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Research Article

Complex Dynamics in Nonlinear Triopoly Market with Different Expectations

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A dynamic triopoly game characterized by firms with different expectations is modeled by three-dimensional nonlinear difference equations, where the market has quadratic inverse demand function and the firm possesses cubic total cost function. The local stability of Nash equilibrium is studied. Numerical simulations are presented to show that the triopoly game model behaves chaotically with the variation of the parameters. We obtain the fractal dimension of the strange attractor, bifurcation diagrams, and Lyapunov exponents of the system.

1. Introduction

An oligopoly is a market form in which a market or industry is dominated by a small number of sellers (oligopolists). Because there are few sellers, each oligopolist is likely to be aware of the actions of the others. The decisions of one firm influence, and are influenced by, the decisions of other firms. Strategic planning by oligopolists needs to take into account the likely responses of the other market participants.

The classic model of oligopolies was proposed by the French mathematician, Cournot [1]. Recently, the dynamics of the oligopoly game have been studied. Puu [2] studied the adjustment process by three Cournot oligopolists based on an isoelastic demand function and constant marginal costs. Ahmed et al. [3] built the dynamical system model of bounded rationality. Yassen and Agiza [4] analyzed a duopoly game with delayed bounded rationality, and they used the quadratic cost function form, $C_i(q_i) = c_i q_i^2$. Expectations play an important role in modelling economic phenomena. Agiza et al. [5] studied the complex dynamics and synchronization of a duopoly game with the same expectation strategies. Then, Agiza and Elsadany [6] extended the same expectations strategies to the different expectations strategies case. Bischi and Kopel [7] introduced adaptive expectations in a duopoly game.

Du and Huang [8] obtained that the real-stable region of Nash equilibrium of output game model is smaller than that in general. Brianzoni et al. [9] studied the relationship between corruption in public procurement and economic growth within the Solow framework in discrete time. Ma and Ji [10] established a model on the electricity market. In the model, the inverse demand function and cost functions are all nonlinear, and the three firms take the same expectation strategies, that is, bounded rationality. Du et al. [11] studied an output duopoly competing evolution model by using modern game theory and decision-making analyses about chaos control. Ma et al. [12] analyzed dynamic process of the triopoly games in Chinese 3G telecommunication market basing on a Bertrand model with bounded rationality. Sheng et al. [13] discussed self-adaptive proportional control method in economic chaotic system, and the results showed that performances of the system are improved by controlling chaos. Elabbasy et al. [14] analyzed triopoly game with heterogeneous players which possess liner demand function and parabolic total cost function. Xin et al. [15] presented a nonlinear discrete game model for two oligopolistic firms whose products are adnascent. In microeconomics, however, the total cost function is analogous to the cubic function whose inflection point lies in the first quadrant, that is, the slope of total cost function is always nonnegative in its definitional domain and decreases to zero on the left side of the inflection point, but in gradually increases while on the right side of the inflection point.

By supposing the quadratic inverse demand function and cubic total cost functions, we establish a model on the three oligarchs market basing on the above models.

In this paper, we consider that each firm form a different strategy in order to compute its expected output. We assume that first firm adopts naive expectations and second firm has adaptive expectations, while third firm represents a boundedly rational player. The main aim of this work is to investigate the dynamic behaviors of three firms using different expectations rules. Theoretical analysis and numerical simulations of the system are made in detail.

The structure of the paper is as follows. In Section 2, we describe a nonlinear triopoly game model. In Section 3, we analyze the fixed points and local stability of the model. In Section 4, we study the strange attractor, bifurcation, and Lyapunov exponent by numerical simulations. Finally, a conclusion is drawn in Section 5.

