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Research Article

Global Attractivity of a Family of Max-Type Difference Equations

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We propose to study a generalized family of max-type difference equations and then prove the global attractivity of a particular case of it under some parameter conditions. Through some numerical results of other cases, we finally pose a generic conjecture.

1. Introduction

The study of max-type difference equations is a hotspot in the area of discrete dynamics because such equations are often closely related to automatic control theory and competitive dynamics. For recent advances in this direction see [1–8] and the references therein.

Motivated by [9], Liu et al. [10] studied the following nonautonomous max-type difference equation:

$$y_n = \frac{p + r y_{n-s}}{q + \phi_n(y_{n-1}, \dots, y_{n-m}) + y_{n-s}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $p \geq 0$, $r, q > 0$, $s, m \in \mathbb{N}$, and $\phi_n : (\mathbb{R}^+)^m \to \mathbb{R}^+$, $n \in \mathbb{N}_0$ are mappings satisfying the condition $\beta \min\{x_1, \dots, x_m\} \leq \phi_n(x_1, x_2, \dots, x_m) \leq \beta \max\{x_1, \dots, x_m\}$, for some fixed $\beta \in (0, +\infty)$. When $p = 0$, $\beta \in (0, 1)$, they proved that every positive solution to (1.1) converges to zero if $r \leq q$, while $(r-q)/(1+\beta)$ if $r > q$. If $p > 0$ and $rq \geq p$, then each positive solution to (1.1) converges to $(\sqrt{(q-r)^2 + 4p(1+\beta)} - (q-r))/(2(1+\beta))$, for some $\beta \in (0, +\infty)$, except for the case $q < r$, $\beta \in (\beta_0, +\infty)$, where $\beta_0 = 4p/(q-r)^2 + 1$. Note that the behavior of positive solutions to (1.1) for the case $q < r$, $\beta \in (\beta_0, +\infty)$, is still an unsolved open problem as was mentioned in [10].

Here, we propose to investigate the asymptotic behavior of positive solutions to the generalized family of max-type difference equations

$$x_n = \max_{1\leq i\leq k}\left\{\frac{p_i + r_i x_{n-s}}{q_i + x_{n-s} + f_i(x_{n-1}, \ldots, x_{n-m})}\right\}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $p_i \geq 0$, $r_i, q_i > 0$, $s, m, k \in \mathbb{N}$, $k \geq 2$ and the functions $f_i : [0, +\infty)^m \to [0, +\infty)$, $i = 1, 2, \ldots, k$ satisfy the condition

$$\beta \min\{u_1, \ldots, u_m\} \leq f_i(u_1, u_2, \ldots, u_m) \leq \beta \max\{u_1, \ldots, u_m\}, \tag{1.3}$$

for some fixed $\beta \in (0, 1)$.

In this paper, we mainly consider the particular case that all p_i are zero, and then obviously (1.2) reduces to the following form:

$$x_n = x_{n-s} \times \max_{1\leq i\leq k}\left\{\frac{r_i}{q_i + x_{n-s} + f_i(x_{n-1}, \ldots, x_{n-m})}\right\}, \quad n \in \mathbb{N}_0. \tag{1.4}$$

Let x^* be a nonnegative equilibrium point of (1.4), then we have

$$x^* = x^* \times \max_{1\leq i\leq k}\left\{\frac{r_i}{q_i + (1+\beta)x^*}\right\}. \tag{1.5}$$

It follows directly from (1.5) that if $0 < r_i \leq q_i$ for all $i = 1, 2, \ldots, k$, then (1.4) has the unique nonnegative equilibrium $x^* = 0$, while if there exists at least one $j \in \{1, 2, \ldots, k\}$ such that $r_j > q_j$, then (1.4) has a zero equilibrium $x^* = 0$ and a unique positive equilibrium $x^* = \max_{1\leq i\leq k}\{r_i - q_i\}/(1+\beta)$.

Finally, the following two beautiful theorems are derived.

Theorem 1.1. *Consider* (1.4) *with condition* (1.3). *If $0 < r_i \leq q_i$ for all $i = 1, 2, \ldots, k$, then every positive solution to* (1.4) *converges to the unique nonnegative equilibrium zero.*

Theorem 1.2. *Consider* (1.4) *with positive initial values and positive r_i and q_i. Let $f_i : [0, +\infty)^m \to [0, +\infty)$ be functions such that for some fixed $\beta \in (0, 1)$, there hold*

$$\beta \min\{u_1, \ldots, u_m\} \leq f_i(u_1, \ldots, u_m) \leq \beta \max\{u_1, \ldots, u_m\}, \quad i = 1, 2, \ldots, k. \tag{1.6}$$

If there exists at least one $j \in \{1, 2, \ldots, k\}$ such that $r_j > q_j$, then the unique positive equilibrium of (1.4) *is a global attractor.*

2. Preliminary Lemmas

For the purpose of establishing the main results, some auxiliary lemmas are essential.

