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Research Article

Homotopy Perturbation Method for Solving Wave-Like Nonlinear Equations with Initial-Boundary Conditions

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The homotopy perturbation method is employed to obtain approximate analytical solutions of the wave-like nonlinear equations with initial-boundary conditions. An efficient way of choosing the auxiliary operator is presented. The results demonstrate reliability and efficiency of the method.

1. Introduction

In this paper, we consider the equation

$$u_{tt} - c^2 u_{xx} = f(x, t, u, u_x, u_y, u_{xt}) + g(x, t), \quad 0 < x < \infty \tag{1.1}$$

with initial conditions

$$u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \psi(x), \tag{1.2}$$

and boundary condition

$$u(0, t) = h(t), \tag{1.3}$$

where f, g, φ, ψ, and h are known functions.

Problems like (1.1)-(1.2)-(1.3) model many problems in classical and quantum mechanics, solitons, and matter physics [1, 2]. If f is a function of u only, we obtain a Klein-Gordon or sine-Gordon-type equations.

In the last decade, some various approximate methods have been developed, such as the homotopy perturbation method (HPM) [3–13] and Adomian's decomposition method (ADM) [14–20] to solve linear and nonlinear differential equations.

Unlike the various approximation techniques for solving nonlinear wave type problems, which are usually valid for initial value problems (without boundary conditions) or some special type of problems (homogenous, etc.), our technique is applicable for all initial-boundary problems of type (1.1)-(1.2)-(1.3). Chowdhury and Hashim [9] applied the HPM for solving Klein-Gordon and sine-Gordon equations, with initial conditions (1.2). El-Sayed [19] and Wazwaz and Gorguis [20] used ADM for solving wave-like and heat-like problems. Their approaches cannot be applied for all wave-like equations with initial-boundary conditions since the operator $L = u_{tt}$ cannot control the boundary condition (1.3)—see Example 2.1 below.

The central idea here is that the problem

$$\begin{gathered} u_{tt} - c^2 u_{xx} = g(x,t), \\ u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad u(0,t) = h(t), \end{gathered} \tag{1.4}$$

has a unique solution (see, e.g., [21]) and therefore there exists an inverse of the operator $L = u_{tt} - c^2 u_{xx}$.

The main idea of HPM is to introduce a homotopy parameter, say p, which takes values from 0 to 1. When $p = 0$ the equation usually reduces to a sufficiently simplified form (linear or very easy nonlinear). As p increases to 1, the equation goes through a sequence of "deformations" (homotopics) and at $p = 1$ takes the original form of the equation.

We rewrite (1.1) as

$$Lu = f(x,t,u,u_x,u_y,u_{xy}) + g(x,t), \tag{1.5}$$

and construct the following homotopy:

$$Lu - Lv_0 + p\left[Lv_0 - f(x,t,u,u_x,u_y,u_{xy}) - g(x,t)\right] = 0. \tag{1.6}$$

Usually we take v_0 as a solution of the problem (1.1)-(1.2)-(1.3) with $f = 0$ or simply $v_0 = 0$. Assume that the solution of (1.6) is in the form

$$u = u_0 + pu_1 + pu_2 + \cdots, \tag{1.7}$$

and substituting (1.7) into (1.6) and equating terms of the same powers of p we obtain a system of equations for $u_0, u_1, u_2, \ldots$. Solving these system of equations we obtain a solution in the form

$$u = u_0 + u_1 + u_2 + \cdots. \tag{1.8}$$

2. Applications

The HPM and ADM offer excellent choices for obtaining the closed-form analytical solutions of wave-like equations. Chowdhury and Hashim [9], El-Sayed [19], and Wazwaz and Gorguis [20] recently showed how the HPM and ADM can be applied to find an analytic approximate solution of wave-like equations with initial conditions (1.2). They mainly used the operator $L = u_{tt}$ for solving the wave-like problems. But in case of inhomogeneous nonlinear or even linear equations with initial-boundary conditions, these approaches have some difficulties. If we construct the standard homotopy with $L = u_{tt}$, for solving wave-like equations with initial-boundary conditions, usually in the second or even in the first stage of HPM, we obtain an "overdetermined" or very difficult problem. To explain these difficulties we consider the following example.

