Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 870164, 7 pages
doi:10.1155/2011/870164

Research Article

Existence of Periodic Positive Solutions for Abstract Difference Equations

Shugui Kang, Yaqiong Cui, and Jianmin Guo

Institute of Applied Mathematics, Shanxi Datong University Datong, Shanxi 037009, China

Correspondence should be addressed to Shugui Kang, dtkangshugui@126.com

Received 30 April 2011; Accepted 9 June 2011

Academic Editor: M. De la Sen

We will consider the existence of multiple positive periodic solutions for a class of abstract difference equations by using the well-known fixed point theorem (due to Krasnoselskii).

In the past several years, the existence of periodic solutions for first-order functional differential equations

$$y'(t) = -a(t)y(t) + f\left(t, y(t - \tau(t))\right) \tag{1}$$

has been extensively investigated (see [1–3], and the references therein). In [4–6], the existence of periodic positive solutions for difference equations

$$x_{n+1} = a_n x_n + \lambda h_n f\left(x_{n-\tau(n)}\right) \tag{2}$$

has been considered. To the best of our knowledge, however, little has been done for the abstract difference equations (see [7–9]). In this note, we will consider this problem. To this end, let X be a real Banach space and let $K \subset X$ be a cone, then a Banach space X with a partial ordering $\leq$ induced by a cone K is called an ordered Banach space. On the other hand, we will denote the identity operator defined on X by I.

In [7–9], the authors considered the existence of periodic solutions for the abstract equation

$$x_{n+1} = A_n x_n + F_n(x_n). \tag{3}$$

In this note, we will consider the equation

$$x_{n+1} = A_n x_n + \lambda F_n\left(x_{n-\tau(n)}\right), \quad n \in Z, \tag{4}$$

where $\{A_n\}_{n\in Z}$ is a T-periodic sequence of bounded linear operator defined on X and satisfies $(\prod_{k=0}^{T-1} A_k^{-1} - I)^{-1} A_n (\prod_{k=0}^{T-1} A_k^{-1} - I) = A_n$ for $n \in Z$, $(\prod_{k=0}^{T-1} A_k^{-1} - I)x \in K$ and $(\prod_{k=0}^{T-1} A_k^{-1} - I)^{-1} x \in K$ for any $x \in K$, $A_k x \in K$ and $A_k^{-1} x \in K$ for any $x \in K$ $(k = 0, 1, \ldots, T-1)$, $\{\tau(n)\}_{n\in Z}$ is an integer valued T-periodic sequence, and $\{F_n\}_{n\in Z}$ is a T-periodic sequence of bounded functions from X to K, and λ is a positive constant.

If (4) has a T-periodic solution in X, then we have

$$\prod_{k=0}^{n} A_k^{-1} x_{n+1} - \prod_{k=0}^{n-1} A_k^{-1} x_n = \prod_{k=0}^{n} A_k^{-1}\left(\lambda F_n\left(x_{n-\tau(n)}\right)\right). \tag{5}$$

Summing the above equation from n to $n + T - 1$, we have

$$\prod_{k=0}^{n-1} A_k^{-1}\left(\prod_{k=n}^{n+T-1} A_k^{-1} - I\right) x_n = \sum_{s=n}^{n+T-1} \prod_{k=0}^{s} A_k^{-1}\left(\lambda F_s\left(x_{s-\tau(s)}\right)\right). \tag{6}$$

That is,

$$x_n = \lambda \sum_{s=n}^{n+T-1} G(n,s) F_s\left(x_{s-\tau(s)}\right), \quad n \in Z, \tag{7}$$

where

$$G(n,s) = \left(\prod_{k=0}^{T-1} A_k^{-1} - I\right)^{-1} \prod_{k=n}^{s} A_k^{-1}. \tag{8}$$

If (7) has a T-periodic solution in X, then we have

$$\begin{aligned} x_{n+1} - x_n = {} & \left(\prod_{k=0}^{T-1} A_k^{-1} - I\right)^{-1} \sum_{s=n+1}^{n+T} \prod_{k=n+1}^{s} A_k^{-1}\left(\lambda F_s\left(x_{s-\tau(s)}\right)\right) \\ & - \left(\prod_{k=0}^{T-1} A_k^{-1} - I\right)^{-1} \sum_{s=n}^{n+T-1} \prod_{k=n}^{s} A_k^{-1}\left(\lambda F_s\left(x_{s-\tau(s)}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\left(\prod_{k=0}^{T-1} A_k^{-1} - I \right)^{-1} A_n \left(\prod_{k=0}^{T-1} A_k^{-1} - I \right) - I \right) \sum_{s=n}^{n+T-1} G(n,s) \left(\lambda F_s \left(x_{s-\tau(s)} \right) \right) \\
&\quad + \left(\prod_{k=0}^{T-1} A_k^{-1} - I \right)^{-1} \left(\prod_{k=n+1}^{n+T} A_k^{-1} - I \right) \left(\lambda F_n \left(x_{n-\tau(n)} \right) \right) \\
&= A_n x_n - x_n + \lambda F_n \left(x_{n-\tau(n)} \right).
\end{aligned}
\tag{9}
$$

This equation is equivalent to (4). Thus, we have the following result.

