

Research Article

Control and Synchronization of Fractional-Order Financial System Based on Linear Control

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Control and synchronization of the financial systems with fractional-order are discussed in this paper. Based on the stability theory of fractional-order differential equations, Routh-Hurwitz stability condition, and by using linear control, simpler controllers are designed to achieve control and synchronization of the fractional-order financial systems. The proposed controllers are linear and easy to implement, which have improved the existing results. Theoretical analysis and numerical simulations are shown to demonstrate the validity and feasibility of the proposed method.

1. Introduction

Chaos, as a very interesting nonlinear phenomenon, has been intensively investigated in many fields over the last four decades. Since the pioneering work [1] that chaotic dynamics could exist in an economical model, research on the dynamical behavior of economical model has become one of the most interesting and important topics which have received increasing attention. Many continuous chaotic models have been proposed to study complex economic dynamics, such as the forced Vander-Pol model [2], the IS-ML model [3], Behrens-Feichtinger model [4], and Cournot-Puu model [5]. Just as all the other chaotic systems in engineering, financial chaotic system has complex dynamical behaviors and possess some special features, such as excessive sensitivity to initial conditions, the complex patterns of phase portraits, positive Lyapunov exponents, and bounded and fractal properties of the motion in the phase space. These features are inherent properties of the system itself, rather than caused by external disturbances, which denote some economic behaviors in the fields of finance, stocks, and social economics. In addition, as a nonlinear system, there exist many attractors with

different topology, such as fixed points, limit cycles, quasiperiodic attractors, and chaotic attractors in chaotic financial systems, which make the contradiction in economic operation process complex and changeful. Chaotic behavior in financial systems is undesirable due to threatening the safety of investment. Therefore, to study the chaotic behaviors in nonlinear economical systems plays a very important role in decision-making by policy makers. Once the instability of a periodic solution, bifurcation, multiperiodic bifurcation, or similar phenomena appear in the economic chaotic systems, decision makers should take some measures to intervene. Some researches suggest that controlling chaos should improve the performance of a chaotic economy and stabilizing periodic solutions of a chaotic market model may increase economic efficiency [6–10]. In recent years, some authors have paid much attention to chaos control in economic models and obtained some results. For example, Ahmed et al. and Agiza used OGY method to control chaos in economic systems [11, 12], respectively. Pyragas [7] and Holyst and Urbanowicz [9] proposed the delayed feedback control, which has been widely used for controlling chaos in economic models. Sun et al. adopted impulsive control to control a financial model [13]. Du et al. applied phase space compression to control chaos in economic systems [14].

On the other hand, it has been found that many systems in interdisciplinary fields can be described by fractional differential equations, for example, dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, viscoelastic systems, quantitative finance, and diffusion wave [15–17]. Recently, fractional-order systems have become an active research area, particularly in control and synchronization of chaotic systems. Interest has been growing in fractional calculus not only from physicists and engineers but also from researchers in life science and economics [18–20]. In fact, financial variables possessing long memories make fractional models more appropriate for dynamic behaviors in finance. Research on fractional-order financial models has a wider range of applications. In 2008, Chen studied nonlinear dynamics and chaos in a fractional-order financial system [21]. As he pointed out, one of the major differences between fractional-order and integer-order models is that fractional-order models possess memory; that is, the fractional-order model depends on the history of the system. The magnitude of the financial variables such as foreign exchange rates, gross domestic product, interest rates, production, and stock market prices can have very long memory; the reason for describing financial systems using a fractional nonlinear model is that it simultaneously possesses memory and chaos. From then on, researchers set out to investigate the fractional financial models, for instance, in [22] a sliding mode controller was designed for a fractional-order chaotic financial system and in [23] control of a fractional-order chaotic financial system by nonlinear feedback control was discussed. As we know, linear feedback control is especially attractive and has been successfully applied to practical implementation, which was adopted in [24–26] to realize control and synchronization of integer-order chaotic systems. However, there exists substantial difference between fractional-order differential systems and integer-order differential ones. Most of the properties, conclusions, and methods to deal with integer-order systems cannot be simply extended to the case of fractional-order ones. Therefore, results about fractional-order chaotic systems are much less than those of integer-order systems. To the best of our knowledge, there are few results about control and synchronization of fractional-order systems via linear feedback control [27, 28].

In this paper, based on the stability theory of fractional-order differential equations, Routh-Hurwitz stability condition, we investigate control and synchronization of fractional-order financial systems proposed by Chen via linear control. Sufficient conditions are established and easy to verify. Compared with sliding mode control and nonlinear feedback

control used for discussing fractional-order financial systems in [22, 23], the linear control is economic and easy to implement, through which control and synchronization of fractional-order financial systems will be obtained only by choosing suitable feedback gains. The main job of this paper lies in two aspects. One is to remove chaotic phenomenon from fractional financial system by controlling, which makes prediction impossible in the financial world. The other is to realize harmonious and sustainable development between drive financial systems and response ones by investigating synchronization. The obtained results have a certain value to the theoretical guidance and application.

The remainder of this paper is organized as follows. In Section 2, preliminary results are presented and fractional-order financial system is described. In Section 3, some sufficient criteria for control of the fractional-order financial system are given. In Section 4, we discussed synchronization of the fractional-order financial system via linear error feedback. In Section 5, numerical simulations are given to illustrate the effectiveness of the main results. Finally, conclusions are drawn in Section 6.