Example 2.1. Consider the linear problem

$$\begin{gathered} u_{tt} - u_{xx} = 6x, \\ u(x,0) = -x^3, \qquad u_t(x,0) = 0, \qquad u(0,t) = \sin^2 t. \end{gathered} \tag{2.1}$$

The exact solution is

$$\begin{aligned} u &= -x^3 + \sin^2(t-x) \quad \text{for } x < t \\ &= -x^3 \quad \text{for } x > t. \end{aligned} \tag{2.2}$$

Using our HPM technique we can easily find this solution. Indeed, let us take $L = u_{tt} - u_{xx}$, $v_0 = 0$ and construct the homotopy

$$Lu - Lv_0 + p[Lv_0 - 6x] = 0. \tag{2.3}$$

Now substituting (1.7) into u and equating the coefficients of like powers of p, we get a system of linear equations

$$\begin{gathered} Lu_0 = 0, \qquad u_0(x,0) = -x^3, \qquad u_{0t}(x,0) = 0, \qquad u_0(0,t) = \sin^2 t, \\ Lu_1 = 6x, \qquad u_1(x,0) = 0, \qquad u_{1t}(x,0) = 0, \qquad u_1(0,t) = 0, \\ Lu_2 = 0, \qquad u_2(x,0) = 0, \qquad u_{2t}(x,0) = 0, \qquad u_2(0,t) = 0, \ldots . \end{gathered} \tag{2.4}$$

Solving correspondingly we obtain

$$\begin{aligned}
u_0 &= -\frac{1}{2}\left((x+t)^3 + (x-t)^3\right) = -3t^2x - x^3 \quad \text{for } x \geq t \\
&= -3t^2x - x^3 + \sin^2(t-x) \quad \text{for } x < t, \\
u_1 &= \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} 6y\,dy\,ds = 3t^2x, \quad \text{for } x > t, \\
u_1 &= \frac{1}{2}\int_0^{t-x} \int_{t-x-s}^{x+t-s} 6y\,dy\,ds + \frac{1}{2}\int_{t-x}^{t} \int_{x-t+s}^{x+t-s} 6y dy\,ds = 3t^2x, \quad \text{for } x < t.
\end{aligned} \tag{2.5}$$

$u_2 = 0, \ldots$. Thus we obtain an exact solution $u_{\text{exact}} = u_1 + u_2$.

Now let us show that this problem can not be solved when L is taken as $L = u_{tt}$.

Indeed if we choose $v_0 = 0$ or $v_0 = -x^3$ (which seems most natural and appropriate) and $L = u_{tt}$ and construct the homotopy

$$Lu - Lv_0 + p[Lv_0 - u_{xx} - 6x] = 0, \tag{2.6}$$

we obtain an overdetermined problem for u_0 in the form

$$Lu_0 = 0, \qquad u_0(x,0) = -x^3, \qquad u_{0t}(x,0) = 0, \qquad u_0(0,t) = \sin^2 t \tag{2.7}$$

which has no solution.

If in the first stage we consider the boundary conditions $u_0(x,0) = -x^3$, $u_{0t}(x,0) = 0$, $u_0(0,t) = 0$, we obtain $u_0 = -x^3$ and in the second stage we need to solve the problem

$$Lu_1 = u_{0xx} + 6x, \qquad u_1(x,0) = 0, \qquad u_{1t}(x,0) = 0, \qquad u_1(0,t) = \sin^2 t, \tag{2.8}$$

which is overdetermined again and has no solution.

Example 2.2. Consider the problem

$$\begin{gathered}
u_{tt} - u_{xx} = u_x u - e^{2t}x + e^t x, \\
u(x,0) = x, \qquad u_t(x,0) = x, \qquad u(0,t) = 0.
\end{gathered} \tag{2.9}$$

The exact solution is $u = e^t x$. We take again $L = u_{tt} - u_{xx}$, $v_0 = 0$, and construct the homotopy

$$Lu - Lv_0 + p\left[Lv_0 - u_x u + e^{2t}x - e^t x\right] = 0. \tag{2.10}$$

Substituting (1.7) into (2.10) we obtain

$$\begin{aligned} & L\left(u_0 + pu_1 + p^2u_2 + \cdots\right) - Lv_0 \\ & \quad + p\left[Lv_0 - \left(u_{0x} + pu_{1x} + p^2u_{2x} + \cdots\right)\left(u_0 + pu_1 + \cdots\right) + e^{2t}x - e^t x\right] = 0, \end{aligned} \tag{2.11}$$