2. The Triopoly Game Model

We consider a Cournot triopoly game where q_i denotes the quantity supplied by firm i, $i = 1, 2, 3$. The firms offer goods at discrete-time periods $t = 0, 1, 2$, a common market. Suppose that the t-output of firm i is $q_i(t)$. At one period t, each firm must form an expectations of the rival's output in the next time period in order to determine the corresponding profit-maximizing quantities for period $t + 1$. The total outputs are

$$Q(t) = q_1(t) + q_2(t) + q_3(t), \tag{2.1}$$

and the inverse demand function [16] is

$$P = P(Q(t)) = m - nQ^2(t). \tag{2.2}$$

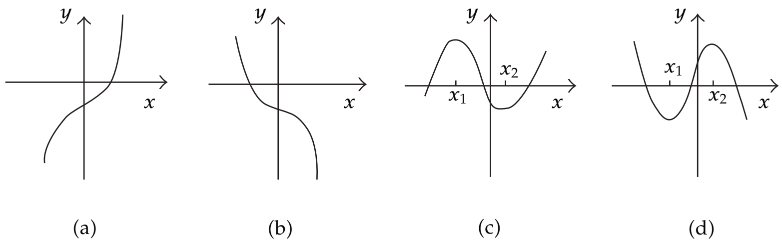

Figure 1: The four forms of cubic function graphs.

In microeconomics, the total cost curve is the analogy of cubic function, so we employ

$$C_i(q_i(t)) = a_i + b_i q_i(t) + c_i q_i^2(t) + d_i q_i^3(t), \quad i = 1,2,3. \tag{2.3}$$

The derivative of total cost function is

$$C_i'(q_i(t)) = 3d_i q_i^2(t) + 2c_i q_i(t) + b_i, \tag{2.4}$$

and the discriminant is

$$\Delta = (2c_i)^2 - 4(3d_i)b_i = 4c_i^2 - 12b_i d_i. \tag{2.5}$$

There are four forms of cubic function graph (Figure 1): if $d > 0$ and $\Delta \le 0$, that is, Figure 1(a); if $d < 0$ and $\Delta \le 0$, that is, Figure 1(b); if $d > 0$ and $\Delta > 0$, that is, Figure 1(c); if $d < 0$ and $\Delta > 0$, that is, Figure 1(d).

In Figure 1(a), when $\Delta \le 0$, $C_i'(q_i(t)) \ge 0$, $q_i(t) \in \Re$, always established, also the inflection point $(-c_i/3d_i, C_i(-c_i/3d_i))$ falls in the first quadrant, at the same time $a_i > 0$ (fixed cost is positive), $d_i > 0$, that is,

$$\begin{gathered} d_i > 0, \\ a_i > 0, \\ \Delta = 4c_i^2 - 12b_i d_i \le 0, \\ -\frac{c_i}{3d_i} > 0, \\ C_i\left(-\frac{c_i}{3d_i}\right) = 2c_i^3 - 9b_i c_i d_i + 27a_i d_i^2 > 0, \end{gathered} \tag{2.6}$$

the cubic function becomes total cost function in microeconomics. Hence, the profit of firm i in period t is given by

$$\pi_i(t) = q_i(t)\left[m - nQ^2(t)\right] - \left[a_i + b_i q_i(t) + c_i q_i^2(t) + d_i q_i^3(t)\right], \quad i = 1,2,3. \tag{2.7}$$

In this game, the firm makes the optimal output decision for the maximal profit. One of the methods is to calculate the partial differentiation of the profit and let it be equal to 0:

$$\frac{\partial \pi_i(t)}{\partial q_i(t)} = m - nQ^2(t) - 2nq_i(t)Q(t) - b_i - 2c_iq_i(t) - 3d_iq_i^2(t) = 0, \quad i = 1,2,3. \tag{2.8}$$

Based on (2.8), we can find out the firm's response function (2.9) for its competitors of a certain period in triopoly market. Also (2.9) expresses a firm's optimal output from the every given possible speculated productions of other two firms in a fixed time, thus the maximum benefit is obtained:

$$\begin{aligned} q_1^*(t) &= \frac{1}{3n + 3d_1}\left[-2n\left(q_2(t) + q_3(t)\right) - c_1 + \sqrt{M}\right], \\ q_2^*(t) &= \frac{1}{3n + 3d_2}\left[-2n\left(q_1(t) + q_3(t)\right) - c_2 + \sqrt{N}\right], \\ q_3^*(t) &= \frac{1}{3n + 3d_3}\left[-2n\left(q_1(t) + q_2(t)\right) - c_3 + \sqrt{T}\right]. \end{aligned} \tag{2.9}$$