Lemma 2.1. *Consider the first-order difference equation*

$$x_n = x_{n-1} \times \max_{1\le i\le k}\left\{\frac{r_i}{q_i + x_{n-1}}\right\}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

with positive initial value x_{-1} and positive r_i and q_i. If there exists at least one $j \in \{1,2,\ldots,k\}$ such that $r_j > q_j$, then

$$\lim_{n\to\infty} x_n = \max\{r_i - q_i : i = 1,2,\ldots,k\}. \tag{2.2}$$

Proof. Suppose that $\max\{r_i - q_i : i = 1,2,\ldots,k\} = r_\tau - q_\tau$, which is positive, for some $\tau \in \{1,2,\ldots,k\}$. By making the variable change $x_n = (r_\tau - q_\tau)y_n$ into (2.1) and then canceling the positive term $r_\tau - q_\tau$ from the resulting equation, we can derive

$$y_n = y_{n-1} \times \max_{1\le i\le k}\left\{\frac{r_i}{q_i + (r_\tau - q_\tau)y_{n-1}}\right\}, \quad n \in \mathbb{N}_0. \tag{2.3}$$

Note that $\min\{a_1/b_1, a_2/b_2\} \le (a_1 + a_2)/(b_1 + b_2) \le \max\{a_1/b_1, a_2/b_2\}$ for $a_i, b_i > 0$, $i = 1,2$. Then it follows from (2.3) that

$$y_{n+1} = \max_{1\le i\le k}\left\{\frac{q_i y_n + (r_i - q_i)y_n}{q_i + (r_\tau - q_\tau)y_n}\right\} \le \max_{1\le i\le k}\left\{\frac{q_i y_n + (r_\tau - q_\tau)y_n}{q_i + (r_\tau - q_\tau)y_n}\right\} \le \max\{y_n, 1\}. \tag{2.4}$$

In addition, the following two inequalities hold:

$$y_{n+1} - 1 = \max_{1\le i\le k}\left\{\frac{r_i y_n}{q_i + (r_\tau - q_\tau)y_n} - 1\right\} \ge \frac{r_\tau y_n}{q_\tau + (r_\tau - q_\tau)y_n} - 1 = \frac{q_\tau(y_n - 1)}{q_\tau + (r_\tau - q_\tau)y_n}, \tag{2.5}$$

$$y_{n+1} - y_n = \max_{1\le i\le k}\left\{\frac{r_i y_n}{q_i + (r_\tau - q_\tau)y_n} - y_n\right\} \ge \frac{r_\tau y_n}{q_\tau + (r_\tau - q_\tau)y_n} - y_n = \frac{(r_\tau - q_\tau)y_n(1 - y_n)}{q_\tau + (r_\tau - q_\tau)y_n}. \tag{2.6}$$

In the following, we are confronted with three possibilities.

Case 1. If there exists $n_0 \ge -1$ such that $y_{n_0} = 1$, then it follows from (2.4) and (2.5) that $y_n = 1$ holds for all $n \ge n_0$.

Case 2. If there exists $n_0 \ge -1$ such that $y_{n_0} > 1$, then it follows from (2.5) and (2.6) that

$$y_{n_0} \ge y_{n_0+1} \ge y_{n_0+2} \ge \cdots > 1. \tag{2.7}$$

Thus there is a finite limit $\gamma = \lim_{n\to\infty} y_n \geq 1$. By taking the limits on both sides of (2.3) and canceling the positive factor γ from the resulting equation, we obtain

$$1 = \max_{1\leq i\leq k}\left\{\frac{r_i}{q_i + (r_\tau - q_\tau)\gamma}\right\}, \tag{2.8}$$

which implies $\gamma = 1$. Because if $\gamma > 1$, then

$$1 = \max_{1\leq i\leq k}\left\{\frac{r_i}{q_i + (r_\tau - q_\tau)\gamma}\right\} < \max_{1\leq i\leq k}\left\{\frac{r_i}{q_i + (r_\tau - q_\tau)}\right\} = 1, \tag{2.9}$$

leading to a contradiction.

Case 3. If $y_n < 1$ for all $n \geq -1$, then it follows from (2.5) and (2.6) that

$$y_{-1} < y_0 < y_1 < \cdots < y_n < \cdots < 1. \tag{2.10}$$

Therefore, the limit of y_n exists, denoted by $0 < \gamma = \lim_{n\to\infty} y_n \leq 1$. By taking the limits on both sides of (2.3) and canceling the nonzero factor γ from the resulting equation, there hold

$$1 = \max_{1\leq i\leq k}\left\{\frac{r_i}{q_i + (r_\tau - q_\tau)\gamma}\right\}, \tag{2.11}$$

which implies $\gamma = 1$. Because if $0 < \gamma < 1$, then

$$1 = \max_{1\leq i\leq k}\left\{\frac{r_i}{q_i + (r_\tau - q_\tau)\gamma}\right\} > \max_{1\leq i\leq k}\left\{\frac{r_i}{q_i + (r_\tau - q_\tau)}\right\} = 1, \tag{2.12}$$

which is a contradiction.