and equating terms of the coefficients of like powers of p gives

$$\begin{gathered} Lu_0 = Lv_0, \quad u_0(x,0) = x, \quad u_{0t}(x,0) = x, \quad u_0(0,t) = 0, \\ Lu_1 + Lv_0 - u_{0x}u_0 + e^{2t}x - e^t x = 0, \quad u_1(x,0) = 0, \quad u_{1t}(x,0) = 0, \quad u_1(0,t) = 0, \\ Lu_2 - u_{0x}u_1 - u_{1x}u_0 = 0, \quad u_2(x,0) = 0, \quad u_{2t}(x,0) = 0, \quad u_2(0,t) = 0, \\ Lu_3 - u_{2x}u_0 - u_{1x}u_1 - u_{0x}u_2 = 0, \quad u_3(x,0) = 0, \quad u_{3t}(x,0) = 0, \quad u_3(0,t) = 0, \ldots. \end{gathered} \tag{2.12}$$

Solving these equations we obtain (see [21, Chapter 3.4])

$$\begin{gathered} u_0 = \frac{1}{2}[(x+t) + (x-t)] + \frac{1}{2}\int_{x-t}^{x+t} y\,dy = x + \frac{(x+t)^2}{4} - \frac{(x-t)^2}{4} = x(t+1), \\ u_1 = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} \left(y(s+1)^2 - e^{2s}y + e^s y\right) dy\,ds = \frac{x}{12}\left(t^4 + 4t^3 + 6t^2 - 6t - 3e^{2t} + 12e^t - 9\right), \end{gathered} \tag{2.13}$$

for $x > t$ and

$$\begin{aligned} u_1 &= \frac{1}{2}\int_0^{t-x} \int_{t-x-s}^{x+t-s} g(y,s)\,dy\,ds + \frac{1}{2}\int_{t-x}^{t} \int_{x-t+s}^{x+t-s} g(y,s)\,dy\,ds \\ &= x^2e^{t-x} - \frac{1}{4}xe^{2t-2x} - \frac{3}{4}x + \frac{1}{2}t^2x - tx^3 + \frac{1}{3}t^3x + \frac{2}{3}tx^4 + \frac{1}{12}t^4x - \frac{1}{2}x^2e^{2t-2x} \\ &\quad + xe^{t-x} - \frac{1}{2}t^2x^3 - \frac{1}{2}tx - \frac{1}{2}x^3 + \frac{2}{3}x^4 - \frac{1}{4}x^5 + tx^3 - \frac{2}{3}tx^4 - \frac{1}{4}xe^{2t} + \frac{1}{2}t^2x^3 \\ &\quad + xe^t + \frac{1}{2}x^3 - \frac{2}{3}x^4 + \frac{1}{4}x^5 + \frac{1}{2}x^2\frac{e^{2t}}{e^{2x}} - x\frac{e^t}{e^x} - x^2\frac{e^t}{e^x} + \frac{1}{4}x\frac{e^{2t}}{e^{2x}} \\ &= \frac{1}{12}xt^4 + \frac{1}{3}xt^3 + \frac{1}{2}xt^2 - \frac{1}{2}xt - \frac{1}{4}xe^{2t} + xe^t - \frac{3}{4}x, \end{aligned} \tag{2.14}$$

Table 1: Maximum errors for Example 2.2.

$t \setminus x$	0.1	0.2	0.3	0.4	0.5	0.6	0.8	0.9
0.1	1×10^{-9}	3×10^{-9}	4×10^{-9}	6×10^{-9}	7×10^{-9}	9×10^{-9}	10^{-8}	10^{-8}
0.2	3×10^{-9}	2×10^{-7}	3×10^{-7}	4×10^{-7}	5×10^{-7}	6×10^{-7}	8×10^{-7}	9×10^{-7}
0.4	6×10^{-9}	4×10^{-7}	5×10^{-6}	3×10^{-5}	4×10^{-5}	4×10^{-5}	6×10^{-5}	7×10^{-5}
0.5	7×10^{-9}	5×10^{-7}	6×10^{-6}	4×10^{-5}	10^{-4}	2×10^{-4}	2×10^{-4}	3×10^{-4}
0.7	10^{-8}	7×10^{-7}	9×10^{-6}	5×10^{-5}	2×10^{-4}	7×10^{-4}	2×10^{-3}	2×10^{-3}
0.8	7×10^{-4}	10^{-3}	2×10^{-3}	3×10^{-3}	3×10^{-3}	4×10^{-3}	5×10^{-3}	6×10^{-3}
1.0	3×10^{-3}	6×10^{-3}	9×10^{-3}	10^{-2}	10^{-2}	0.018 4	2×10^{-2}	3×10^{-2}