In (2.9),

$$\begin{aligned} M &= \left(n^2 - 3nd_1\right)\left(q_2(t) + q_3(t)\right)^2 + 4nc_1\left(q_2(t) + q_3(t)\right) + c_1^2 + 3(mn - nb_1 + md_1 - b_1d_1), \\ N &= \left(n^2 - 3nd_2\right)\left(q_1(t) + q_3(t)\right)^2 + 4nc_2\left(q_1(t) + q_3(t)\right) + c_2^2 + 3(mn - nb_2 + md_2 - b_2d_2), \\ T &= \left(n^2 - 3nd_3\right)\left(q_1(t) + q_2(t)\right)^2 + 4nc_3\left(q_1(t) + q_2(t)\right) + c_3^2 + 3(mn - nb_3 + md_3 - b_3d_3). \end{aligned} \tag{2.10}$$

The first firm adopts naive expectations, that is,

$$q_i(t+1) = q_i^*(t). \tag{2.11}$$

The second firm has adaptive expectations, that is,

$$q_i(t+1) = q_i(t) + \alpha\left[q_i(t) - q_i^*(t)\right], \quad -1 < \alpha < 0, \tag{2.12}$$

where α is feedback parameter. The third firm represents a boundedly rational player, that is,

$$q_i(t+1) = q_i(t) + \beta q_i(t)\frac{\partial \pi_i(t)}{\partial q_i(t)}, \quad 0 < \beta < 1, \tag{2.13}$$

where β is the output modification speed parameter. Hence, the dynamical triopoly game in this case is formed from combining (2.11)–(2.13). Then, the dynamical system of different expectations is described by

$$\begin{aligned}
q_1(t+1) &= \frac{1}{3n+3d_1}\left[-2n\left(q_2(t)+q_3(t)\right)-c_1+\sqrt{M}\right],\\
q_2(t+1) &= q_2(t)+\alpha\left\{q_2(t)-\frac{1}{3n+3d_2}\left[-2n\left(q_1(t)+q_3(t)\right)-c_2+\sqrt{N}\right]\right\},\\
q_3(t+1) &= q_3(t)+\beta q_3(t)\Big[-3(n+d_3)q_3^2(t)-\left(4nq_1(t)+4nq_2(t)+2c_3\right)q_3(t)\\
&\qquad -n\left(q_1(t)+q_2(t)\right)^2+m-b_3\Big].
\end{aligned} \tag{2.14}$$

In the next sections, we study the rich dynamical behaviors of this model.

3. The Fixed Points and Local Stability

To investigate the local stability of the fixed points, we find the Jacobian matrix for the system of (2.14) as the following form:

$$\mathbf{J}=\begin{pmatrix} J_{11} & J_{12} & J_{13}\\ J_{21} & J_{22} & J_{23}\\ J_{31} & J_{32} & J_{33}\end{pmatrix}. \tag{3.1}$$

In the Jacobian matrix, all the elements are

$$\begin{aligned}
J_{11} &= 0,\\
J_{12} &= \frac{1}{3n+3d_1}\left[-2n+\frac{\left(n^2-3nd_1\right)\left(q_2(t)+q_3(t)\right)+2nc_1}{\sqrt{M}}\right],\\
J_{13} &= \frac{1}{3n+3d_1}\left[-2n+\frac{\left(n^2-3nd_1\right)\left(q_2(t)+q_3(t)\right)+2nc_1}{\sqrt{M}}\right],\\
J_{21} &= \frac{\alpha}{3n+3d_2}\left[2n-\frac{\left(n^2-3nd_2\right)\left(q_1(t)+q_3(t)\right)+2nc_2}{\sqrt{N}}\right],\\
J_{22} &= 1+\alpha,
\end{aligned}$$

$$J_{23} = \frac{\alpha}{3n+3d_2}\left[2n - \frac{\left(n^2 - 3nd_2\right)\left(q_1(t) + q_3(t)\right) + 2nc_2}{\sqrt{N}}\right],$$

$$J_{31} = -2n\beta q_3(t)\left(2q_3(t) + q_1(t) + q_2(t)\right),$$

$$J_{32} = -2n\beta q_3(t)\left(2q_3(t) + q_1(t) + q_2(t)\right),$$

$$J_{33} = -9\beta q_3^2(t)(n + d_3) - 4\beta q_3(t)\left(2nq_1(t) + 2nq_2(t) + c_3\right) - n\beta\left(q_1(t) + q_2(t)\right)^2 + m\beta - \beta b_3 + 1. \tag{3.2}$$