2. Preliminaries and System Description

Fractional calculus is a generalization of integration and differentiation to a noninteger-order integrodifferential operator D_t^α defined by

$$D_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & R(\alpha) > 0, \\ 1, & R(\alpha) = 0, \\ \int_a^t (d\tau)^{-\alpha}, & R(\alpha) < 0. \end{cases} \quad (2.1)$$

Fractional calculus is being used in various fields gradually, such as biophysics, nonlinear dynamics, informatics, and control engineering [29]. There are some definitions for fractional derivatives. The commonly used definitions are Grunwald-Letnikov (GL), Riemann-Liouville (RL), and Caputo (C) definitions.

The Grunwald-Letnikov (GL) derivative with fractional-order α is given by

$${}^G D_t^\alpha f(t) = \lim_{h \rightarrow 0} f_h^{(\alpha)}(t) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{i=0}^{[(t-\alpha)/h]} (-1)^i \binom{\alpha}{i} f(t - ih), \quad (2.2)$$

where $[\cdot]$ means the integer part.

The Riemann-Liouville (RL) fractional derivatives are defined by

$${}^R D_t^\alpha f(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{(\alpha-n+1)}} d\tau, \quad n-1 < \alpha < n, \quad (2.3)$$

where $\Gamma(\cdot)$ is the gamma function, $\Gamma(\tau) = \int_0^\infty t^{\tau-1} e^{-t} dt$.

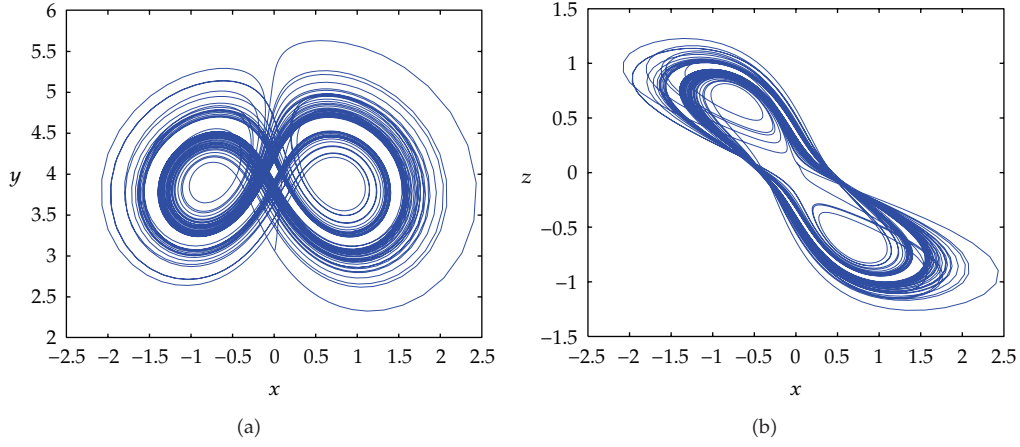


Figure 1: Chaotic attractors of system (2.5) with order $(0.86, 0.92, 0.95)$.

The Caputo (C) fractional derivative is defined as follows:

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t f^{(n)}(\tau) (t-\tau)^{(n-\alpha-1)} d\tau, \quad n-1 < \alpha < n. \quad (2.4)$$

It should be noted that the advantage of Caputo approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as those for integer-order differential, which have well-understood physical meanings. Therefore, in the rest of this paper, the notation D_*^α is chosen as the Caputo fractional derivative operator ${}^C D_\alpha$.

The fractional-order chaotic financial system can be described by

$$\begin{aligned} D_*^{\alpha_1} x &= z + (y - a)x, \\ D_*^{\alpha_2} y &= 1 - by - x^2, \\ D_*^{\alpha_3} z &= -x - cz, \end{aligned} \quad (2.5)$$

where α_1, α_2 , and α_3 are the fractional-order, $\alpha_1, \alpha_2, \alpha_3 \in (0, 1)$. System (2.5) describes the time variation of three state variables: the interest rate x , the investment demand y , and the price index z ; a is the saving amount, b is the cost per investment, c is the elasticity of demand of commercial markets, parameters a, b , and c are positive real constants. When $a = 3, b = 0.1, c = 1$, fractional-order $(\alpha_1, \alpha_2, \alpha_3)$ is taken as $(0.86, 0.92, 0.95)$, the largest Lyapunov exponent of system (2.5) is greater than 0 [23]. Figure 1 displays chaotic attractors of system (2.5).

To obtain our results, the following lemma is presented.

