Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 929824, 14 pages
doi:10.1155/2011/929824

Research Article

Functional Inequalities Associated with Cauchy Additive Functional Equation in Non-Archimedean Spaces

A. Ebadian,[1] N. Ghobadipour,[1] Th. M. Rassias,[2] and M. Eshaghi Gordji[3, 4, 5]

[1] *Department of Mathematics, Urmia University, Urmia, Iran*
[2] *Department of Mathematics, National Technical University of Athens, 15780 Zografou, Greece*
[3] *Department of Mathematics, Semnan University, P. O. Box 35195-363, Iran*
[4] *Research Group of Nonlinear Analysis and Applications (RGNAA), Semnan University, Iran*
[5] *Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Iran*

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com

Received 30 October 2010; Accepted 19 January 2011

Academic Editor: Rigoberto Medina

We investigate the generalized Hyers-Ulam stability of the functional inequalities $\|f((x+y+z)/4)+f((3x-y-4z)/4)+f((4x+3z)/4)\| \le \|2f(x)\|$ and $\|f((y-x)/3)+f((x-3z)/3)+f((3x+3z-y)/3)\| \le \|f(x)\|$ in non-Archimedean normed spaces in the spirit of the Th. M. Rassias stability approach.

1. Introduction

Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \le \delta \tag{1.1}$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \delta \tag{1.2}$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. In 1978, Rassias [3] proved the following theorem.

Theorem 1.1. *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.3}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.4}$$

for all $x \in E$. If $p < 0$ then inequality (1.3) holds for all $x, y \neq 0$, and (1.4) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in real t for each fixed $x \in E$, then T is linear.

In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations. The reader is referred to [5–13] for a number of results in this domain of research.

In 1994, a generalization of the Rassias theorem was obtained by Găvruţa as follows [14].

Suppose $(G, +)$ is an abelian group, E is a Banach space, and that the so-called admissible control function $\varphi : G \times G \to \mathbb{R}$ satisfies

$$\tilde{\varphi}(x, y) := 2^{-1}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty \tag{1.5}$$

for all $x, y \in G$. If $f : G \to E$ is a mapping with

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x, y) \tag{1.6}$$

for all $x, y \in G$, then there exists a unique mapping $T : G \to E$ such that $T(x+y) = T(x)+T(y)$ and $\|f(x) - T(x)\| \le \tilde{\varphi}(x, x)$ for all $x, y \in G$.

During the last decades, several stability problems of functional equations have been investigated by a number of mathematicians, see [15–17] and references therein for more detailed information.

By a *non-Archimedean* field we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0, 1)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \le \max\{|r|, |s|\}$ for all $r, s \in K$. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\|$ $(r \in \mathbb{K}, x \in X)$;

(iii) the strong triangle inequality (ultrametric), namely,

$$\|x+y\| \le \max\{\|x\|, \|y\|\} \quad (x, y \in X). \tag{1.7}$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean space. Due to the fact that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\| : m \le j \le n-1\} \quad (n > m), \tag{1.8}$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent (see [18–22]).

Gilányi [23] and Rätz [24] showed that if f satisfies the functional inequality

$$\left\|2f(x) + 2f(y) - f\left(xy^{-1}\right)\right\| \le f(xy), \tag{1.9}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f\left(xy^{-1}\right). \tag{1.10}$$

Gilányi [23] and Fechner [25] proved the generalized Hyers-Ulam stability of the functional inequality (1.3).

Cho and Kim [26] proved the generalized Hyers-Ulam stability of the following functional inequalities:

$$\begin{gathered}\left\|f\left(\frac{x-y}{2} - z\right) + f(y) + 2f(z)\right\| \le \left\|f\left(\frac{x+y}{2} + z\right)\right\| + \varphi(x, y, z),\\ \|f(x) + f(y) + 2f(z)\| \le \left\|2f\left(\frac{x+y}{2} + z\right)\right\| + \varphi(x, y, z),\end{gathered} \tag{1.11}$$

which are associated with Jordan-von Neumann-type Cauchy-Jensen additive functional equations.

Now, we consider the following functional inequality:

$$\left\|f\left(\frac{x+y+z}{4}\right) + f\left(\frac{3x-y-4z}{4}\right) + f\left(\frac{4x+3z}{4}\right)\right\| \le \|2f(x)\|, \tag{1.12}$$

$$\left\|f\left(\frac{y-x}{3}\right) + f\left(\frac{x-3z}{3}\right) + f\left(\frac{3x+3z-y}{3}\right)\right\| \le \|f(x)\|, \tag{1.13}$$

which is associated with Cauchy additive functional equation.