It is difficult to obtain the analytical solutions in (2.14), so we assign a value to each parameter. Let $m = 5$, $n = 1$, $b_1 = 0.4$, $c_1 = -0.03$, $d_1 = 0.005$, $b_2 = 0.35$, $c_2 = -0.025$, $d_2 = 0.006$, $b_3 = 0.3$, $c_3 = -0.02$, $d_3 = 0.007$, and $q_i(t+1) = q_i(t),\ i = 1, 2, 3$. We can have at most twelve fixed points:

$$\begin{aligned}
p_1 &= (-11.7603, 22.7082, -11.0250),\\
p_2 &= (-23.3748, 12.1247, 11.3572),\\
p_3 &= (12.1586, -22.7057, 10.6744),\\
p_4 &= (24.0193, -12.2490, -11.8248),\\
p_5 &= (-10.9683, -10.6505, 21.5217),\\
p_6 &= (-0.5387, -0.5548, -0.5709),\\
p_7 &= (0.5450, 0.5581, 0.5712),\\
p_8 &= (11.4286, 10.7135, -21.9975),\\
p_9 &= \left(-16.5828 - i\cdot 1.1934 \times 10^{-39}, 16.5984 + i\cdot 1.0365 \times 10^{-39}, 0\right),\\
p_{10} &= \left(0.7563 - i\cdot 2.9304 \times 10^{-40}, 0.7697 - i\cdot 1.0252 \times 10^{-39}, 0\right),\\
p_{11} &= \left(-0.7471 + i\cdot 2.1154 \times 10^{-40}, -0.7651 + i\cdot 7.7378 \times 10^{-40}, 0\right),\\
p_{12} &= \left(16.9122 + i\cdot 1.3936 \times 10^{-41}, -16.8731 + i\cdot 5.0528 \times 10^{-41}, 0\right).
\end{aligned} \tag{3.3}$$

They are all independent of the parameters α and β apparently. The outputs of zero, negative number and complex number, are meaningless in application, so they are omitted from consideration. Only p_7 is reasonable, and the Jacobian matrix at p_7 is

$$\mathbf{J} = \begin{pmatrix} 0 & -0.5732 & -0.5732 \\ 0.5736\alpha & \alpha + 1 & 0.5736\alpha \\ -2.5653\beta & -2.5653\beta & 1 - 4.4688\beta \end{pmatrix}. \tag{3.4}$$

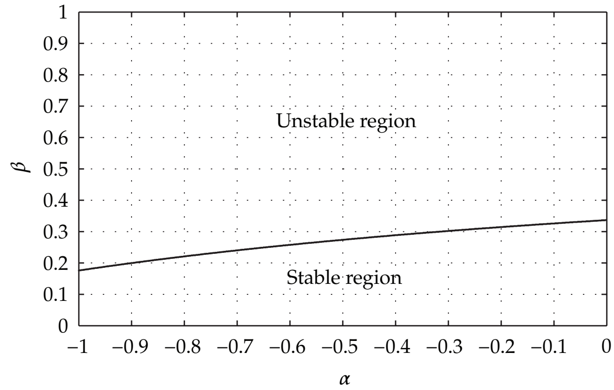

Figure 2: The stable region of the fixed point p_7.

Its characteristic equation is

$$f(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C, \tag{3.5}$$

where

$$\begin{aligned} A &= -2 - \alpha + 4.4688\beta, \\ B &= 1.3288\alpha - 2.9973\alpha\beta - 5.9392\beta + 1, \\ C &= 1.2528\alpha\beta - 0.3288\alpha + 1.4704\beta. \end{aligned} \tag{3.6}$$

According to the Routh-Hurwitz stability criterion, the necessary and sufficient condition of asymptotic stabilization at p_7 is that all zero points of its characteristic polynomial are inside the unit circle in complex plane. So it must satisfy the following four conditions [17]:

$$\begin{aligned} f(1) &= A + B + C + 1 > 0, \\ -f(-1) &= -A + B - C + 1 > 0, \\ C^2 - 1 &< 0, \\ \left(1 - C^2\right)^2 - (B - AC)^2 &> 0. \end{aligned} \tag{3.7}$$

The conditions (3.7) determine a stable region in the plane (α, β) as shown in Figure 2. However, p_7 is asymptotically stable with the values α, β in the stable region, and it shows that the output will reach the Nash equilibrium p_7 by modulating limited times with random initial output.