Consider the following fractional-order system:

$$\begin{aligned} D_*^{\alpha_1} x &= f(x, y, z), \\ D_*^{\alpha_2} y &= g(x, y, z), \\ D_*^{\alpha_3} z &= h(x, y, z). \end{aligned} \quad (2.6)$$

Lemma 2.1 (see [30]). *System (2.6) is asymptotically stable at the equilibrium points if $|\arg(\lambda_i(A))| > \alpha\pi/2$, $\alpha = \max(\alpha_1, \alpha_2, \alpha_3)$, $i = (1, 2, 3)$, for all eigenvalues λ_i of the Jacobian matrix J :*

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix}. \quad (2.7)$$

3. Chaos Control

3.1. Analysis of Equilibrium Points

Let

$$\begin{aligned} z + (y - a)x &= 0, \\ 1 - by - x^2 &= 0, \\ -x - cz &= 0. \end{aligned} \quad (3.1)$$

It is easy to obtain that system (2.5) has three equilibrium points if

$$\begin{aligned} c - b - abc &> 0, \\ P_1 &= \left(0, \frac{1}{b}, 0\right), \\ P_2 &= \left(\sqrt{\frac{(c-b-abc)}{c}}, \frac{(1+ac)}{c}, \left(-\frac{1}{c}\right) \cdot \sqrt{\frac{(c-b-abc)}{c}}\right), \\ P_3 &= \left(-\sqrt{\frac{(c-b-abc)}{c}}, \frac{(1+ac)}{c}, \left(\frac{1}{c}\right) \cdot \sqrt{\frac{(c-b-abc)}{c}}\right). \end{aligned} \quad (3.2)$$

The Jacobian matrix of system (2.5) at equilibrium point (x^*, y^*, z^*) is

$$J = \begin{bmatrix} y^* - a & x^* & 1 \\ -2x^* & -b & 0 \\ -1 & 0 & -c \end{bmatrix}. \quad (3.3)$$

The characteristic equation of the Jacobian matrix J is given by

$$\begin{aligned} \lambda^3 + (a + b + c - y^*)\lambda^2 + (ab + ac + bc - by^* - cy^* + 2(x^*)^2 + 1)\lambda \\ + abc - bcy + b + 2c(x^*)^2 = 0. \end{aligned} \quad (3.4)$$

When $a = 3$, $b = 0.1$, $c = 1$, $P_1 = (0, 10, 0)$. Substituting the parameters a , b , and c into (3.4), one obtains

$$\lambda^3 - 5.9\lambda^2 - 6.6\lambda - 0.6 = 0. \quad (3.5)$$

The roots of (3.5) are $\lambda_1 = 6.8730$, $\lambda_2 = -0.8730$, and $\lambda_3 = -0.1$; obviously, λ_1 is a positive real number, based on the lemma, $\arg(\lambda_1) = 0$, $\arg(\lambda_2) = \arg(\lambda_3) = \pi$. So, the equilibrium point P_1 is unstable.

When $a = 3$, $b = 0.1$, $c = 1$, $P_2 = (\sqrt{0.6}, 4, -\sqrt{0.6})$, $P_3 = (-\sqrt{0.6}, 4, \sqrt{0.6})$. Substituting the parameters a , b , and c into (3.4), the characteristic equation of the Jacobian matrix J at equilibrium points P_2, P_3 is

$$\lambda^3 + 0.1\lambda^2 + 1.2\lambda + 1.2 = 0. \quad (3.6)$$

The roots of (3.6) are $\lambda_1 = -0.7256$, $\lambda_2 = 0.3128 + 1.2474i$, and $\lambda_3 = 0.3128 - 1.2474i$. According to the lemma, λ_1 is a negative real number, $\arg(\lambda_1) = \pi$, $\arg|(\lambda_{2,3})| = 1.3251$. $\alpha = 2|\arg(\lambda_{2,3})|/\pi = 0.8436$. Therefore, we can conclude that if the maximum fractional-order α among $\alpha_1, \alpha_2, \alpha_3$ is less than 0.8436, the equilibrium points P_2, P_3 are stable. On the contrary, the equilibrium points P_2, P_3 are unstable.

According to the above analysis, when the maximum fractional-order α among α_1, α_2 , and α_3 is less than 0.8436, there exist two stable equilibrium points; when $\alpha_1 = \alpha_2 = \alpha_3 = 0.8436$, system (2.5) will admit a limit cycle; when fractional-order α_1, α_2 , and α_3 are all greater than 0.8436, there are no stable equilibria, with all the equilibrium points being unstable, which implies that there may exist chaos for system (2.5). By calculating the values of Lyapunov exponents of system (2.5), it could be found that system (2.5) exhibits chaotic behaviors if maximum fractional-order α among α_1, α_2 , and α_3 is greater than or equal to 0.86 [23].

3.2. Chaos Control

In this subsection, linear state feedback controller is designed to control fractional-order chaotic financial system to its equilibrium.

The controlled fractional-order chaotic financial system is given by

$$\begin{aligned} D_*^{\alpha_1} x &= z + (y - a)x + k_1(x - x^*), \\ D_*^{\alpha_2} y &= 1 - by - x^2 + k_2(y - y^*), \\ D_*^{\alpha_3} z &= -x - cz + k_3(z - z^*), \end{aligned} \quad (3.7)$$

where k_1 , k_2 , and k_3 denote feedback gains and (x^*, y^*, z^*) is the desired equilibrium point. Obviously, system (3.7) has one equilibrium point (x^*, y^*, z^*) .