Theorem 1. *Assume that* $A_0, A_1, \dots, A_{T-1}$ *and* $(\prod_{k=0}^{T-1} A_k^{-1} - I)$ *are invertible and* $A_{n+1}^{-1} A_{n+2}^{-1} \cdots A_{n+T}^{-1} = A_0^{-1} A_1^{-1} \cdots A_{T-1}^{-1}$ $(n \in Z)$. *Then* $\{x_n\}_{n \in Z}$ $(x_n \in X)$ *is a* T*-periodic solution of* (4) *if and only if it is a* T*-periodic solution of* (7).

We now assume that $0 < N \le \|G(n,s)\| \le M < +\infty$ for $n \in Z$ and $n \le s \le n+T-1$ and that $\sigma = N/M$. To obtain our main results, we firstly give a lemma. The proof of that lemma can be found in [10].

Lemma 1. *Let* E *be a Banach space, and let* $P \subset E$ *be a cone. Assume* Ω_1, Ω_2 *are bounded open subsets of* E *such that* $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. *Suppose that* $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ *is a completely continuous operator such that*

(1) $\|Tu\| \le \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geqslant \|u\|$ *for* $u \in P \cap \partial\Omega_2$ *or that*

(2) $\|Tu\| \geqslant \|u\|$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|$ *for* $u \in P \cap \partial\Omega_2$.

Then T *has a fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

For the sake of convenience, the conditions needed for our criteria are listed as follows.

(H_1) $F_n \in C(X,X)$, and there exists $\{u_k\} \subset X$ with $\|u_k\| \to 0$ such that $F_n(u_k) > \theta$ $(u_k \geqslant \theta)$ for $n = 1,2,\dots,T$ and $k = 1,2,\dots$.

(H_2) $F_n \in C(X,X)$ and $F_n(u) > \theta$ for $u > \theta$ and $n = 1,2,\dots,T$.

(L_1) $\lim_{\|u\| \to 0} \|F_n(u)\|/\|u\| = \infty$ for $n = 1,2,\dots,T$.

(L_2) $\lim_{\|u\| \to \infty} \|F_n(u)\|/\|u\| = \infty$ for $n = 1,2,\dots,T$.

(L_3) $\lim_{\|u\| \to 0} \|F_n(u)\|/\|u\| = 0$ for $n = 1,2,\dots,T$.

(L_4) $\lim_{\|u\| \to \infty} \|F_n(u)\|/\|u\| = 0$ for $n = 1,2,\dots,T$.

(L_5) $\lim_{\|u\| \to 0} \|F_n(u)\|/\|u\| = l$ for $n = 1,2,\dots,T$ and $0 < l < \infty$.

(L_6) $\lim_{\|u\| \to \infty} \|F_n(u)\|/\|u\| = L$ for $n = 1,2,\dots,T$ and $0 < L < \infty$.

Now let $\widehat{Y}$ be the set of all T-periodic sequences in X, endowed with the usual linear structure and the norm

$$
\|u\| = \max_{0 \le n \le T-1} \|u_n\|. \tag{10}
$$

Then $\widehat{Y}$ is a Banach space with cone

$$\Omega = \left\{ u = \{u_n\} \in \widehat{Y} : u_n \geqslant \theta,\ \|u_n\| \geqslant \sigma \|u\|,\ n \in Z \right\}. \tag{11}$$

Define a mapping $H : \widehat{Y} \to \widehat{Y}$ by

$$(Hu)_n = \lambda \sum_{s=n}^{n+T-1} G(n,s)\left(F_s\left(u_{s-\tau(s)}\right)\right), \quad n \in Z. \tag{12}$$

Then it is easily seen that H is completely continuous on (bounded) subset of Ω, and for $u \in \Omega$,

$$\begin{aligned} \|(Hu)_n\| &\leq \lambda \sum_{s=n}^{n+T-1} \|G(n,s)\| \cdot \left\|F_s\left(u_{s-\tau(s)}\right)\right\| \\ &\leq \lambda M \sum_{s=n}^{n+T-1} \left\|F_s\left(u_{s-\tau(s)}\right)\right\| \end{aligned} \tag{13}$$

so that

$$\|(Hu)_n\| \geqslant \lambda N \sum_{s=n}^{n+T-1} \left\|F_s\left(u_{s-\tau(s)}\right)\right\| \geqslant \sigma \|Hu\| \tag{14}$$

That is, $H\Omega$ is contained in Ω.