From Figure 2, it is clear that the outputs are asymptotically stable which the firm adopts adaptive expectations of negative feedback mechanism $(-1 < \alpha < 0)$, but market will loose of stability with the change of β.

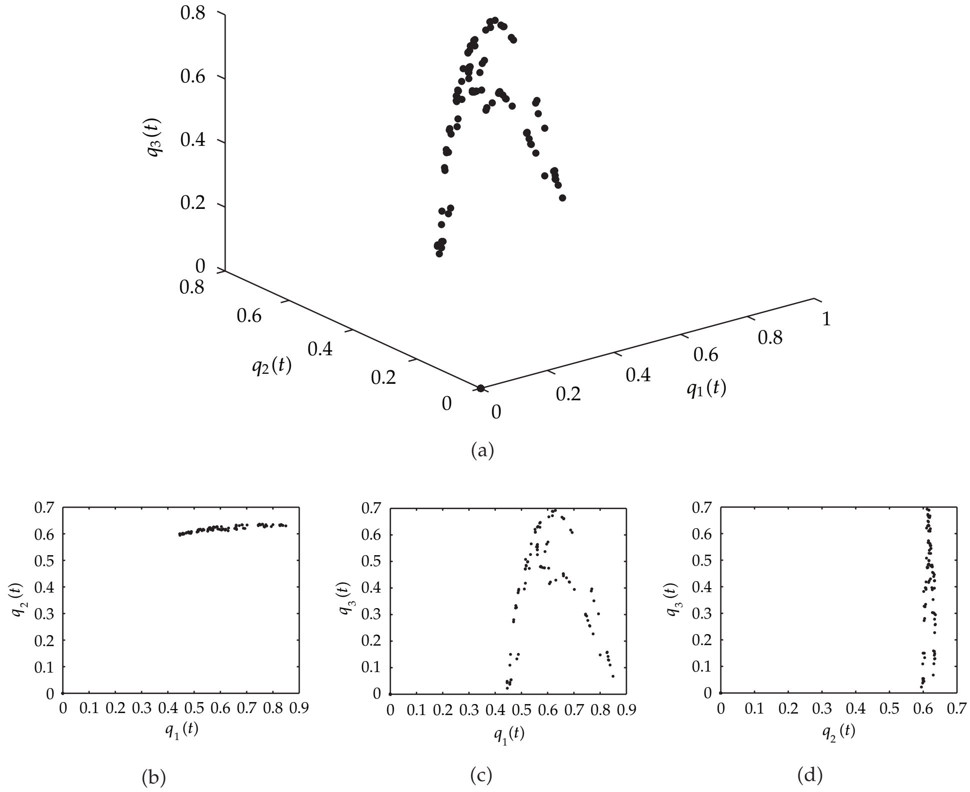

Figure 3: Three-dimensional and two-dimensional view of strange attractors.

4. Numerical Simulations of the System

4.1. The Strange Attractor and Fractal Dimension

In the phase space, the chaotic motion is stochastic and its trajectory never closed in a given region. When the parameters take the values of $m = 5$, $n = 1$, $b_1 = 0.4$, $c_1 = -0.03$, $d_1 = 0.005$, $b_2 = 0.35$, $c_2 = -0.025$, $d_2 = 0.006$, $b_3 = 0.3$, $c_3 = -0.02$, $d_3 = 0.007$, $\alpha = -0.1$, $\beta = 0.57$, and the initial outputs are 0.2, 0.5, 0.8, the chaotic attractors of system map (2.14) is shown in Figure 3.