The Jacobian matrix of system (3.7) at equilibrium point (x^*, y^*, z^*) is

$$J = \begin{bmatrix} y^* - a + k_1 & x^* & 1 \\ -2x^* & k_2 - b & 0 \\ -1 & 0 & k_3 - c \end{bmatrix}. \quad (3.8)$$

The characteristic equation of the Jacobian matrix (3.8) is

$$\begin{aligned} \lambda^3 &+ (a + b + c - k_1 - k_2 - k_3 - y^*)\lambda^2 \\ &+ (ab + ac + bc + k_1k_2 + k_2k_3 + k_1k_3 - (b + c)k_1 \\ &+ (y - a - c)k_2 - bcy^* + (y - a - b)k_3 + 1 - by^* + 2(x^*)^2)\lambda \\ &- bck_1 + ck_1k_2 + (cy^* - ac - 1)k_2 - (by^* + ab + 2(x^*)^2)k_3 + bk_1k_3 \\ &+ (a - y)k_2k_3 - k_1k_2k_3 + abc + b + c - y^* + 2c(x^*)^2 = 0. \end{aligned} \quad (3.9)$$

Our goal is to find suitable feedback gains such that all the state trajectories of system (3.7) are controlled to its equilibrium point, that is to say, roots of (3.9) should satisfy the conditions in the lemma.

Theorem 3.1. *When $a = 3$, $b = 0.1$, $c = 1$, $P_1 = (0, 10, 0)$; system (3.7) stabilizes to equilibrium point P_1 , if state feedback gains k_1 , k_2 , and k_3 satisfy the following conditions:*

$$-10 \leq k_1 - k_3 < -6, \quad k_1 + k_3 < -6, \quad k_2 < 0.1. \quad (3.10)$$

Proof. Substituting the parameters a , b , and c into (3.9), one obtains

$$(10\lambda + 1 - 10k_2)\left(\lambda^2 - (k_1 + k_3 + 6)\lambda + k_1k_3 - k_1 + 7k_3 - 6\right) = 0. \quad (3.11)$$

It is very easy to obtain the roots of (3.11):

$$\lambda_1 = k_2 - 0.1, \quad \lambda_{2,3} = \frac{1}{2}\left(k_1 + k_3 + 6 \pm \sqrt{(k_1 - k_3 + 10)(k_1 - k_3 + 6)}\right). \quad (3.12)$$

Note that λ_1 is a negative real number, $\lambda_{2,3}$ are a pair of conjugate imaginary roots, and the real parts of imaginary root are negative, that is, $\arg(\lambda_1) = \pi$, $\arg(\lambda_{2,3}) > \pi/2$. Therefore, the trajectory of the controlled fractional-order system (3.7) is asymptotically stable at equilibrium point P_1 . \square

In Theorem 3.1, we designed three simple linear feedback controllers to ensure controlled system (3.7) stabilized to P_1 . In practice, two linear feedback controllers or single linear feedback controller in controlled system (3.7) will do the same thing. Then we have the following corollaries.

Corollary 3.2. *Controlled system (3.7) will approach asymptotically to P_1 with one of the following conditions about feedback gains:*

$$\begin{aligned} -10 \leq k_1 < -6, \quad k_2 < 0.1, \quad k_3 = 0, \\ k_1 + k_3 < -6, \quad -10 \leq k_1 - k_3 < -6, \quad k_2 = 0. \end{aligned} \quad (3.13)$$

Corollary 3.3. *Controlled system (3.7) will approach asymptotically to P_1 if feedback gains satisfy*

$$-10 \leq k_1 < -6, \quad k_2 = 0, \quad k_3 = 0. \quad (3.14)$$

Theorem 3.4. *When $a = 3$, $b = 0.1$, $c = 1$, $P_2 = (\sqrt{0.6}, 4, -\sqrt{0.6})$; if feedback gains k_1 , k_2 , and k_3 satisfy one of the following conditions, system (3.7) will approach and stabilizes to equilibrium point P_2 asymptotically:*

$$\begin{aligned} k_1 = 0, \quad k_2 < -0.9, \quad k_3 = 0, \\ k_1 < \frac{121 - \sqrt{62161}}{220}, \quad k_2 = 0, \quad k_3 = 0. \end{aligned} \quad (3.15)$$

Proof. (1) Substitute the parameters $a = 3$, $b = 0.1$, $c = 1$, and $k_1 = k_3 = 0$ into (3.9), one obtains

$$\lambda^3 + \left(\frac{1}{10} - k_2\right)\lambda^2 + \frac{6}{5}\lambda + \frac{6}{5} = 0. \quad (3.16)$$

According to the Routh-Hurwitz criterion, real parts of these eigenvalues $\lambda_{1,2,3}$ of (3.16) are all negative if

$$\frac{1}{10} - k_2 > 0, \quad \left(\frac{1}{10} - k_2\right)\frac{6}{5} > \frac{6}{5}, \quad (3.17)$$

that is, $k_2 < -0.9$. That implies that the trajectory of the controlled fractional-order system (3.7) is asymptotically stable at equilibrium point P_2 .