for $x < t$. In a similar manner we have

$$\begin{aligned} Lu_2 = u_{0x}u_1 + u_{1x}u_0 &= (t+1)\frac{1}{12}x\left(t^4 + 4t^3 + 6t^2 - 6t - 3e^{2t} + 12e^t - 9\right) \\ &\quad + \frac{1}{12}\left(t^4 + 4t^3 + 6t^2 - 6t - 3e^{2t} + 12e^t - 9\right)(t+1)x \\ &= \frac{1}{6}x(t+1)\left(t^4 + 4t^3 + 6t^2 - 6t - 3e^{2t} + 12e^t - 9\right), \end{aligned} \tag{2.15}$$

and therefore

$$\begin{aligned} u_2 &= L^{-1}\left(\frac{1}{6}x(t+1)\left(t^4 + 4t^3 + 6t^2 - 6t - 3e^{2t} + 12e^t - 9\right)\right) \\ &= \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} \left(\frac{1}{6}y(s+1)\left(s^4 + 4s^3 + 6s^2 - 6s - 3e^{2s} + 12e^s - 9\right)\right) dy\, ds \\ &= \frac{x}{504}\left(2t^7 + 14t^6 + 42t^5 - 210t^3 - 378t^2 - 63te^{2t} + 1008te^t + 63t - 1008e^t + 1008\right), \end{aligned} \tag{2.16}$$

for $x > t$. In a similar manner we obtain

$$\begin{aligned} Lu_2 &= u_{0x}u_1 + u_{1x}u_0 = \frac{xt^5}{6} + \frac{5}{6}xt^4 + \frac{5}{3}xt^3 - \frac{xte^{2t}}{2} + 2xte^t - \frac{5xt}{2} - \frac{xe^{2t}}{2} + 2xe^t - \frac{3x}{2}, \\ u_2 &= \frac{1}{2}\int_0^{t-x} \int_{t-x-s}^{x+t-s} g(y,s)\, dy\, ds + \frac{1}{2}\int_{t-x}^{t} \int_{x-t+s}^{x+t-s} g(y,s)\, dy\, ds, \end{aligned} \tag{2.17}$$

for $x < t$, where $g(y,s) = (1/6)ys^5 + (5/6)ys^4 + (5/3)ys^3 - (1/2)yse^{2s} + 2yse^s - (5/2)ys - (1/2)ye^{2s} + 2ye^s - (3/2)y$ and so

$$u_2 = \frac{1}{252}xt^7 + \frac{1}{36}xt^6 + \frac{1}{12}xt^5 - \frac{5}{12}xt^3 - \frac{3}{4}xt^2 - \frac{1}{8}xte^{2t} + 2xte^t + \frac{1}{8}xt - 2xe^t + 2x. \tag{2.18}$$

The absolute errors between the exact and three term approximation of the series solution for some values of $(x,t) \in [0,1] \times [0,1]$ are shown in Table 1.

Example 2.3. Now we consider the problem

$$\begin{aligned} u_{tt} - u_{xx} &= u_{xt}^2 - 16t^2x^2 + 16t^2x - 2t^2 - 2x^2 + 2x, \\ u(x,0) = x, \qquad & u_t(x,0) = 0, \qquad u(0,t) = 0. \end{aligned} \tag{2.19}$$

The exact solution is $u = x + t^2x - t^2x^2$. We take $v_0 = 0$, and construct the homotopy

$$Lu - Lv_0 + p\left[Lv_0 - u_{xt}^2 + 16t^2x^2 - 16t^2x + 2t^2 + 2x^2 - 2x\right] = 0. \tag{2.20}$$

Now substituting (1.7) into (2.20) and equating terms of the coefficients of like powers of p, we obtain

$$\begin{gathered} Lu_0 = 0, \qquad u_0(x,0) = x, \qquad u_{0t}(x,0) = u_0(0,t) = 0, \\ Lu_1 + Lv_0 - u_{0xt}^2 + 16t^2x^2 - 16t^2x + 2t^2 + 2x^2 - 2x = 0, \qquad u_1(x,0) = u_{1t}(x,0) = u_1(0,t) = 0, \\ Lu_2 - 2u_{0xt}u_{1xt} = 0, \qquad u_2(x,0) = u_{2t}(x,0) = u_2(0,t) = 0, \\ Lu_3 - (u_{1xt})^2 - 2u_{0xt}u_{2xt} = 0, \qquad u_3(x,0) = u_{3t}(x,0) = u_3(0,t) = 0. \end{gathered} \tag{2.21}$$