An attractor is informally described as strange if it has non integer dimension. This is often the case when the dynamics on it are chaotic, and the trajectory may be periodic or chaotic. The obvious character of the chaotic attractor is the exponential separation of two adjacent trajectories, which shows the sensitive dependence on the initial conditions of the chaotic system. The Lyapunov exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. It is common to refer to the largest one as the Maximal Lyapunov exponent (MLE), because it determines a notion of predictability for a dynamical system. A positive MLE is usually taken as an indication that the system is chaotic. The Lyapunov exponents of the system map (2.14) on the above conditions are $\lambda_1 = 0.300850$, $\lambda_2 = -0.062698$, and $\lambda_3 = -0.913312$, respectively. The MLE λ_1 is positive, which shows the chaotic character in the outputs game model of the triopoly market.

Strange attractors are typically characterized by fractal dimension. Fractal dimension illustrates that the chaotic motion has self-similar structure, that is to say, the chaotic motion follows a definite rule. In particular from, the knowledge of the Lyapunov exponents, it is possible to obtain the so-called Kaplan-Yorke dimension D_{KY}, which is defined as follows:

$$D_{\mathrm{KY}} = k + \frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|}, \tag{4.1}$$

where k is the maximum integer such that the sum of the k largest exponents is still non negative, that is, k is the the maximum i satisfying $\sum_{i=1}^{k} \lambda_i \geq 0$ and $\sum_{i=1}^{k+1} \lambda_i < 0$. The λ_i is the Lyapunov exponents series, arranged in descending order by numerical value. D_{KY} represents an upper bound for the information dimension of the system [18]. Therefore, in the system map (2.14), $k = 2$ and the Kaplan-Yorke dimension is

$$D_{\mathrm{KY}} = 2 + \frac{0.300850 - 0.062698}{|-0.913312|} = 2.260756, \tag{4.2}$$

which is hyperchaotic behavior. This shows that the economic system is a chaotic system of fractal dimensional structure at this time, so that the evolution of system becomes more complex. When the system sinks into chaotic state, the firms will be difficult to make long-term strategic planning and cannot obtain a stable profit. At the same time, because of sharp market fluctuations, it is also difficult for firms to keep pace with market changes.

4.2. The Outputs Bifurcation and Lyapunov Exponent Spectrum

To provide some numerical evidences for the chaotic behavior of system map (2.14), we present outputs bifurcations diagrams with respect to α and β (Figures 4 and 5) and Laypunov exponent spectrum with respect to α and β (Figures 6 and 7). Figures 4 and 6 are fixed $\beta = 0.25$, $\alpha \in [-1, 0]$. Figures 5 and 7 are fixed $\alpha = -0.1$, $\beta \in [0, 0.6]$. The parameters take the values of $m = 5$, $n = 1$, $b_1 = 0.4$, $c_1 = -0.03$, $d_1 = 0.005$, $b_2 = 0.35$, $c_2 = -0.025$, $d_2 = 0.006$, $b_3 = 0.3$, $c_3 = -0.02$, $d_3 = 0.007$, and the initial outputs are 0.2, 0.5, 0.8.

Figure 4 shows that the trajectories, through inverse period-doubling bifurcations, reach Nash equilibrium $p_7(0.5450, 0.5581, 0.5712)$ with the increase of α, and the chaotic phenomenon does not emerge. This can also be discovered in Figure 6 that there is no positive Lyapunov exponent. The bifurcation diagram is in good agreement with Lyapunov exponent spectrum. It indicates that when the firm takes adaptive expectations, the smaller the absolute value of the negative feedback factor α is, the more stable of the market will be.

Figure 5 shows that the trajectories converge to the Nash equilibrium p_7 when $\beta < 0.3225$, and the Nash equilibrium becomes unstable when $\beta > 0.3225$. Then, the period doubling bifurcations appears, that is, period-doubling, period four, period eight, and the chaotic behaviors occur when $\beta > 0.5525$. It can be obtained from Figure 7 that the Lyapunov exponents are positive corresponding to the chaotic region. This means that the market becomes unstable and easily access to the chaotic state for a large value of adjustment speed. In a word, the adjustment speed of the bounded rational firm on the market can cause the outputs game model to demonstrate complicated characters.