(2) Substituting the parameters $a = 3$, $b = 0.1$, $c = 1$, and $k_2 = k_3 = 0$ into (3.9), one obtains

$$\lambda^3 + \left(\frac{1}{10} - k_1\right)\lambda^2 + \left(\frac{6}{5} - \frac{11}{10}k_1\right)\lambda + \frac{6}{5} - \frac{1}{10}k_1 = 0. \quad (3.18)$$

By applying the Routh-Hurwitz criterion, if the following conditions for feedback gains are met,

$$\frac{1}{10} - k_1 > 0, \quad \frac{6}{5} - \frac{11}{10}k_1 > 0, \quad \left(\frac{1}{10} - k_1\right)\left(\frac{6}{5} - \frac{11}{10}k_1\right) > \frac{6}{5} - \frac{1}{10}k_1, \quad (3.19)$$

then real parts of these eigenvalues $\lambda_{1,2,3}$ are all negative. It follows that $k_1 < (1/220)(121 - \sqrt{62161})$. Thus, the trajectory of the controlled fractional-order system (3.7) is asymptotically stable at equilibrium point P_2 . \square

Remark 3.5. Actually, we adopt single linear feedback controller to stabilize P_2 of controlled system in Theorem 3.4, namely, stabilizing P_2 by adding single linear feedback controller on the first state or the second state, but we cannot do it via adding single linear feedback controller on the third state; the reasons are described as follows.

Substituting the parameters $a = 3$, $b = 0.1$, $c = 1$, and $k_1 = k_2 = 0$ into (3.9), one obtains

$$\lambda^3 + \left(\frac{1}{10} - k_3\right)\lambda^2 + \left(\frac{6}{5} + \frac{9}{10}k_3\right)\lambda - \frac{11}{10}k_3 + \frac{6}{5} = 0. \quad (3.20)$$

By using the Routh-Hurwitz criterion, real parts of these eigenvalues $\lambda_{1,2,3}$ are all negative if

$$\frac{1}{10} - k_3 > 0, \quad -\frac{11}{10}k_3 + \frac{6}{5} > 0, \quad \left(\frac{1}{10} - k_3\right)\left(\frac{6}{5} + \frac{9}{10}k_3\right) > -\frac{11}{10}k_3 + \frac{6}{5}. \quad (3.21)$$

By using simple calculation, such k_3 is obviously absent. So the trajectory of the controlled fractional-order system (3.7) is not asymptotically stable at equilibrium point P_2 .

Theorem 3.6. *When $a = 3$, $b = 0.1$, $c = 1$, $P_2 = (-\sqrt{0.6}, 4, \sqrt{0.6})$; if feedback gains k_1 , k_2 , and k_3 satisfy one of the following conditions, system (3.7) will approach and stabilize to equilibrium point P_2 asymptotically:*

$$\begin{aligned} k_1 &= 0, & k_2 &< -0.9, & k_3 &= 0, \\ k_1 &< \frac{121 - \sqrt{62161}}{220}, & k_2 &= 0, & k_3 &= 0. \end{aligned} \quad (3.22)$$

Proof. The proof is the same as that of Theorem 3.4 and so we omit it here. \square

4. Chaos Synchronization

In this section, we will investigate synchronization of fractional-order financial system (2.5). Three and two simple linear feedback controllers are designed to achieve synchronization,

which simplify the existing synchronization schemes and reduce the synchronization cost. Drive system and response system are described as follows, respectively:

$$\begin{aligned} D_*^{\alpha_1} x_m &= z_m + (y_m - a)x_m, \\ D_*^{\alpha_2} y_m &= 1 - by_m - x_m^2, \end{aligned} \quad (4.1)$$

$$\begin{aligned} D_*^{\alpha_3} z_m &= -x_m - cz_m, \\ D_*^{\alpha_1} x_s &= z_s + (y_s - a)x_s + u_1, \\ D_*^{\alpha_2} y_s &= 1 - by_s - x_s^2 + u_2, \\ D_*^{\alpha_3} z_s &= -x_s - cz_s + u_3, \end{aligned} \quad (4.2)$$

where u_1 , u_2 , and u_3 denote the external control inputs, to be designed later. It follows from systems (4.1) and (4.2) that the following error dynamical system is

$$\begin{aligned} D_*^{q_1} e_1 &= (y_m - a)e_1 + x_s e_2 + e_3 - u_1, \\ D_*^{q_2} e_2 &= (x_m + x_s)e_1 - be_2 - u_2, \\ D_*^{q_3} e_3 &= -e_1 - ce_3 - u_3, \end{aligned} \quad (4.3)$$

where $e_1 = x_m - x_s$, $e_2 = y_m - y_s$, and $e_3 = z_m - z_s$. Our aim is to find suitable control laws $u_i (i = 1, 2, 3)$ for stabilizing the error dynamics system (4.3). To this end, the following theorem is proposed.

Theorem 4.1. *For any given initial conditions, synchronization between systems (4.1) and (4.2) will occur if control schemes are defined as follows:*

$$\begin{aligned} u_1 &= k_1 e_1, \\ u_2 &= k_2 e_2, \\ u_3 &= k_3 e_3, \end{aligned} \quad (4.4)$$

where k_1 , k_2 , and k_3 are feedback gains and satisfy the following conditions:

$$\begin{aligned} k_1 &> L - a, \\ k_2 &< -b - \frac{9L^2}{4(L - a - k_1)}, \\ k_3 &< -c. \end{aligned} \quad (4.5)$$

Proof. Combining (4.3) with (4.4), the error system (4.3) is given by

$$\begin{aligned} D_*^{q_1} e_1 &= (y_m - a - k_1)e_1 + x_s e_2 + e_3, \\ D_*^{q_2} e_2 &= (x_m + x_s)e_1 - (b + k_2)e_2, \\ D_*^{q_3} e_3 &= -e_1 - (c + k_3)e_3. \end{aligned} \quad (4.6)$$