Lemma 2. *Assume that there exist two positive numbers a and b such that $a \neq b$,*

$$\max_{0 \leq \|x\| \leq a, 0 \leq n \leq T-1} \|F_n(x)\| \leq \frac{a}{\lambda A}, \tag{15}$$

$$\min_{\sigma b \leq \|x\| \leq b, 0 \leq n \leq T-1} \|F_n(x)\| \geq \frac{b}{\lambda B}, \tag{16}$$

where

$$A = \max_{0 \leq n \leq T-1} \sum_{s=n}^{n+T-1} \|G(n,s)\|, \tag{17}$$

$$B = \min_{0 \leq n \leq T-1} \sum_{s=n}^{n+T-1} \|G(n,s)\|. \tag{18}$$

Then there exists $\overline{u} \in \Omega$ which is a fixed point of H and satisfies $\min\{a,b\} \leq \|\overline{u}\| \leq \max\{a,b\}$.

Proof. Let $\Omega_\xi = \{w \in \Omega \mid \|w\| < \xi\}$. Assume that $a < b$, then, for any $u \in \Omega$ which satisfies $\|u\| = a$, in view of (15), we have

$$\|(Hu)_n\| \leq \left\{\lambda \sum_{s=n}^{n+T-1} \|G(n,s)\|\right\} \cdot \frac{a}{\lambda A} \leq \lambda A \cdot \frac{a}{\lambda A} = a. \tag{19}$$

That is, $\|Hu\| \leq \|u\|$ for $u \in \partial\Omega_a$. For any $u \in \Omega$ which satisfies $\|u\| = b$, we have

$$\|(Hu)_n\| \geqslant \left\{\lambda \sum_{s=n}^{n+T-1} \|G(n,s)\|\right\} \cdot \frac{b}{\lambda B} \geqslant \lambda B \cdot \frac{b}{\lambda B} = b. \tag{20}$$

That is, we have $\|Hu\| \geqslant \|u\|$ for $u \in \partial\Omega_b$. In view of Theorem 1, there exists $\overline{u} \in \Omega$, which satisfies $a \leq \|\overline{u}\| \leq b$ such that $H\overline{u} = \overline{u}$. If $a > b$, (19) is replaced by $\|(Hu)_n\| \geqslant b$ in view of (16) and (20) is replaced by $\|(Hu)_n\| \leq a$ in view of (15). The same conclusion is proved. The proof is complete. □

Theorem 2. *Suppose (H_1), (L_1), and (L_2) hold. Then for any $\lambda \in (0, \lambda^*)$,* (4) *has at least two positive periodic solutions, where*

$$\lambda^* = \frac{1}{A} \sup_{r>0} \frac{r}{\max_{0\leq\|u\|\leq r, 0\leq n\leq T-1} \|F_n(u)\|}. \tag{21}$$

Proof. In view of (H_1), we can let $q(r) = r/(A \max_{0\leq\|u\|\leq r, 0\leq n\leq T-1}\|F_n(u)\|)$. By ($L_1$) and ($L_2$), we see further that $\lim_{r\to 0} q(r) = \lim_{r\to\infty} q(r) = 0$. Thus, there exists $r_0 > 0$ such that $q(r_0) = \max_{r>0} q(r) = \lambda^*$. For any $\lambda \in (0, \lambda^*)$, by the intermediate value theorem, there exist $a_1 \in (0, r_0)$ and $a_2 \in (r_0, \infty)$ such that $q(a_1) = q(a_2) = \lambda$. Thus, we have $\|F_n(u)\| \leq a_1/(\lambda A)$ for $\|u\| \in [0, a_1]$ and $n = 0, 1, 2, \ldots, T-1$, and $\|F_n(u)\| \leq a_2/(\lambda A)$ for $\|u\| \in [0, a_2]$ and $n = 0, 1, 2, \ldots, T-1$. On the other hand, in view of (L_1) and (L_2), we see that there exist $b_1 \in (0, a_1)$ and $b_2 \in (a_2, \infty)$ such that $\|F_n(u)\|/\|u\| \geqslant 1/(\lambda\sigma B)$ for $\|u\| \in (0, b_1] \cup [b_2\sigma, \infty)$. That is, $\|F_n(u)\| \geqslant b_1/(\lambda B)$ for $\|u\| \in [b_1\sigma, b_1]$ and $\|F_n(u)\| \geqslant b_2/(\lambda B)$ for $\|u\| \in [b_2\sigma, b_2]$). An application of Lemma 2 leads to two distinct solutions of (4). □