In either of the above three cases, we get $\lim_{n\to\infty} y_n = 1$, implying $\lim_{n\to\infty} x_n = r_\tau - q_\tau$. □

From Lemma 2.1, we have the following result.

Lemma 2.2. *Consider the s-order difference equation*

$$x_n = x_{n-s} \times \max_{1\leq i\leq k}\left\{\frac{r_i}{q_i + x_{n-s}}\right\}, \quad n \in \mathbb{N}_0, \tag{2.13}$$

with positive initial values and $r_i, q_i > 0$. *If there exists at least one* $j \in \{1, 2, \ldots, k\}$ *such that* $r_j > q_j$, *then*

$$\lim_{n\to\infty} x_n = \max\{r_i - q_i : i = 1, 2, \ldots, k\}. \tag{2.14}$$

Proof. Let $\{x_n\}_{n\geq -s}$ be an arbitrary positive solution to (2.13). Apparently we know that the sequence $\{x_n\}_{n\geq -s}$ can be divided into s subsequences $\{x_{j+sk}\}_{k\geq 0}$, $j = -s, -s+1, \dots, -1$, which are, respectively, positive solutions to the first-order equation (2.1) with positive initial values $x_{-s}, x_{-s+1}, \dots, x_{-1}$. According to Lemma 2.1, we derive $\lim_{k\to\infty} x_{j+sk} = \max\{r_i - q_i : i = 1,2,\dots,k\}$ for all $j = -s, -s+1, \dots, -1$, which directly lead to $\lim_{n\to\infty} x_n = \max\{r_i - q_i : i = 1,2,\dots,k\}$. □

Lemma 2.3. *Let* $a > b > 0$, $0 < \beta < 1$, *and* $0 < \epsilon < ((1-\beta)/(1+\beta))(a-b)$. *Define two sequences* $\{m_k\}$ *and* $\{M_k\}$ *in the following way:*

$$\begin{aligned} M_1 &= a - b, \\ m_k &= M_1 - \beta\left(M_k + \frac{\epsilon}{k}\right), \quad k = 1,2,\dots, \\ M_k &= M_1 - \beta\left(m_{k-1} - \frac{\epsilon}{(k-1)}\right), \quad k = 2,3,\dots. \end{aligned} \tag{2.15}$$

Then $\lim_{k\to\infty} m_k = \lim_{k\to\infty} M_k$.

Proof. Observe that

$$\begin{aligned} M_2 - M_1 &= -\beta((1-\beta)(a-b) - (\beta+1)\epsilon) < 0, \\ m_{k+1} - m_k &= \beta\left[M_k - M_{k+1} + \frac{\epsilon}{k(k+1)}\right], \quad k = 1,2,\dots, \\ M_{k+1} - M_k &= -\beta\left[m_k - m_{k-1} + \frac{\epsilon}{k(k-1)}\right], \quad k = 2,3,\dots. \end{aligned} \tag{2.16}$$

It follows by induction that $\{m_k\}$ is increasing and $\{M_k\}$ is decreasing. Again by induction we derive $m_k < a-b$ and $M_k > 0$, $k = 1,2,\dots$. Hence there are two finite limits $\xi = \lim_{k\to\infty} m_k$ and $\eta = \lim_{k\to\infty} M_k$. By taking limits on both sides of (2.15), we derive

$$\xi = a - b - \beta\eta, \qquad \eta = a - b - \beta\xi, \tag{2.17}$$

which imply $(1-\beta)(\xi - \eta) = 0$. Therefore $\xi = \eta = (a-b)/(1+\beta)$. □

3. Proofs of Main Theorems

In this section, we are in a position to prove the main theorems presented in Section 1.

Proof of Theorem 1.1. Note that for the case $r_i < q_i$, $i = 1,2,\dots,k$, the behavior of positive solutions to (1.4) is quite simple. In this case, we have that

$$x_n \leq x_{n-s} \times \max_{1\leq i\leq k}\left\{\frac{r_i}{q_i}\right\} = \mu x_{n-s}, \tag{3.1}$$

where $\mu = \max_{1\le i\le k}\{r_i/q_i\} < 1$. Easily the subsequences $\{x_{ls+j}\}_{l\in\mathbb{N}_0}$, $j \in \{0,1,\dots,s-1\}$ converge to zero, hence the sequence $\{x_n\}$ also converges to zero.

For the case $r_i \le q_i$, $i = 1,2,\dots,k$ with at least one $j \in \{1,2,\dots,k\}$ such that $r_j = q_j$, we can obtain that

$$x_n \le x_{n-s} \times \max_{1\le i\le k}\left\{\frac{r_i}{q_i}\right\} = x_{n-s}. \tag{3.2}$$

In this case, the subsequences $\{x_{ls+j}\}_{l\in\mathbb{N}_0}$, $j = 0,1,\dots,s-1$ are all positive and nonincreasing, thus they converge, respectively, to some nonnegative limits $\psi_j := \lim_{l\to\infty} x_{ls+j}$, $j=0,1,\dots,s-1$.