Error dynamical system (4.6) can be rewritten as the following matrix form:

$$\begin{bmatrix} D_*^{q_1} e_1 \\ D_*^{q_2} e_2 \\ D_*^{q_3} e_3 \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad (4.7)$$

where

$$A = \begin{bmatrix} y_m - a - k_1 & x_s & 1 \\ x_m + x_s & -(b + k_2) & 0 \\ -1 & 0 & -(c + k_3) \end{bmatrix}. \quad (4.8)$$

Suppose that λ is one of the eigenvalues of matrix A and the corresponding non-eigenvector is $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^T$, that is,

$$A\varepsilon = \lambda\varepsilon. \quad (4.9)$$

Taking conjugate transposal on both sides of (4.9), one obtains

$$(A\varepsilon)^H = \bar{\lambda}\varepsilon^H. \quad (4.10)$$

Equation (4.9) multiplied left by $1/2\varepsilon^H$ plus (4.10) multiplied right by $1/2\varepsilon$, we have

$$\varepsilon^H \left(\frac{1}{2}A + \frac{1}{2}A^H \right) \varepsilon = \frac{1}{2}(\lambda + \bar{\lambda})\varepsilon^H \varepsilon. \quad (4.11)$$

Because a chaotic system has bounded trajectories, there exists a positive constant L , such that $|x| < L$, $|y| < L$. Thus,

$$\begin{aligned} \varepsilon^H \frac{1}{2} (A + A^H) \varepsilon &= \begin{bmatrix} \varepsilon_1^H, \varepsilon_2^H, \varepsilon_3^H \end{bmatrix} \begin{bmatrix} y_m - a - k_1 & \frac{x_m + 2x_s}{2} & 0 \\ \frac{x_m + 2x_s}{2} & -(b + k_2) & 0 \\ 0 & 0 & -(c + k_3) \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \\ &\leq \begin{bmatrix} |\varepsilon_1^H|, |\varepsilon_2^H|, |\varepsilon_3^H| \end{bmatrix} \begin{bmatrix} L - a - k_1 & \frac{3}{2}L & 0 \\ \frac{3}{2}L & -(b + k_2) & 0 \\ 0 & 0 & -(c + k_3) \end{bmatrix} \begin{bmatrix} |\varepsilon_1| \\ |\varepsilon_2| \\ |\varepsilon_3| \end{bmatrix}. \end{aligned} \quad (4.12)$$

From (4.11), we have

$$\begin{aligned} \frac{1}{2} (\lambda + \bar{\lambda}) &= \frac{\varepsilon^H ((1/2)A + (1/2)A^H) \varepsilon}{\varepsilon^H \varepsilon} \\ &\leq \frac{1}{\varepsilon^H \varepsilon} \begin{bmatrix} |\varepsilon_1^H|, |\varepsilon_2^H|, |\varepsilon_3^H| \end{bmatrix} P \begin{bmatrix} |\varepsilon_1| \\ |\varepsilon_2| \\ |\varepsilon_3| \end{bmatrix}^T, \end{aligned} \quad (4.13)$$

where

$$P = \begin{bmatrix} L - a - k_1 & \frac{3}{2}L & 0 \\ \frac{3}{2}L & -(b + k_2) & 0 \\ 0 & 0 & -(c + k_3) \end{bmatrix}. \quad (4.14)$$

It is obvious that real parts of all eigenvalues λ are negative and matrix P should be negative definite, namely, the following inequalities hold:

$$\begin{aligned} L - a - k_1 &< 0, \\ -(b + k_2)(L - a - k_1) - \frac{9}{4}L^2 &< 0, \\ (b + k_2)(c + k_3)(L - a - k_1) + \frac{9}{4}L^2(c + k_3) &< 0. \end{aligned} \quad (4.15)$$

Simplifying the above inequalities, one has

$$\begin{aligned} k_1 &> L - a, \\ k_2 &< -b - \frac{9L^2}{4(L - a - k_1)}, \\ k_3 &< -c. \end{aligned} \quad (4.16)$$

Therefore, based on stability theorem of fractional-order systems, error system (4.6) is asymptotically stable at the origin, which implies that synchronization between systems (4.1) and (4.2) will be achieved. \square

Based on the above analysis, it is easy to obtain that two linear feedback controllers could also achieve synchronization between systems (4.1) and (4.2). Then, we have the following corollary.

Corollary 4.2. *For any given initial condition, if control schemes are described as $u_1 = k_1 e_1$, $u_3 = k_3 e_3$ and feedback gains satisfy*

$$L - a < k_1 < L - a + \frac{9}{4b}L^2, \quad k_3 < -c, \quad (4.17)$$

then the response system (4.2) can synchronize the drive system (4.1).