Theorem 3. *Suppose (H_2), (L_3), and (L_4) hold. Then for any $\lambda > \lambda^{**}$,* (4) *has at least two positive periodic solutions, where*

$$\lambda^{**} = \frac{1}{B} \inf_{r>0} \frac{r}{\min_{\sigma r\leq\|u\|\leq r, 0\leq n\leq T-1} \|F_n(u)\|}, \tag{22}$$

and B is defined by (18).

Proof. Let $p(r) = r/(B \min_{\sigma r\leq\|u\|\leq r, 0\leq n\leq T-1}\|F_n(u)\|)$. Clearly, $p \in C((0,\infty),(0,\infty))$. From ($L_3$) and ($L_4$), we see that $\lim_{r\to 0} p(r) = \lim_{r\to\infty} p(r) = \infty$. Thus, there exists $r_0 > 0$ such that $p(r_0) = \min_{r>0} p(r) = \lambda^{**}$. For any $\lambda > \lambda^{**}$, there exist $b_1 \in (0, r_0)$ and $b_2 \in (r_0, \infty)$ such that $p(b_1) = p(b_2) = \lambda$. Thus we have $\|F_n(u)\| \geqslant b_1/(\lambda B)$ for $\|u\| \in [\sigma b_1, b_1]$ and $n = 0, 1, \ldots, T-1$, and $\|F_n(u)\| \geqslant b_2/(\lambda B)$ for $\|u\| \in [\sigma b_2, b_2]$ and $n = 0, 1, \ldots, T-1$. On the other hand, in view of (L_3), we see that there exists $a_1 \in (0, b_1)$ such that $\|F_n(u)\|/\|u\| \leq 1/(\lambda A)$ for $\|u\| \in (0, a_1]$ and

$n = 0, 1, \ldots, T-1$. Thus we have $\|F_n(u)\| \leq a_1/(\lambda A)$ for $0 \leq \|u\| \leq a_1$ and $n = 0, 1, \ldots, T-1$. In view of (L_4), we see that there exists $a \in (b_2, \infty)$ such that $\|F_n(u)\|/\|u\| \leq 1/(\lambda A)$ for $\|u\| \in (a, \infty)$ and $n = 0, 1, \ldots, T-1$. Let $\delta = \max_{0 \leq \|u\| \leq a, 0 \leq n \leq T-1} \|F_n(u)\|$. Then we have $\|F_n(u)\| \leq a_2/(\lambda A)$ for $\|u\| \in [0, a_2]$ and $n = 0, 1, \ldots, T-1$, where $a_2 > a$ and $a_2 \geqslant \lambda \delta A$. An application of Lemma 2 leads to two distinct solutions of (4). □

Theorem 4. *Assume that (H_2), (L_5), and (L_6) hold. Then, for each λ satisfying*

$$\frac{1}{\sigma BL} < \lambda < \frac{1}{Al} \tag{23}$$

or

$$\frac{1}{\sigma Bl} < \lambda < \frac{1}{AL}, \tag{24}$$

equation (4) *has a positive periodic solution.*

Proof. Suppose (23) holds. Let $\varepsilon > 0$ such that

$$\frac{1}{\sigma B(L-\varepsilon)} \leq \lambda \leq \frac{1}{A(l+\varepsilon)}. \tag{25}$$

Note that $l > 0$, then there exists $H_1 > 0$ such that $\|F_n(u)\| \leq (l+\varepsilon)\|u\|$ for $0 < \|u\| \leq H_1$ and $n = 0, 1, \ldots, T-1$. So, for $u \in \Omega$ with $\|u\| = H_1$, we have

$$\begin{aligned} \|(Hu)_n\| &\leq \lambda(l+\varepsilon) \sum_{s=n}^{n+T-1} \|G(n,s)\| \cdot \left\|u_{s-\tau(s)}\right\| \\ &\leq \lambda(l+\varepsilon)\|u\| \sum_{s=n}^{n+T-1} \|G(n,s)\| \\ &\leq \lambda A(l+\varepsilon)\|u\| \leq \|u\|. \end{aligned} \tag{26}$$