If we replace n in (1.4) by $sl+j$, $l \in \mathbb{N}_0$ for an arbitrary fixed $j \in \{0,1,\dots,s-1\}$ and let $l \to \infty$, we can get

$$\psi_j = \psi_j \times \max_{1\le i\le k}\left\{\frac{r_i}{q_i + \psi_j + f_i(\psi_{v_1},\dots,\psi_{v_m})}\right\}, \tag{3.3}$$

where $v_i \in \{0,1,\dots,s-1\}$, $i = 1,\dots,m$. Without loss of generality, assume that $\psi_j \ne 0$, then we obtain that

$$1 = \frac{r_\tau}{q_\tau + \psi_j + f_\tau(\psi_{v_1},\dots,\psi_{v_m})}, \tag{3.4}$$

with some fixed number $\tau \in \{1,2,\dots,k\}$. Because $r_\tau \le q_\tau$, then it follows from (3.4) that

$$q_\tau + \psi_j + f_\tau(\psi_{v_1},\dots,\psi_{v_m}) = r_\tau \le q_\tau. \tag{3.5}$$

Therefore we have

$$\psi_j + f_\tau(\psi_{v_1},\dots,\psi_{v_m}) = 0, \tag{3.6}$$

leading to $\psi_j = 0$, which is a contradiction. Hence we have that $\psi_j = 0$, $j = 0,1,\dots,s-1$, and every positive solution to (1.4) converges to zero, if $r_i \le q_i$ for all $i = 1,2,\dots,k$. □

Proof of Theorem 1.2. Suppose that $\max\{r_i - q_i : i = 1,2,\dots,k\} = r_\tau - q_\tau > 0$ for some $\tau \in \{1,2,\dots,k\}$. Let ϵ be an arbitrary fixed real number with $0 < \epsilon < ((1-\beta)/(1+\beta))(r_\tau - q_\tau)$. Define two sequences $\{M_k\}$ and $\{m_k\}$ in the way shown in (2.15) with $a = r_\tau$, $b = q_\tau$. □

Let $\{x_n\}$ be an arbitrary positive solution to (1.4). Next, we proceed by proving two claims.

Claim 1. There exists $N_1 \in \mathbb{N}$ such that $m_1 - \epsilon \le x_n \le M_1 + \epsilon$ for all $n \ge N_1$.

Proof of Claim 1. Note that

$$x_n \le x_{n-s} \times \max_{1\le i\le k}\left\{\frac{r_i}{q_i + x_{n-s}}\right\}, \quad n = 0,1,2,\dots. \tag{3.7}$$

Consider the following difference equation:

$$z_n^{(1)} = z_{n-s}^{(1)} \times \max_{1\le i\le k}\left\{\frac{r_i}{q_i + z_{n-s}^{(1)}}\right\}, \quad n = 0,1,2,\dots. \tag{3.8}$$

Let $\{z_n^{(1)}\}$ be a positive solution to (3.7) with the initial values $z_{-1}^{(1)} = x_{-1}, z_{-2}^{(1)} = x_{-2},\dots,z_{-s}^{(1)} = x_{-s}$.

Note that the mapping $h(x) = rx/(q+x)$ is strictly increasing on the interval $(0,+\infty)$. It follows by induction that $x_n \le z_n^{(1)}$ for all $n \ge -s$. By Lemma 2.2, we have $\lim_{n\to\infty} z_n^{(1)} = r_\tau - q_\tau = M_1$. Hence there is an integer $N_1' \in \mathbb{N}$ such that $x_n \le M_1 + \epsilon$ for $n \ge N_1'$.

Let $t = \max\{s, m\}$. Note that

$$x_n \ge x_{n-s} \times \max_{1\le i\le k}\left\{\frac{r_i}{q_i + x_{n-s} + \beta(M_1+\epsilon)}\right\}, \quad n \ge N_1' + t. \tag{3.9}$$

Consider the difference equation

$$y_n^{(1)} = y_{n-s}^{(1)} \times \max_{1\le i\le k}\left\{\frac{r_i}{q_i + y_{n-s}^{(1)} + \beta(M_1+\epsilon)}\right\}, \quad n \ge N_1' + t, \tag{3.10}$$

with $y_{N_1'+t-1}^{(1)} = x_{N_1'+t-1}, y_{N_1'+t-2}^{(1)} = x_{N_1'+t-2},\dots,y_{N_1'}^{(1)} = x_{N_1'}$. Note the monotonicity of $h(x)$, it follows by induction that $x_n \ge y_n^{(1)}$ for all $n \ge N_1'$. By Lemma 2.2, we get that $\lim_{n\to\infty} y_n^{(1)} = m_1$. Thus there exists an integer $N_1 \ge N_1'$ such that $x_n \ge m_1 - \epsilon$ for all $n \ge N_1$. □

Working inductively, we will reach the following claim.