The purpose of this paper is to prove that if f satisfies the inequalities (1.12) and (1.13), which satisfies certain conditions, then f is Cauchy additive, and thus we prove the generalized Hyers-Ulam stability of the functional inequalities (1.12) and (1.13) in non-Archimedean normed spaces.

2. Stability of Functional Inequality (1.12)

In this section, we prove the generalized Hyers-Ulam stability of the functional inequality (1.12). Throughout this section, we assume that S is an additive semigroup and X is a complete non-Archimedean space.

We need the following lemma in the main results.

Lemma 2.1. *Let $f : S \to X$ be a mapping such that*

$$\left\| f\left(\frac{x+y+z}{4}\right) + f\left(\frac{3x-y-4z}{4}\right) + f\left(\frac{4x+3z}{4}\right) \right\| \le \|2f(x)\| \tag{2.1}$$

for all $x, y, z \in S$. If $|3| > |2|$, then the mapping f is Cauchy additive.

Proof. Letting $x = y = z = 0$ in (2.1), we get $|3|\|f(0)\| \le |2|\|f(0)\|$. So, $f(0) = 0$. Letting $x = z = 0$ and replacing y by $4y$ in (2.1), we get $\|f(y) + f(-y)\| \le |2|\|f(0)\|$ for all $y \in S$. So, $f(-y) = -f(y)$ for all $y \in S$. Setting $x = 0$ in (2.1), we obtain

$$\left\| f\left(\frac{y+z}{4}\right) + f\left(\frac{-y-4z}{4}\right) + f\left(\frac{3z}{4}\right) \right\| \le |2|\|f(0)\|. \tag{2.2}$$

So,

$$f\left(\frac{y+z}{4}\right) + f\left(\frac{-y-4z}{4}\right) + f\left(\frac{3z}{4}\right) = 0 \tag{2.3}$$

for all $y, z \in S$. Replacing y by $2z$ in (2.3), we get

$$2f\left(\frac{3z}{4}\right) = f\left(\frac{3z}{2}\right) \tag{2.4}$$

for all $z \in S$. Using (2.4), we obtain $f(2z) = 2f(z)$ and $f(4z) = 4f(z)$ for all $z \in S$. Letting $x = 0$, $w_1 = (y+z)/4$ and $w_2 = (y-4z)/4$, in (2.1) we get

$$f(w_1) + f(w_2) = f(w_1 + w_2) \tag{2.5}$$

for all $w_1, w_2 \in S$. Hence, f is additive. □

Theorem 2.2. *Let $\varphi : S \times S \times S \to \mathbb{R}^+ \cup \{0\}$ be a function such that*

$$\lim_{n\to\infty} \max\left[\frac{\varphi(2^n x, 2^n y, 2^n z)}{|2|^n}, \frac{\varphi(-2^n x, -2^n y, -2^n z)}{|2|^n}\right] = 0 \tag{2.6}$$

for all $x, y, z \in S$ and let the limit

$$\tilde{\varphi}(z) := \lim_{n\to\infty} \max\left\{\max\left[\frac{\varphi(0, 2.2^i z, 2^i z)}{|2|^i}, \frac{\varphi(0, -2.2^i z, -2^i z)}{|2|^i}\right];\ 0 \le i < n\right\} \tag{2.7}$$

exists for all $z \in S$. Suppose that $f : S \to X$ with $f(0) = 0$ is a mapping satisfying

$$\left\| f\left(\frac{x+y+z}{4}\right) + f\left(\frac{3x-y-4z}{4}\right) + f\left(\frac{4x+3z}{4}\right)\right\| \le |2|\|f(x)\| + \varphi(x, y, z) \tag{2.8}$$

for all $x, y, z \in S$. Then there exists an additive mapping $h : S \to X$ such that

$$\left\|\frac{f(z) - f(-z)}{2} - h(z)\right\| \le \frac{1}{|2|^2}\tilde{\varphi}(z) \tag{2.9}$$

for all $z \in S$. Moreover, if

$$\lim_{k\to\infty}\lim_{n\to\infty} \max\left\{\max\left[\frac{\varphi(0, 2.2^i z, 2^i z)}{|2|^i}, \frac{\varphi(0, -2.2^i z, -2^i z)}{|2|^i}\right];\ k \le i < n+k\right\} = 0 \tag{2.10}$$

then h is the unique additive mapping satisfying (2.9).