Solving equations yields

$$\begin{aligned} u_0 &= \frac{1}{2}(x + t + x - t) = x, \\ u_1 &= \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} \left(-16s^2y^2 + 16s^2y - 2s^2 - 2y^2 + 2y\right) dy\, ds \\ &= -\frac{4}{45}t^6 - \frac{4}{3}t^4x^2 + \frac{4}{3}t^4x - \frac{1}{3}t^4 - t^2x^2 + t^2x \quad \text{for } x > t. \end{aligned} \tag{2.22}$$

In a similar manner we get

$$\begin{aligned} u_1 &= \frac{1}{2}\int_0^{t-x} \int_{t-x-s}^{x+t-s} \left(-16s^2y^2 + 16s^2y - 2s^2 - 2y^2 + 2y\right) dy\, ds \\ &+ \frac{1}{2}\int_{t-x}^{t} \int_{x-t+s}^{x+t-s} \left(-16s^2y^2 + 16s^2y - 2s^2 - 2y^2 + 2y\right) dy\, ds = -\frac{8}{15}t^5x - \frac{4}{3}t^4x \\ &- \frac{16}{9}t^3x^3 - \frac{4}{3}t^3x + \frac{4}{3}t^2x^4 + t^2x^2 + t^2x - \frac{8}{15}tx^5 - \frac{4}{3}tx^3 + \frac{4x^6}{45} + \frac{x^4}{3}, \end{aligned} \tag{2.23}$$

Table 2: Maximum errors for Example 2.3.

$t \setminus x$	0.1	0.2	0.4	0.5	0.7	0.8	0.9
0.1	2×10^{-5}	10^{-5}	10^{-6}	9×10^{-8}	5×10^{-6}	10^{-5}	2×10^{-5}
0.2	3×10^{-4}	2×10^{-4}	2×10^{-5}	5×10^{-6}	9×10^{-5}	2×10^{-4}	3×10^{-4}
0.4	3×10^{-3}	3×10^{-3}	7×10^{-4}	3×10^{-4}	2×10^{-3}	3×10^{-3}	6×10^{-3}
0.5	6×10^{-3}	6×10^{-3}	2×10^{-3}	1×10^{-3}	4×10^{-3}	9×10^{-3}	10^{-2}
0.7	10^{-2}	10^{-2}	10^{-2}	9.9×10^{-3}	2×10^{-2}	4×10^{-2}	6×10^{-2}
0.8	2×10^{-2}	2×10^{-2}	2×10^{-2}	2×10^{-2}	4×10^{-2}	7×10^{-2}	0.1106
0.9	2×10^{-2}	3×10^{-2}	3×10^{-2}	4×10^{-2}	8×10^{-2}	0.12594	0.1862

for $x < t$. Taking only two-term approximation for u we have

$$\begin{aligned}
u &= u_0 + u_1 = x - \frac{4}{45}t^6 - \frac{4}{3}t^4x^2 + \frac{4}{3}t^4x - \frac{1}{3}t^4 - t^2x^2 + t^2x \quad \text{for } x > t, \\
u &= x - \frac{8}{15}t^5x + \frac{4}{3}t^4x - \frac{16}{9}t^3x^3 - \frac{4}{3}t^3x + \frac{4}{3}t^2x^4 \\
&\quad + t^2x^2 + t^2x - \frac{8}{15}tx^5 - \frac{4}{3}tx^3 + \frac{4}{45}x^6 + \frac{1}{3}x^4,
\end{aligned} \tag{2.24}$$

for $x < t$. The absolute errors between the exact and two-term approximation of the series solution for some values of $(x, t) \in [0, 1] \times [0, 1]$ are shown in Table 2. A higher accuracy level can be attained by evaluating some more terms.