Claim 2. For every $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that

$$m_k - \frac{\epsilon}{k} \le x_n \le M_k + \frac{\epsilon}{k}, \tag{3.11}$$

for all $n \ge N_k$.

Proof of Claim 2. Obviously, the case $k = 1$ follows directly from Claim 1. In the following, we proceed by induction. Assume that the assertion is true for $k = \omega(\omega \ge 1)$. Then it suffices to prove the assertion is also true for $k = \omega + 1$.

Note that

$$x_n \le x_{n-s} \times \max_{1\le i\le k}\left\{\frac{r_i}{q_i + x_{n-s} + \beta(m_\omega - \epsilon/\omega)}\right\}, \quad n \ge N_\omega + t. \tag{3.12}$$

Consider the difference equation

$$z_n^{(\omega+1)} = z_{n-s}^{(\omega+1)} \times \max_{1\le i\le k}\left\{\frac{r_i}{q_i + z_{n-s}^{(\omega+1)} + \beta(m_\omega - \epsilon/\omega)}\right\}, \quad n \ge N_\omega + t, \tag{3.13}$$

with $z_{N_\omega+t-1}^{(\omega+1)} = x_{N_\omega+t-1}, z_{N_\omega+t-2}^{(\omega+1)} = x_{N_\omega+t-2}, \ldots, z_{N_\omega}^{(\omega+1)} = x_{N_\omega}$. Note the monotonicity of $h(x)$, it follows by induction that $x_n \leq z_n^{(\omega+1)}$ for all $n \geq N_\omega$. By Lemma 2.2, we have that $\lim_{n\to\infty} z_n^{(\omega+1)} = M_{\omega+1}$. So there is an integer $N'_{\omega+1} \in \mathbb{N}$ such that $x_n \leq M_{\omega+1} + \epsilon/(\omega+1)$ for all $n \geq N'_{\omega+1}$. Then note that

$$x_n \geq x_{n-s} \times \max_{1\leq i\leq k}\left\{\frac{r_i}{q_i + x_{n-s} + \beta(M_{\omega+1} + \epsilon/(\omega+1))}\right\}, \quad n \geq N'_{\omega+1} + t. \tag{3.14}$$

Consider the following difference equation

$$y_n^{(\omega+1)} = y_{n-s}^{(\omega+1)} \times \max_{1\leq i\leq k}\left\{\frac{r_i}{q_i + y_{n-s}^{(\omega+1)} + \beta(M_{\omega+1} + \epsilon/(\omega+1))}\right\}, \quad n \geq N'_{\omega+1} + t, \tag{3.15}$$

with $y_{N'_{\omega+1}+t-1}^{(\omega+1)} = x_{N'_{\omega+1}+t-1}, z_{N'_{\omega+1}+t-2}^{(\omega+1)} = x_{N'_{\omega+1}+t-2}, \ldots, z_{N'_{\omega+1}}^{(\omega+1)} = x_{N'_{\omega+1}}$. By the monotonicity of $h(x)$, it follows by induction that $x_n \geq y_n^{(\omega+1)}$ for all $n \geq N'_{\omega+1}$. By Lemma 2.2, we have that $\lim_{n\to\infty} y_n^{(\omega+1)} = m_{\omega+1}$. So there is an integer $N_{\omega+1} \geq N'_{\omega+1}$ such that $x_n \geq m_{\omega+1} - \epsilon/(\omega+1)$ for all $n \geq N_{\omega+1}$. □

From Claim 2, we derive

$$\lim_{k\to\infty} m_k = \lim_{k\to\infty}\left(m_k - \frac{\epsilon}{k}\right) \leq \underline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} x_n \leq \lim_{k\to\infty}\left(M_k + \frac{\epsilon}{k}\right) = \lim_{k\to\infty} M_k. \tag{3.16}$$

This plus Lemma 2.3 leads to that

$$\lim_{n\to\infty} x_n = \lim_{k\to\infty} m_k = \lim_{k\to\infty} M_k = \frac{r_\tau - q_\tau}{1+\beta}. \tag{3.17}$$