Next, since $L > 0$, there exists a $\overline{H_2} > 0$ such that $\|F_n(u)\| \geqslant (L-\varepsilon)\|u\|$ for $\|u\| \geqslant \overline{H_2}$ and $n = 0, 1, \ldots, T-1$. Let $H_2 = \max\{2H_1, \overline{H_2}\}$. Then for $u \in \Omega$ with $\|u\| = H_2$,

$$\begin{aligned} \|(Hu)_n\| &\geqslant \lambda(L-\varepsilon) \sum_{s=n}^{n+T-1} \|G(n,s)\| \cdot \left\|u_{s-\tau(s)}\right\| \\ &\geqslant \lambda(L-\varepsilon)\sigma\|u\| \sum_{s=n}^{n+T-1} \|G(n,s)\| \\ &\geqslant \lambda(L-\varepsilon)\sigma B\|u\| \geqslant \|u\|. \end{aligned} \tag{27}$$

In view of Lemma 1, we see that (4) has a positive periodic solution.

The other case is similarly proved. □

Our Theorems 1–4 generalize the main results from [5, 6].

If $T = 2$, X is a Hilbert space, A_0, A_1, and $A_0^{-1}A_1^{-1} - I$ are invertible self-conjugate operator defined on X, $A_0A_1, (A_0^{-1}A_1^{-1} - I)A_0$, $(A_0^{-1}A_1^{-1} - I)A_1$ are self-conjugate operator defined on X, then A_0, A_1 satisfy conditions of this paper.

As an example, let both $\{\lambda_n\}$ and $\{\lambda'_n\}$ be real bounded sequence, $\{\mu_n\}$ and $\{\mu'_n\}$ are also real bounded sequence, where

$$\mu_n = \begin{cases} \dfrac{1}{\lambda_n}, & \lambda_n \neq 0, \\ 0, & \lambda_n = 0, \end{cases} \qquad \mu'_n = \begin{cases} \dfrac{1}{\lambda'_n}, & \lambda'_n \neq 0, \\ 0, & \lambda'_n = 0. \end{cases} \tag{28}$$

$\{e_n\}$ is complete orthonormal set of space l^2 : $e_n = \{0, \dots, 0, \overset{(n)}{1}, 0, \dots 0\}$ $(n = 1, 2, \dots)$. Let

$$A_0 x = \sum_{n=1}^{\infty} \xi_n \lambda_n e_n, \qquad A_1 x = \sum_{n=1}^{\infty} \xi_n \lambda'_n e_n \tag{29}$$

for any $x = \sum_{n=1}^{\infty} \xi_n e_n$, then A_0 and A_1 are both self-conjugate operator, and satisfy all of above conditions.

Acknowledgments

The project is partially supported by the Natural Science Foundation of Shanxi Province (2008011002-1) and Shanxi Datong University (2010-B-01, 2009-Y-15) and by High Science and Technology Foundation of Shanxi Province (20101109).

References

[1] S. Kang, B. Shi, and G. Wang, "Existence of maximal and minimal periodic solutions for first-order functional differential equations," *Applied Mathematics Letters*, vol. 23, no. 1, pp. 22–25, 2010.

[2] S. Kang and S. S. Cheng, "Existence and uniqueness of periodic solutions of mixed monotone functional differential equations," *Abstract and Applied Analysis*, vol. 2008, Article ID 162891, 13 pages, 2009.

[3] S. Kang and G. Zhang, "Existence of nontrivial periodic solutions for first order functional differential equations," *Applied Mathematics Letters*, vol. 18, no. 1, pp. 101–107, 2005.

[4] R. Y. Zhang, Z. C. Wang, Y. Chen, and J. Wu, "Periodic solutions of a single species discrete population model with periodic harvest/stock," *Computers & Mathematics with Applications*, vol. 39, no. 1-2, pp. 77–90, 2000.

[5] S. Cheng and G. Zhang, "Positive periodic solutions of a discrete population model," *Functional Differential Equations*, vol. 7, no. 3-4, pp. 223–230, 2000.

[6] Y. Gao, G. Zhang, and W. G. Ge, "Existence of periodic positive solutions for delay difference equations," *Journal of Systems Science and Mathematical Sciences*, vol. 23, no. 2, pp. 155–162, 2003.

[7] M. I. Gil' and S. S. Cheng, "Periodic solutions of a perturbed difference equation," *Applicable Analysis*, vol. 76, no. 3-4, pp. 241–248, 2000.

[8] M. Gil', "Periodic solutions of abstract difference equations," *Applied Mathematics E-Notes*, vol. 1, pp. 18–23, 2001.

[9] M I. Gil', S. Kang, and G. Zhang, "Positive periodic solutions of abstract difference equations," *Applied Mathematics E-Notes*, vol. 4, pp. 54–58, 2004.

[10] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Orlando, Fla, USA, 1988.