Proof. Putting $x = 0$ and $y = 2z$ in (2.8), we get

$$\left\|2f\left(\frac{3z}{4}\right) + f\left(\frac{-3z}{2}\right)\right\| \le \varphi(0, 2z, z) \tag{2.11}$$

for all $z \in S$. Replacing z by $4z/3$ in (2.11), we obtain

$$\|2f(z) + f(-2z)\| \le \varphi\left(0, \frac{8z}{3}, \frac{4z}{3}\right) \tag{2.12}$$

for all $z \in S$. Replacing z by $-z$ in (2.12), we get

$$\|2f(-z) + f(2z)\| \le \varphi\left(0, \frac{-8z}{3}, \frac{-4z}{3}\right) \tag{2.13}$$

for all $z \in S$. Let $g(z) := (f(z) - f(-z))/2$. It follows from (2.12) and (2.13) that

$$\|g(2z) - 2g(z)\| \le \frac{1}{|2|}\max\left[\varphi\left(0, \frac{8z}{3}, \frac{4z}{3}\right), \varphi\left(0, \frac{-8z}{3}, \frac{-4z}{3}\right)\right] \tag{2.14}$$

for all $z \in S$. Replacing z by $2^{n-1}z$ in (2.14), we get

$$\left\| \frac{g(2^n z)}{2^n} - \frac{g(2^{n-1}z)}{2^{n-1}} \right\| \le \frac{1}{|2|^{n+1}} \max\left[\varphi\left(0, \frac{8.2^{n-1}z}{3}, \frac{4.2^{n-1}z}{3}\right), \varphi\left(0, \frac{-8.2^{n-1}z}{3}, \frac{-4.2^{n-1}z}{3}\right)\right] \tag{2.15}$$

for all $z \in S$. It follows from (2.6) and (2.15) that the sequence $\{g(2^n z)/2^n\}$ is Cauchy. Since X is complete, we conclude that $\{g(2^n z)/2^n\}$ is convergent. Set $h(z) := \lim_{n\to\infty}(g(2^n z)/2^n)$ for all $z \in S$. Using induction one can show that

$$\begin{aligned}
&\left\| \frac{g(2^n z)}{2^n} - g(z) \right\| \\
&\quad \le \frac{1}{|2|^2} \max\left\{ \max\left[\frac{\varphi(0, 8.2^k z/3, 4.2^k z/3)}{|2|^k}, \frac{\varphi(0, -8.2^k z/3, -4.2^k z/3)}{|2|^k} \right];\ 0 \le k < n \right\}
\end{aligned} \tag{2.16}$$

for all $z \in S$ and $n \in \mathbb{N}$. By taking n to approach infinity in (2.16) and using (2.7) one obtains (2.9).

It follows from (2.8) that

$$\begin{aligned}
&\left\| h\left(\frac{x+y+z}{4}\right) + h\left(\frac{3x-y-4z}{4}\right) + h\left(\frac{4x+3z}{4}\right) \right\| \\
&\quad = \lim_{n\to\infty} \frac{1}{|2|^n} \left\| g\left(2^n\left(\frac{x+y+z}{4}\right)\right) + g\left(2^n\left(\frac{3x-y-4z}{4}\right)\right) + g\left(2^n\left(\frac{4x+3z}{4}\right)\right) \right\| \\
&\quad = \lim_{n\to\infty} \frac{1}{|2|^{n+1}} \left\| \frac{f(2^n(x+y+z))}{4} + \frac{f(2^n(3x-y-4z))}{4} + \frac{f(2^n(4x+3z))}{4} \right. \\
&\qquad\qquad \left. - \frac{f(2^n(-x-y-z))}{4} - \frac{f(2^n(-3x+y+4z))}{4} - \frac{f(2^n(-4x-3z))}{4} \right\| \\
&\quad \le \frac{1}{|2|^n} \| f(2^n z) - f(-2^n z) \| + \lim_{n\to\infty} \frac{1}{|2|} \max\left[\frac{\varphi(2^n x, 2^n y, 2^n z)}{|2|^n}, \frac{\varphi(-2^n x, -2^n y, -2^n z)}{|2|^n} \right] \\
&\quad = |2| \| h(z) \|
\end{aligned} \tag{2.17}$$

for all $x, y, z \in S$. So,

$$\left\| h\left(\frac{x+y+z}{4}\right) + h\left(\frac{3x-y-4z}{4}\right) + h\left(\frac{4x+3z}{4}\right) \right\| \le |2| \| h(z) \| \tag{2.18}$$

for all $x, y, z \in S$. By Lemma 2.1, the mapping $h : S \to X$ is additive.