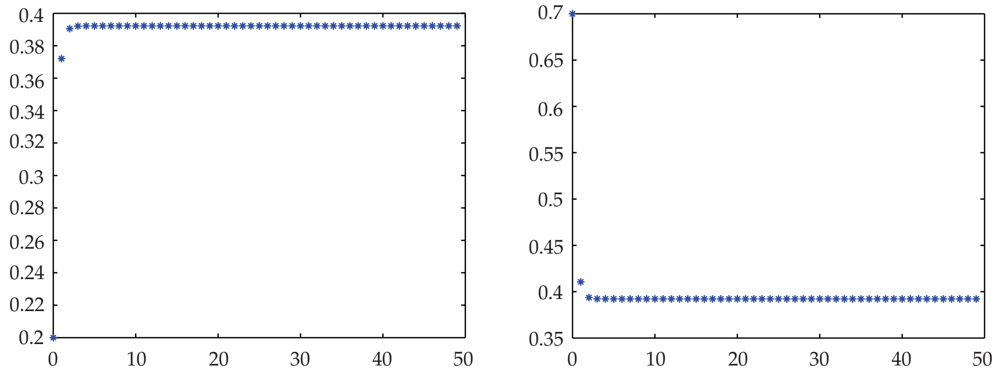
4. Simulations and Future Work

In the previous section, we proved the global attractivity of (1.2) when all p_i are zero. In this section, we investigate the dynamic behavior of (1.2) provided that all p_i are not zero. First, it is trivial to confirm that when all p_i are not zero, (1.2) has the following unique positive equilibrium point $x^* = \max_{1\leq i\leq k}\{\sqrt{(q_i - r_i)^2 + 4p_i(1+\beta)} + r_i - q_i\}/(2(1+\beta))$. In the following, some numerical results are presented.

Experiment 1. Consider the first-order difference equation

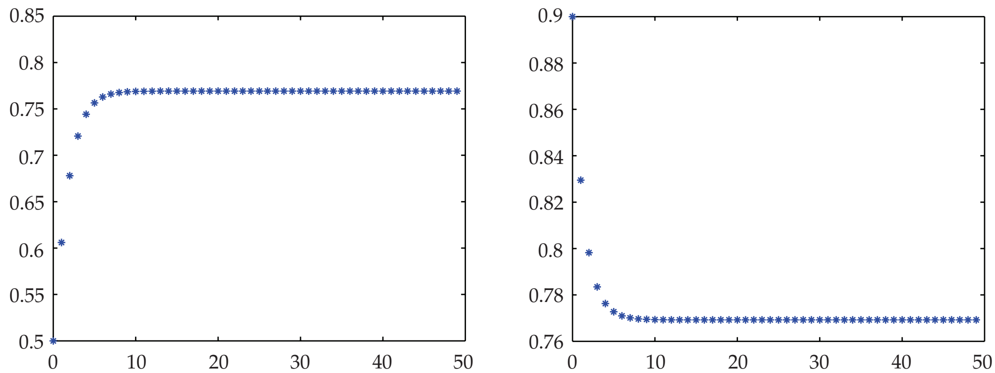
$$x_n = \max\left\{\frac{0.2 + 0.6x_{n-1}}{0.6 + x_{n-1} + 0.3x_{n-1}}, \frac{r x_{n-1}}{q + x_{n-1} + 0.3x_{n-1}}\right\}, \quad n \in \mathbb{N}, \tag{4.1}$$

where $r, q > 0$ and the initial value $x_0 > 0$. (See Figures 1 and 2).



(a) Initial value $x_0 = 0.2$ (b) Initial value $x_0 = 0.7$

Figure 1: $r = 1$, $q = 2$; $x^* = \sqrt{26}/13 \approx 0.3922$.



(a) Initial value $x_0 = 0.5$ (b) Initial value $x_0 = 0.9$

Figure 2: $r = 2$, $q = 1$; $x^* = 10/13 \approx 0.7692$.

Experiment 2. Consider the second-order difference equation

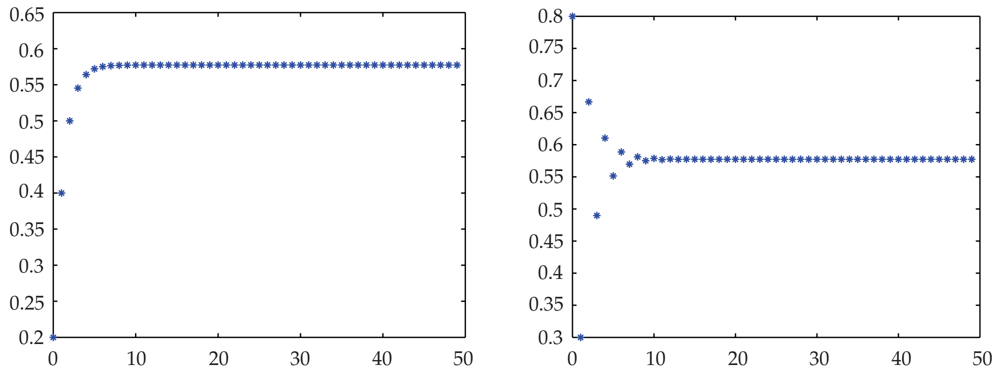
$$x_n = \max\left\{\frac{0.5 + x_{n-2}}{1 + x_{n-2} + 0.5x_{n-1}}, \frac{0.8 + rx_{n-2}}{q + x_{n-2} + 0.5x_{n-1}}\right\}, \quad n \geq 2, \tag{4.2}$$

where $r, q > 0$ and the initial values $x_0, x_1 > 0$. (See Figures 3 and 4).