Example 2.4. Now we consider the problem

$$\begin{gathered}
u_{tt} - u_{xx} = u_x + u_t - 2x - 2, \\
u(x, 0) = x^2, \qquad u_t(x, 0) = 0, \qquad u(0, t) = \sin^2 t.
\end{gathered} \tag{2.25}$$

It is easy to show that the HPM can not be applied if L is taken as u_{tt}. The exact solution is

$$\begin{aligned}
u &= x^2 + \sin^2(t - x) \quad \text{for } x < t \\
&= x^2 \quad \text{for } x \geq t.
\end{aligned} \tag{2.26}$$

We construct the homotopy

$$Lu - Lv_0 + p[Lv_0 - u_x - u_t + 2x + 2] = 0, \tag{2.27}$$

with $L = u_{tt} - u_{xx}$. Substituting (1.7) into u and equating the coefficients of like powers of p, we get a system of linear equations

$$\begin{gathered}
Lu_0 = 0, \qquad u_0(x,0) = x^2, \qquad u_{0t}(x,0) = 0, \qquad u_0(0,t) = \sin^2 t, \\
Lu_1 = u_{0x} + u_{0t} - 2x - 2, \qquad u_1(x,0) = 0, \qquad u_{1t}(x,0) = 0, \qquad u_1(0,t) = 0, \\
Lu_2 = u_{1x} + u_{1t}, \qquad u_2(x,0) = 0, \qquad u_{2t}(x,0) = 0, \qquad u_2(0,t) = 0, \ldots .
\end{gathered} \tag{2.28}$$

Solving we obtain

$$\begin{aligned}
u_0 &= \frac{1}{2}\Big((x+t)^2 - (t-x)^2\Big) + \sin^2(t-x) = 2tx + \sin^2(t-x) \quad \text{for } x < t, \\
u_0 &= \frac{1}{2}\Big((x+t)^2 + (x-t)^2\Big) = x^2 + t^2 \quad \text{for } x > t, \\
Lu_1 &= u_{0x} + u_{0t} - 2x - 2 = 2t + 2x - 2x - 2 = 2t - 2, \\
u_1 &= \frac{1}{2}\int_0^{t-x}\int_{t-x-s}^{x+t-s}(2s-2)\,dy\,ds + \frac{1}{2}\int_{t-x}^{t}\int_{x-t+s}^{x+t-s}(2s-2)\,dy\,ds \\
&= t^2x - tx^2 - 2tx + \frac{1}{3}x^3 + x^2 \quad \text{for } x < t, \\
Lu_1 &= u_{0x} + u_{0t} - 2x - 2 = 2t - 2, \\
u_1 &= \frac{1}{2}\int_0^{t}\int_{x-t+s}^{x+t-s}(2s-2)\,dy\,ds = \frac{1}{3}t^2(t-3) \quad \text{for } x > t.
\end{aligned} \tag{2.29}$$

Continuing we obtain

$$\begin{aligned}
Lu_2 &= u_{1x} + u_{1t} = t^2 - 2tx - 2t + x^2 + 2x + 2tx - x^2 - 2x = t^2 - 2t, \\
u_2 &= \frac{1}{2}\int_0^{t-x}\int_{t-x-s}^{x+t-s}\Big(s^2 - 2s\Big)dy\,ds + \frac{1}{2}\int_{t-x}^{t}\int_{x-t+s}^{x+t-s}\Big(s^2 - 2s\Big)dy\,ds \\
&= \frac{7}{3}t^3x - \frac{5}{2}t^2x^2 - 5t^2x + \frac{4}{3}tx^3 + 3tx^2 - \frac{1}{4}x^4 - \frac{2}{3}x^3 \quad \text{for } x < t, \\
u_2 &= \frac{1}{2}\int_0^{t}\int_{x-t+s}^{x+t-s}\Big(s^2 - 2s\Big)dy\,ds = \frac{1}{12}t^4 - \frac{1}{3}t^3 \quad \text{for } x > t.
\end{aligned} \tag{2.30}$$

Thus the three-term approximation is

$$\begin{aligned}
u &= \sin^2(t-x) + x^2 + \frac{7}{3}t^3x - \frac{5}{2}t^2x^2 - 4t^2x + \frac{4}{3}tx^3 + 2tx^2 - \frac{x^4}{4} - \frac{x^3}{3} \quad \text{for } x < t, \\
u &= x^2 + \frac{1}{12}t^4 \quad \text{for } x > t,
\end{aligned} \tag{2.31}$$

and the error is less than $(1/12)t^4$ for $x > t$.

3. Conclusion

Homotopy perturbation method has been successful for solving many linear and nonlinear wave-type problems. However, it has difficulties in dealing with initial boundary problems, namely, in including all initial and boundary conditions together into the process of homotopy perturbation and computation. Our main goal is to construct the homotopy perturbation scheme containing all initial and boundary conditions together. The goal is achieved by involving an auxiliary operator which includes both variables x and t.

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