Now, let $T : S \to X$ be another additive mapping satisfying (2.9). Then we have

$$\begin{aligned}
\|h(z) - T(z)\| &= \frac{1}{|2|^k}\left\|h\left(2^k z\right) - T\left(2^k z\right)\right\| \\
&\le \frac{1}{|2|^k}\max\left[\left\|T\left(2^k z\right) - g\left(2^k z\right)\right\|, \left\|g\left(2^k z\right) - h\left(2^k z\right)\right\|\right] \\
&\le \frac{1}{|2|^2}\lim_{k\to\infty}\lim_{n\to\infty}\max\left\{\max\left[\frac{\varphi(0, 2.2^i z, 2^i z)}{|2|^i}, \frac{\varphi(0, -2.2^i z, -2^i z)}{|2|^i}\right];\ k \le i < i+k\right\} \\
&= 0
\end{aligned} \tag{2.19}$$

for all $z \in S$. Therefore $h = T$. This completes the proof of the uniqueness of h. □

Corollary 2.3. *Let $p > 1$ and θ be positive real numbers, and let $f : S \to X$ be a mapping satisfying*

$$\left\|f\left(\frac{x+y+z}{4}\right) + f\left(\frac{3x-y-4z}{4}\right) + f\left(\frac{4x+3z}{4}\right)\right\| \le |2|\|f(x)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.20}$$

for all $x, y, z \in S$. If $|2| < 1$ then there exists a unique additive mapping $h : S \to X$ such that

$$\left\|\frac{f(z) - f(-z)}{2} - h(z)\right\| \le \frac{2\theta}{|2|^2}\|z\|^p \tag{2.21}$$

for all $z \in S$.

Proof. Defining $\varphi : S \times S \times S \to X$ by $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ we have

$$\begin{aligned}
&\lim_{n\to\infty}\max\left[\frac{\varphi(2^n x, 2^n y, 2^n z)}{|2|^n}, \frac{\varphi(-2^n x, -2^n y, -2^n z)}{|2|^n}\right] \\
&\quad = \lim_{n\to\infty}\frac{\theta|2|^{np}}{|2|^n}(\|x\|^p + \|y\|^p + \|z\|^p) = 0, \\
&\widetilde{\varphi}(z) := \lim_{n\to\infty}\max\left\{\max\left[\frac{\varphi(0, 2.2^i z, 2^i z)}{|2|^i}, \frac{\varphi(0, -2.2^i z, -2^i z)}{|2|^i}\right];\ 0 \le i < n\right\} \\
&= \lim_{n\to\infty}\max\left\{\frac{|2|^{(i+1)p} + |2|^{ip}}{|2|^i}\theta\|z\|^p;\ 0 \le i < n\right\} \le 2\theta\|z\|^p, \\
&\lim_{k\to\infty}\lim_{n\to\infty}\max\left\{\max\left[\frac{\varphi(0, 2.2^i z, 2^i z)}{|2|^i}, \frac{\varphi(0, -2.2^i z, -2^i z)}{|2|^i}\right];\ k \le i < n+k\right\} \\
&\quad \le \lim_{k\to\infty}|2|^{kp}\theta\|z\|^p = 0
\end{aligned} \tag{2.22}$$

for all $z \in S$.

Applying Theorem 2.2, we conclude the required result. □

Corollary 2.4. *Let $\psi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ be a function satisfying*

$$\psi(|2|r) \leq \psi(|2|)\psi(r) \quad (r \geq 0), \ \psi(|2|) < |2|. \tag{2.23}$$

Let $\theta > 0$, let S be a normed space and let $f : S \to X$ fulfill the inequality

$$\begin{aligned} &\left\| f\left(\frac{x+y+z}{4}\right) + f\left(\frac{3x-y-4z}{4}\right) + f\left(\frac{4x+3z}{4}\right) \right\| \\ &\quad \leq |2| \| f(x) \| + \theta [\psi(\|x\|) + \psi(\|y\|) + \psi(\|z\|)] \end{aligned} \tag{2.24}$$

for all $x, y, z \in S$. Then there exists a unique additive mapping $h : S \to X$ such that

$$\left\| \frac{f(z) - f(-z)}{2} - h(z) \right\| \leq \frac{2\theta}{|2|^2} \psi(\|z\|) \tag{2.25}$$

for all $z \in S$.