Experiment 3. Consider the third-order difference equation

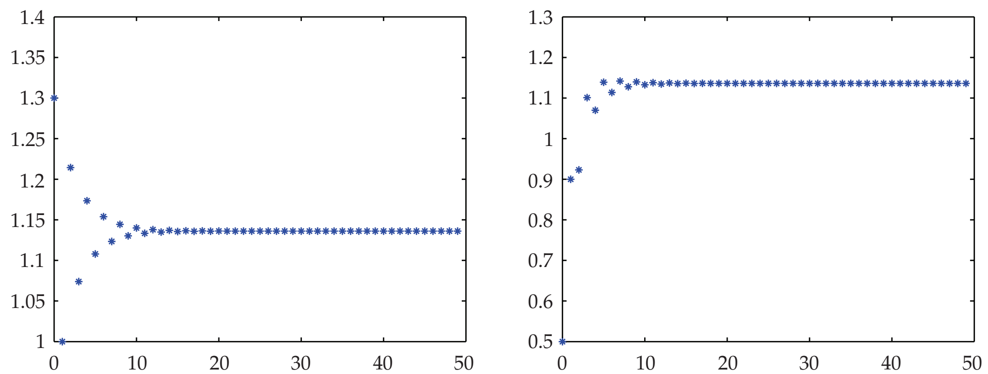
$$x_n = \max\left\{\frac{0.5 + x_{n-3}}{1 + x_{n-3} + 0.9\sqrt{(x_{n-1}^2 + x_{n-2}^2)/2}}, \frac{3x_{n-3}}{2 + x_{n-3} + 0.9\sqrt{(x_{n-1}^2 + x_{n-2}^2)/2}}\right\}, \quad n \geq 3, \tag{4.3}$$

where the initial values $x_0, x_1, x_2 > 0$. (See Figure 5).



(a) Initial values $x_0 = 0.2$, $x_1 = 0.4$ (b) Initial values $x_0 = 0.8$, $x_1 = 0.3$

Figure 3: $r = 1$, $q = 2$; $x^* = \sqrt{3}/3 \approx 0.5774$.



(a) Initial values $x_0 = 1.3$, $x_1 = 1.0$ (b) Initial values $x_0 = 0.5$, $x_1 = 0.9$

Figure 4: $r = 2$, $q = 1$; $x^* = (\sqrt{5.8} + 1)/3 \approx 1.1361$.

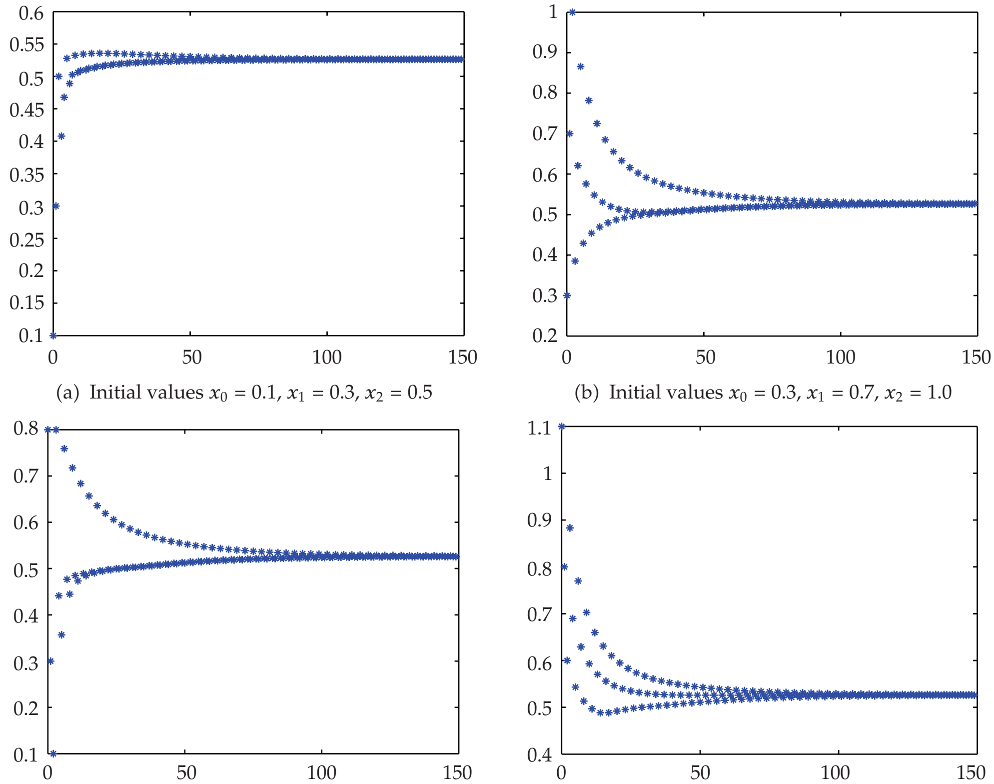
Inspired by this work and the results of [10], here we pose the following conjecture.