Proof. The proof is similar to that of Theorem 4.1. After some computations, we have

$$\begin{aligned} \frac{1}{2}(\lambda + \bar{\lambda}) &= \frac{\varepsilon^H((1/2)A + 1/2A^H)\varepsilon}{\varepsilon^H\varepsilon} \\ &\leq \frac{1}{\varepsilon^H\varepsilon} \left[\left| \varepsilon_1^H \right|, \left| \varepsilon_2^H \right|, \left| \varepsilon_3^H \right| \right] P \left[\left| \varepsilon_1 \right|, \left| \varepsilon_2 \right|, \left| \varepsilon_3 \right| \right]^T, \end{aligned} \quad (4.18)$$

where

$$P = \begin{bmatrix} L - a - k_1 & \frac{3}{2}L & 0 \\ \frac{3}{2}L & -b & 0 \\ 0 & 0 & -(c + k_3) \end{bmatrix}. \quad (4.19)$$

Matrix P must be negative definite; if the following inequalities hold:

$$\begin{aligned} L - a - k_1 &< 0, \\ -b(L - a - k_1) - \frac{9}{4}L^2 &< 0, \\ b(c + k_3)(L - a - k_1) + \frac{9}{4}L^2(c + k_3) &< 0, \end{aligned} \quad (4.20)$$

then one obtains

$$L - a < k_1 < L - a + \frac{9}{4b}L^2, \quad k_3 < -c. \quad (4.21)$$

Therefore, real parts of all eigenvalues λ are negative; according to the stability theorem of fractional-order systems, error system (4.6) is asymptotically stable. This means that the slave system (4.2) can asymptotically synchronize the master system (4.1). \square

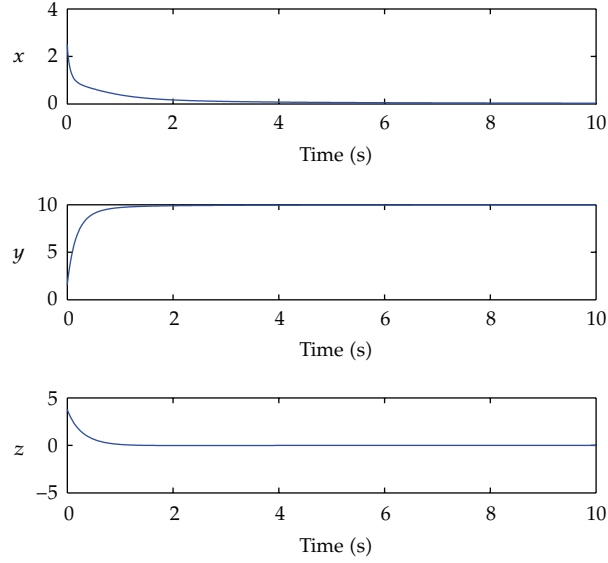


Figure 2: The time response of the states for stabilizing system (3.7) to P_1 ($k_1 = -9$, $k_2 = -5$, $k_3 = -2$).

Remark 4.3. References [22, 23] presented control of the model by using sliding mode control and nonlinear control, respectively, but with no consideration of synchronization. In contrast, control cost of linear control is low.

Remark 4.4. Response time of control and synchronization could be adjusted with suitable state feedback gains and error feedback gains in allowed limits.

5. Numerical Simulations

In this section, to verify theoretical results obtained in the previous section, the corresponding numerical simulations will be performed and an improved predictor-corrector algorithm is applied (see the appendix). In all simulations, fractional-order (α_1 , α_2 , and α_3) is chosen as (0.86, 0.92, and 0.95) to ensure the existence of chaos in system (2.5).

Chaos Control

The parameters of system (2.5) are selected as $a = 3$, $b = 0.1$, and $c = 1$. The initial values of controlled system (3.7) are chosen as $(x(0), y(0), z(0)) = (3, 1, 4)$, respectively. Based on Theorem 3.1, we chose state feedback gain (k_1, k_2, k_3) as $(-9, -5, -2)$, under these conditions, roots of (3.11) are $\lambda_1 = -5.1$, $\lambda_2 = -2.5 + 0.866i$, and $\lambda_3 = -2.5 - 0.866i$. Controlled system (3.7) is asymptotically stable at P_1 . Figure 2 shows that the controlled system (3.7) can be stabilized to P_1 . When state feedback gain $(k_1, k_2, k_3) = (0, -3, 0)$ and $(-2, 0, 0)$, roots $(\lambda_1, \lambda_2, \lambda_3)$ of (3.16) and (3.18) are $(-2.8256, -0.1372 - 0.6371i, -0.1372 + 0.6371i)$ and $(-2.8256, -0.1372 + 0.6371i, -0.1372 - 0.6371i)$, respectively. According to Theorem 3.4, controlled system (3.7) will stabilize at P_2 . Figures 3 and 4 display the simulation results, respectively. When the parameters of the controlled system (3.7) are selected as above, based on Theorem 3.6, P_3 is stabilized. Figures 5 and 6 display the stabilization of the equilibrium point P_3 for state feedback gain $(0, -8, 0)$ and $(-2, 0, 0)$, respectively.