Proof. Defining $\varphi : S \times S \times S \to X$ by $\varphi(x, y, z) := \theta[\psi(\|x\|) + \psi(\|y\|) + \psi(\|z\|)]$ we have

$$\begin{aligned} &\lim_{n\to\infty} \max\left[\frac{\varphi(2^n x, 2^n y, 2^n z)}{|2|^n}, \frac{\varphi(-2^n x, -2^n y, -2^n z)}{|2|^n}\right] \\ &\qquad \leq \theta \lim_{n\to\infty} \max\left[\left(\frac{\psi(|2|)}{|2|}\right)^n (\varphi(x, y, z), \varphi(-x, -y, -z))\right] = 0, \\ &\widetilde{\varphi}(z) := \lim_{n\to\infty} \max\left\{\max\left[\frac{\varphi(0, 2.2^i z, 2^i z)}{|2|^i}, \frac{\varphi(0, -2.2^i z, -2^i z)}{|2|^i}\right]; \ 0 \leq i < n\right\} \\ &\quad = 2\theta\psi(\|z\|), \\ &\lim_{k\to\infty} \lim_{n\to\infty} \max\left\{\max\left[\frac{\varphi(0, 2.2^i z, 2^i z)}{|2|^i}, \frac{\varphi(0, -2.2^i z, -2^i z)}{|2|^i}\right]; \ k \leq i < n+k\right\} \\ &\qquad \leq \lim_{k\to\infty} \left(\frac{\psi(|2|)}{|2|}\right)^k \psi(\|z\|) = 0 \end{aligned} \tag{2.26}$$

for all $z \in S$.

Applying Theorem 2.2, we conclude the required result. □

Theorem 2.5. *Let $\varphi : S \times S \times S \to \mathbb{R}^+ \cup \{0\}$ be a function such that*

$$\lim_{n\to\infty} \max\{|2|^n \varphi(2^{-n}x, 2^{-n}y, 2^{-n}), |2|^n \varphi(-2^{-n}x, -2^{-n}y, -2^{-n}z)\} = 0 \tag{2.27}$$

for all $x, y, z \in S$ and let the limit

$$\tilde{\varphi}(z) := \lim_{n\to\infty} \max\left\{\max\left[|2|^n \varphi\left(0, 2.2^{-i}z, 2^{-i}z\right), |2|^n \varphi\left(0, -2.2^{-i}z, -2^{-i}z\right)\right];\ 0 \le i < n\right\} \tag{2.28}$$

exist for all $z \in S$. Suppose that $f : S \to X$ with $f(0) = 0$ is a mapping satisfying

$$\left\| f\left(\frac{x+y+z}{4}\right) + f\left(\frac{3x-y-4z}{4}\right) + f\left(\frac{4x+3z}{4}\right)\right\| \le |2| \|f(x)\| + \varphi(x, y, z) \tag{2.29}$$

for all $x, y, z \in S$. Then there exists an additive mapping $h : S \to X$ such that

$$\left\| \frac{f(z) - f(-z)}{2} - h(z)\right\| \le \frac{1}{|2|}\tilde{\varphi}(z) \tag{2.30}$$

for all $z \in S$. Moreover, if

$$\lim_{k\to\infty}\lim_{n\to\infty} \max\left\{\max\left[|2|^i \varphi\left(0, 2.2^{-i}z, 2^{-i}z\right), |2|^i \varphi\left(0, -2.2^{-i}z, -2^{-i}z\right)\right];\ k \le i < n+k\right\} = 0 \tag{2.31}$$

then h is the unique additive mapping satisfying (2.30).

Proof. It follows from (2.14) that

$$\left\| 2g\left(\frac{z}{2}\right) - g(z)\right\| \le \frac{1}{|2|} \max\left\{\varphi\left(0, \frac{4z}{3}, \frac{2z}{3}\right), \varphi\left(0, \frac{-4z}{3}, \frac{-2z}{3}\right)\right\} \tag{2.32}$$

for all $z \in S$. Hence,

$$\left\| 2^n g(2^{-n}z) - 2^{(n+1)} g\left(2^{-(n+1)z}\right)\right\| \le |2|^n \max\left\{\varphi\left(0, \frac{4.2^{-n}z}{3}, \frac{2.2^{-n}z}{3}\right), \varphi\left(0, \frac{-4.2^{-n}z}{3}, \frac{-2.2^{-n}z}{3}\right)\right\} \tag{2.33}$$

for all $z \in S$. It follows from (2.27) and (2.33) that the sequence $\{2^n g(2^{-n}z)\}$ is a Cauchy sequence for all $z \in S$. Since X is complete, the sequence $\{2^n g(2^{-n}z)\}$ converges. So, one can define the mapping $h : S \to X$ by $h(z) := \lim_{n\to\infty}\{2^n g(2^{-n}z)\}$ for all $z \in S$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Remark 2.6. We can obtain similar results to Corollary 2.3 for $p < 1$ and Corollary 2.4.