Conjecture 4.1. *Consider* (1.2) *with nonnegative* p_i *and positive* r_i *and* q_i. *Let* $f_i : [0, +\infty)^m \to [0, +\infty)$, $i = 1, 2, \ldots, k$ *be* k *functions such that for some fixed* $\beta \in (0, 1)$, *there hold*

$$\beta \min\{u_1, \ldots, u_k\} \leq f_i(u_1, \ldots, u_k) \leq \beta \max\{u_1, \ldots, u_k\}. \tag{4.4}$$

If $r_i q_i \geq p_i$ *for all* $i = 1, 2, \ldots, k$, *then every positive solution to* (1.2) *converges to the equilibrium point*

$$x^* = \frac{1}{2(1+\beta)} \max_{1 \leq i \leq k} \left\{ \sqrt{(q_i - r_i)^2 + 4p_i(1+\beta)} + r_i - q_i \right\}. \tag{4.5}$$



(a) Initial values $x_0 = 0.1$, $x_1 = 0.3$, $x_2 = 0.5$

(b) Initial values $x_0 = 0.3$, $x_1 = 0.7$, $x_2 = 1.0$

(c) Initial values $x_0 = 0.8$, $x_1 = 0.3$, $x_2 = 0.1$

(d) Initial values $x_0 = 1.1$, $x_1 = 0.8$, $x_2 = 0.6$

Figure 5: $x^* = 10/19 \approx 0.5263$.

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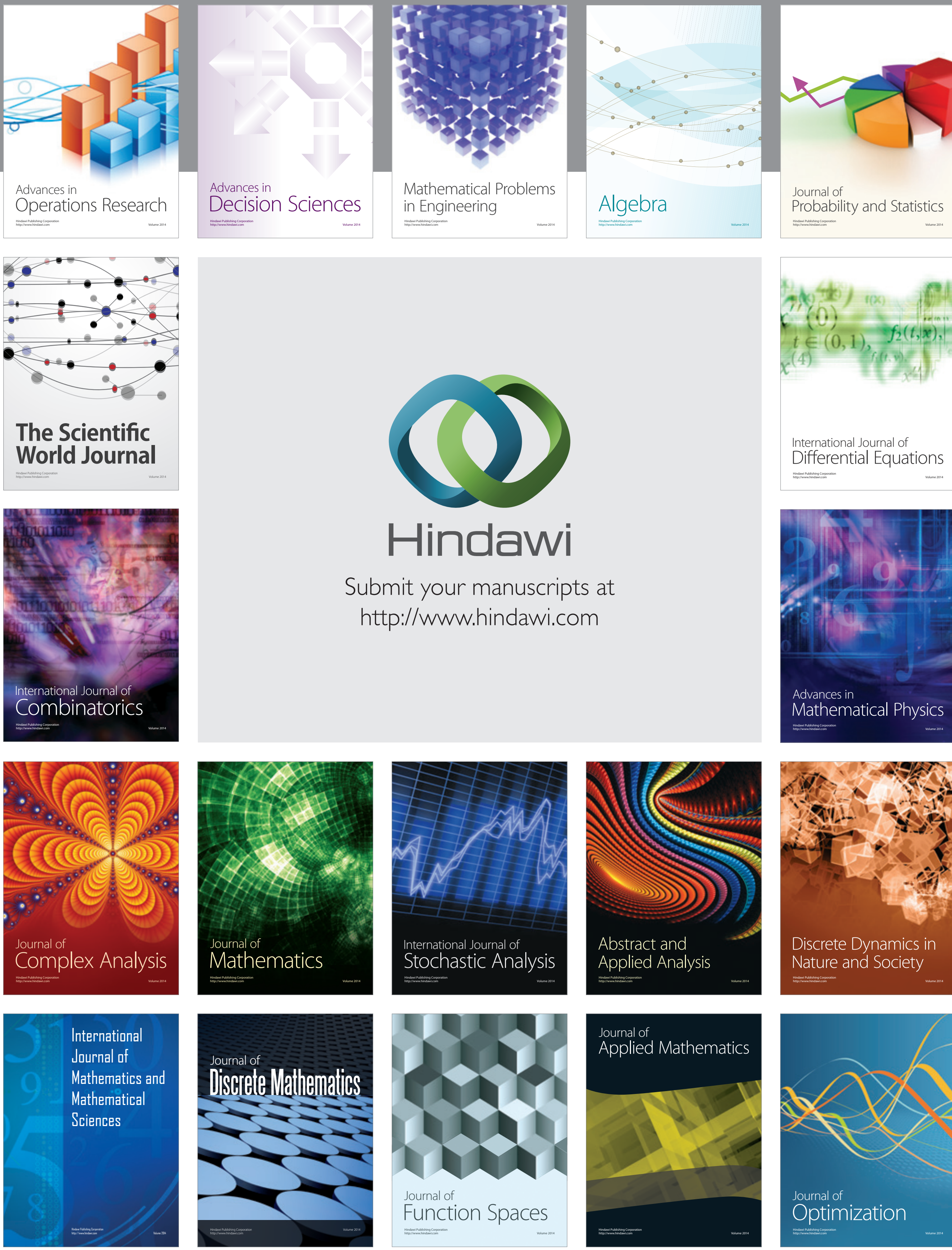
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