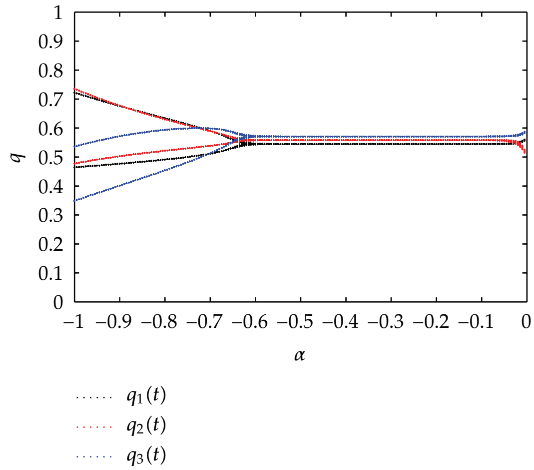

Figure 4: Bifurcation with $\alpha \in [-1, 0]$, $\beta = 0.25$.

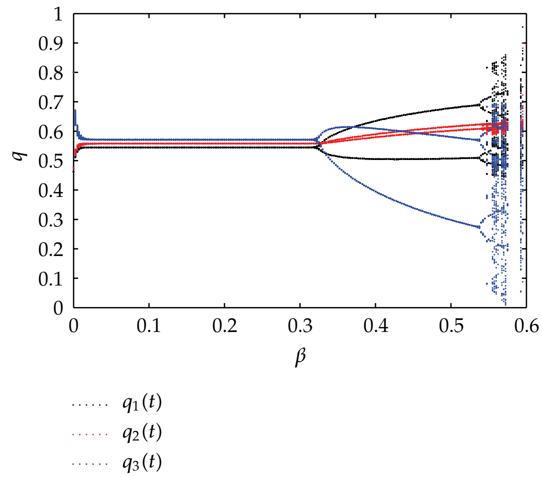

Figure 5: Bifurcation with $\alpha = -0.1$, $\beta \in [0, 0.6]$.

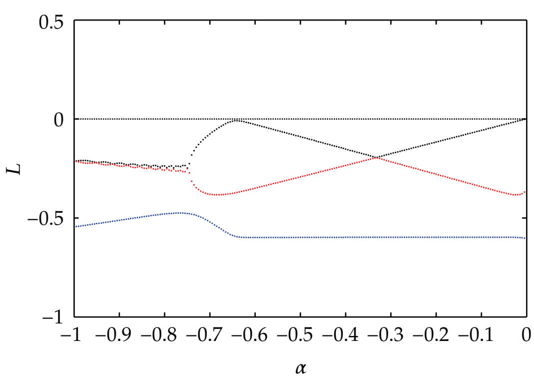

Figure 6: Lyapunov exponent with $\alpha \in [-1, 0]$, $\beta = 0.25$.

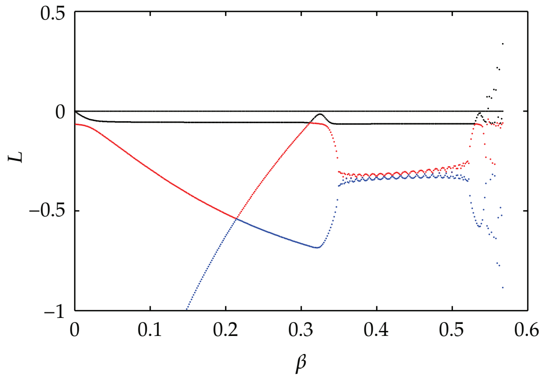

Figure 7: Lyapunov exponent with $\alpha = -0.1$, $\beta \in [0, 0.6]$.

5. Conclusion

In this paper, assuming that the inverse demand function is quadratic and the total cost function is cubic, we analyze the dynamic behaviors of triopoly market model with different expectations. Then the stability of the Nash equilibrium, bifurcation, and chaotic behavior of the repeated game are investigated. We think that the cubic total cost function is more reasonable than parabolic total cost function in microeconomics. The fractal dimension of strange attractors is 2.260756, which shows that the economic system is a chaotic system of fractal dimensional structure. By theoretical analysis and numerical simulation, we reveal that the firm of adaptive expectations has a stabilizing effect on the system, that is, the smaller the absolute value of the negative feedback factor is, the more stable of the market will be. However, the fast adjustment speed of the boundedly rational firm causes instability, even chaos. Hence, the different expectations may lead to rich dynamical behaviors and complexity.

Acknowledgments

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