3. Stability of Functional Inequality (1.13)

We prove the generalized Hyers-Ulam stability of the functional inequality (1.13). Throughout this section, we assume that S is an additive semigroup and X is a complete non-Archimedean space.

We need the following lemma in the main results.

Lemma 3.1. *Let $f : S \to X$ be a mapping such that*

$$\left\| f\left(\frac{y-x}{3}\right) + f\left(\frac{x-3z}{3}\right) + f\left(\frac{3x+3z-y}{3}\right) \right\| \le \|f(x)\| \tag{3.1}$$

for all $x, y, z \in S$. If $f(0) = 0$, then the mapping f is Cauchy additive.

Proof. Letting $x = y = 0$ in (3.1), we get

$$\|f(-z) + f(z)\| \le \|f(0)\| = 0 \tag{3.2}$$

for all $z \in S$. Hence, $f(-z) = -f(z)$ for all $z \in S$. Letting $x = 0$ and $y = 6z$ in (3.1), we get

$$\|f(2z) - 2f(z)\| \le \|f(0)\| = 0 \tag{3.3}$$

for all $z \in S$. Hence,

$$f(2z) = 2f(z) \tag{3.4}$$

for all $z \in S$. Letting $x = 0$ and $y = 9z$ in (3.1), we get

$$\|f(3z) - f(z) - 2f(z)\| \le \|f(0)\| = 0 \tag{3.5}$$

for all $z \in S$. Hence,

$$f(3z) = 3f(z) \tag{3.6}$$

for all $z \in S$. Letting $x = 0$ in (3.1), we get

$$\left\| f\left(\frac{y}{3}\right) + f(-z) + f\left(z - \frac{y}{3}\right) \right\| \le \|f(0)\| = 0 \tag{3.7}$$

for all $x, y, z \in S$. So,

$$f\left(\frac{y}{3}\right) + f(-z) + f\left(z - \frac{y}{3}\right) = 0 \tag{3.8}$$

for all $x, y, z \in S$. Let $t_1 = z - y/3$ and $t_2 = y/3$ in (3.8). Then

$$f(t_2) - f(t_1 + t_2) + f(t_1) = 0 \tag{3.9}$$

for all $t_1, t_2 \in S$. So, f is additive. $\square$

Theorem 3.2. *Let $\varphi : S \times S \times S \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z)}{|2|^n} = 0 \tag{3.10}$$

for all $x, y, z \in S$ and let the limit

$$\widetilde{\varphi}(z) := \lim_{n\to\infty} \max\left\{ \max\left[\frac{\varphi(0, 6.2^i z, 2^i z)}{|2|^i}, \frac{|2|\varphi(0, 3.2^i z, 2^i z)}{|2|^i} \right];\ 0 \le i < n \right\} \tag{3.11}$$

exist for all $z \in S$. Suppose that $f : S \to X$ with $f(0) = 0$ is a mapping satisfying

$$\left\| f\left(\frac{y-x}{3}\right) + f\left(\frac{x-3z}{3}\right) + f\left(\frac{3x+3z-y}{3}\right) \right\| \le \|f(x)\| + \varphi(x,y,z) \tag{3.12}$$

for all $x, y, z \in S$. Then there exists an additive mapping $T : S \to X$ such that

$$\|f(z) - T(z)\| \le \frac{1}{|2|} \widetilde{\varphi}(z) \tag{3.13}$$

for all $z \in S$. Moreover, if

$$\lim_{k\to\infty} \lim_{n\to\infty} \max\left\{ \max\left[\frac{\varphi(0, 6.2^i z, 2^i z)}{|2|^i}, \frac{|2|\varphi(0, 3.2^i z, 2^i z)}{|2|^i} \right];\ k \le i < n+k \right\} = 0 \tag{3.14}$$

then T is the unique additive mapping satisfying (3.13).

