 4, pp. 447–459, 2008.

[14] S. Araci, D. Erdal, and J. J. Seo, "A study on the fermionic p-adic q-integral representation on $\mathbb{Z}_p$ associated with weighted q-Bernstein and q-Genocchi polynomials," *Abstract and Applied Analysis*. In press.


Advances in
Operations Research
Advances in
Decision Sciences
Mathematical Problems
in Engineering
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
International Journal of
Stochastic Analysis
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
Journal of
Applied Mathematics
Journal of
Optimization