Proof. Letting $x = 0$ and $y = 6z$ in (3.12), we get

$$\|f(2z) + 2f(-z)\| \le \varphi(0, 6z, z) \tag{3.15}$$

for all $z \in S$. Putting $x = 0$ and $y = 3z$ in (3.12), we get

$$\|2f(z) + 2f(-z)\| \le |2|\varphi(0, 3z, z) \le \varphi(0, 3z, z) \tag{3.16}$$

for all $z \in S$. It follows from (3.15) and (3.16) that

$$\|f(2z) - 2f(z)\| \le \max\{\varphi(0, 6z, z), \varphi(0, 3z, z)\} \tag{3.17}$$

for all $z \in S$. Replacing z by $2^{n-1}z$ in (3.17), we get

$$\left\| \frac{f(2^n z)}{2^n} - \frac{f(2^{n-1} z)}{2^{n-1}} \right\| \le \frac{1}{|2|^n} \max\left[\varphi\left(0, 6.2^{n-1} z, 2^{n-1} z\right), \varphi\left(0, 3.2^{n-1} z, 2^{n-1} z\right) \right] \tag{3.18}$$

for all $z \in S$. It follows from (3.10) and (3.18) that the sequence $\{f(2^n z)/2^n\}$ is Cauchy. Since X is complete, we conclude that $\{f(2^n z)/2^n\}$ is convergent. Set $T(z) := \lim_{n\to\infty}(f(2^n z)/2^n)$ for all $z \in S$. Using induction one can show that

$$\left\|\frac{f(2^n z)}{2^n} - f(z)\right\| \leq \frac{1}{|2|}\max\left\{\max\left[\frac{\varphi(0, 6.2^k z, 2^k z)}{|2|^k}, \frac{|2|\varphi(0, 3.2^k z, 2^k z)}{|2|^k}\right];\ 0 \leq k < n\right\} \tag{3.19}$$

for all $z \in S$ and $n \in \mathbb{N}$. By taking n to approach infinity in (3.19) and using (3.11) one obtains (3.13).

Replacing x, y, and z by $2^n x$,$2^n y$, and $2^n z$, respectively, in (3.12) we get

$$\begin{aligned}
&\left\|f\left(\frac{2^n(y-x)}{3.2^n}\right) + f\left(\frac{2^n(x-3z)}{3.2^n}\right) + f\left(\frac{2^n(3x+3z-y)}{3.2^n}\right)\right\| \\
&\quad \leq \left\|\frac{f(2^n x)}{2^n}\right\| + \frac{1}{|2|^n}\varphi(2^n x, 2^n y, 2^n z)
\end{aligned} \tag{3.20}$$

for all $x, y, z \in S$. Taking the limit as $n \to \inf ty$ and using (3.10) we get

$$\left\|T\left(\frac{y-x}{3}\right) + T\left(\frac{x-3z}{3}\right) + T\left(\frac{3x+3z-y}{3}\right)\right\| \leq \|T(x)\| \tag{3.21}$$

for all $x, y, z \in S$. By Lemma 3.1, the mapping $T : S \to X$ is additive.

If T' is another additive mapping satisfying (3.13), then

$$\begin{aligned}
\|T(z) - T'(z)\| &= \frac{1}{|2|^k}\left\|T\left(2^k z\right) - T'\left(2^k z\right)\right\| \\
&\leq \frac{1}{|2|^k}\max\left[\left\|T\left(2^k z\right) - f\left(2^k z\right)\right\|, \left\|f\left(2^k z\right) - T'\left(2^k z\right)\right\|\right] \\
&\leq \frac{1}{|2|}\lim_{k\to\infty}\lim_{n\to\infty}\max\left\{\max\left[\frac{\varphi(0, 6.2^i z, 2^i z)}{|2|^i}, \frac{|2|\varphi(0, 3.2^i z, 2^i z)}{|2|^i}\right];\ k \leq i < i+k\right\} \\
&= 0
\end{aligned} \tag{3.22}$$

for all $z \in S$. Therefore $T' = T$. This proves the uniqueness of T. Hence, the mapping $T : S \to X$ is a unique additive mapping satisfying (3.13). □

Corollary 3.3. *Let $p > 1$ and θ be positive real numbers, and let $f : S \to X$ be a mapping satisfying*

$$\left\|f\left(\frac{y-x}{3}\right) + f\left(\frac{x-3z}{3}\right) + f\left(\frac{3x+3z-y}{3}\right)\right\| \leq \|f(x)\| + \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \tag{3.23}$$

for all $x, y, z \in S$. If $|2| < 1$ then there exists a unique additive mapping $T : S \to X$ such that

$$\left\| f(z) - T(z) \right\| \le \frac{2\theta}{|2|} \|z\|^p \tag{3.24}$$

for all $z \in S$.

Proof. Letting $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ in Theorem 3.2, we obtain the result. □

Corollary 3.4. *Let $\psi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\psi(|2|r) \le \psi(|2|)\psi(r) \quad (r \ge 0), \;\; \psi(|2|) < |2|. \tag{3.25}$$

Let $\theta > 0$, let S be a normed space, and let $f : S \to X$ fulfill the inequality

$$\left\| f\left(\frac{y - x}{3}\right) + f\left(\frac{x - 3z}{3}\right) + f\left(\frac{3x + 3z - y}{3}\right) \right\| \le \left\| f(x) \right\| + \theta\left[\psi(\|x\|) + \psi(\left\|y\right\|) + \psi(\|z\|)\right] \tag{3.26}$$

for all $x, y, z \in S$. Then there exists a unique additive mapping $T : S \to X$ such that

$$\left\| f(z) - T(z) \right\| \le \frac{2\theta}{|2|} \psi(\|z\|) \tag{3.27}$$

for all $z \in S$.

Proof. If we define $\varphi(x, y, z) := \theta[\psi(\|x\|) + \psi(\|y\|) + \psi(\|z\|)]$ in Theorem 3.2, then we get the result. □

Remark 3.5. We can formulate a similar statement to Theorem 3.2 in which we can define the sequence $T(z) := \lim_{n\to\infty} 2^n f(z/2^n)$ under suitable conditions on the function φ and ψ and then obtain similar results to Corollary 3.3 for $p < 1$ and Corollary 3.4.

References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[4] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[5] M. Eshaghi Gordji and H. Khodaei, "Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 11, pp. 5629–5643, 2009.

[6] A. Grabiec, "The generalized Hyers-Ulam stability of a class of functional equations," *Publicationes Mathematicae Debrecen*, vol. 48, no. 3-4, pp. 217–235, 1996.

[7] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and Their Applications, 34, Birkhäuser, Boston, Mass, USA, 1998.
[8] G. Isac and Th. M. Rassias, "On the Hyers-Ulam stability of ψ-additive mappings," *Journal of Approximation Theory*, vol. 72, no. 2, pp. 131–137, 1993.
[9] A. Najati, "On the stability of a quartic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 1, pp. 569–574, 2008.
[10] C.-G. Park, "On the stability of the orthogonally quartic functional equation," *Bulletin of the Iranian Mathematical Society*, vol. 31, no. 1, pp. 63–70, 2005.
[11] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.
[12] Th. M. Rassias, "On the stability of functional equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.
[13] L. Tan and S. Xiang, "On the Aleksandrov-Rassias problem and the Hyers-Ulam-Rassias stability problem," *Banach Journal of Mathematical Analysis*, vol. 1, no. 1, pp. 11–22, 2007.
[14] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
[15] S. Czerwik, *Stability of Functional Equations of Ulam-Hyers-Rassias Type*, Hadronic Press, Palm Harbor, Fla, USA, 2003.
[16] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.
[17] Th. M. Rassias, Ed., *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.
[18] M. Eshaghi Gordji and M. B. Savadkouhi, "Stability of cubic and quartic functional equations in non-Archimedean spaces," *Acta Applicandae Mathematicae*, vol. 110, no. 3, pp. 1321–1329, 2010.
[19] M. Eshaghi Gordji and M. B. Savadkouhi, "Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1198–1202, 2010.
[20] M. Eshaghi Gordji, M. B. Savadkouhi, and M. Bidkham, "Stability of a mixed type additive and quadratic functional equation in non-Archimedean spaces," *Journal of Computational Analysis and Applications*, vol. 12, no. 2, pp. 454–462, 2010.
[21] M. Eshaghi Gordji, H. Khodaei, and R. Khodabakhsh, "General quartic-cubic-quadratic functional equation in non-Archimedean normed spaces," *Scientific Bulletin-University Politehnica of Bucharest. Series A*, vol. 72, no. 3, pp. 69–84, 2010.
[22] M. Eshaghi Gordji, R. Khodabakhsh, H. Khodaei, and S. M. Jung, "AQCQ-functional equation in non-Archimedean normed spaces," *Abstract and Applied Analysis*, vol. 2010, Article ID 741942, 22 pages, 2010.
[23] A. Gilányi, "Eine zur Parallelogrammgleichung äquivalente Ungleichung," *Aequationes Mathematicae*, vol. 62, no. 3, pp. 303–309, 2001.
[24] J. Rätz, "On inequalities associated with the Jordan-von Neumann functional equation," *Aequationes Mathematicae*, vol. 66, no. 1-2, pp. 191–200, 2003.
[25] W. Fechner, "Stability of a functional inequality associated with the Jordan-von Neumann functional equation," *Aequationes Mathematicae*, vol. 71, no. 1-2, pp. 149–161, 2006.
[26] Y.-S. Cho and H.-M. Kim, "Stability of functional inequalities with Cauchy-Jensen additive mappings," *Abstract and Applied Analysis*, vol. 2007, Article ID 89180, p. 13, 2007.