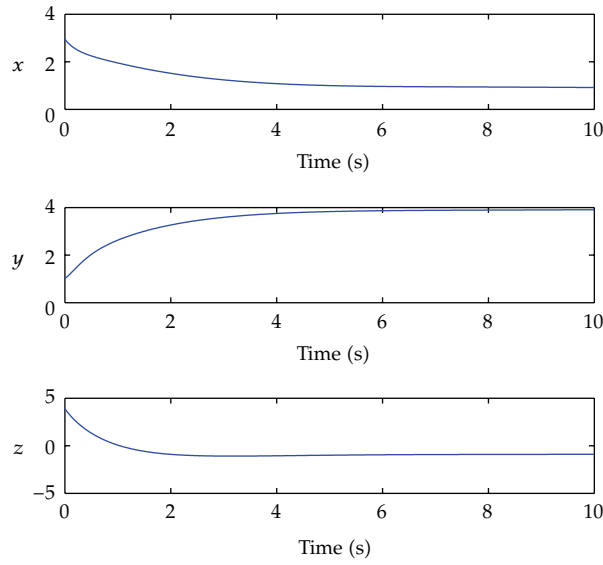


Figure 3: The time response of the states for stabilizing system (3.7) to $P_2(k_1 = 0, k_2 = -3, k_3 = 0)$.

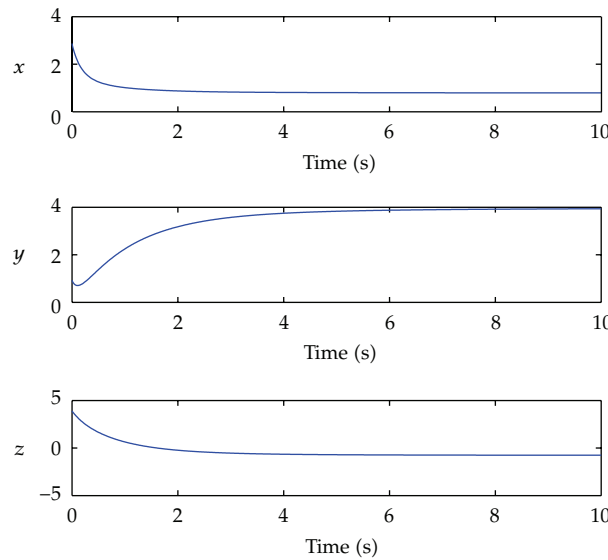


Figure 4: The time response of the states for stabilizing system (3.7) to $P_2(k_1 = -2, k_2 = 0, k_3 = 0)$.

Chaos Synchronization

When the parameters of system (2.5) are chosen as $a = 3, b = 0.1,$ and $c = 1,$ select the initial values of the drive and the response systems as $(x_m(0), y_m(0), z_m(0)) = (4, -3, 2)$ and $(x_s(0), y_s(0), z_s(0)) = (-3, 2, 4),$ respectively. For error feedback gain $(k_1, k_2, k_3) = (12, 9, -2),$ simulation result of the synchronization between systems (4.1) and (4.2) is shown in Figure 7. The synchronization error states between systems (4.1) and (4.2) and displayed in Figure 8. When error feedback gains $k_1 = 8, k_2 = 0,$ and $k_3 = -2,$ it can be seen that the derive

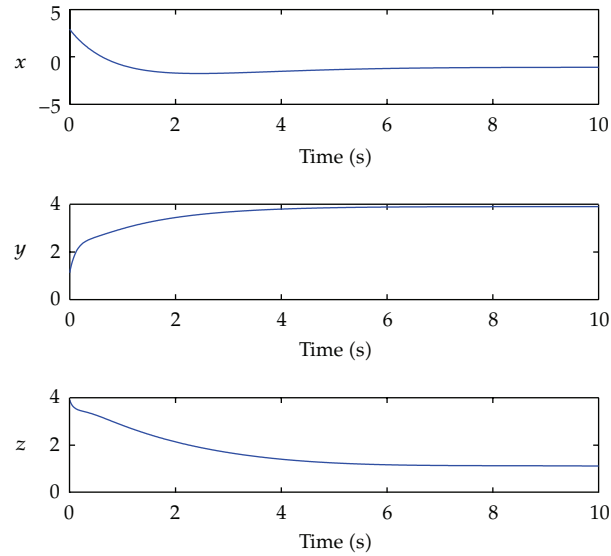


Figure 5: The time response of the states for stabilizing system (3.7) to $P_3(k_1 = 0, k_2 = -8, k_3 = 0)$.

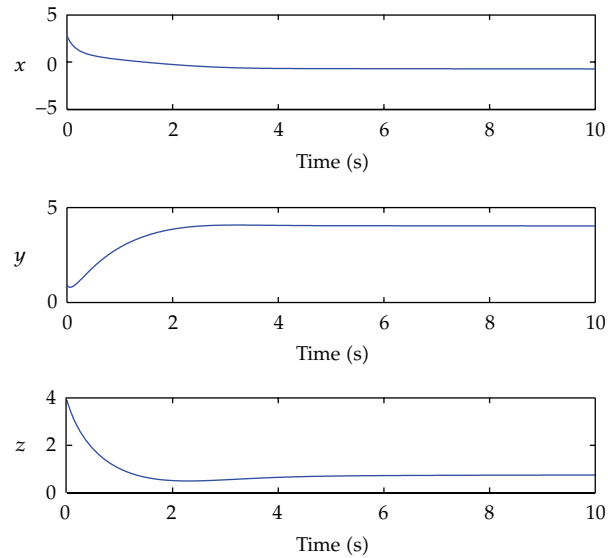


Figure 6: The time response of the states for stabilizing system (3.7) to $P_3(k_1 = -2, k_2 = 0, k_3 = 0)$.

system (4.1) and the response system (4.2) achieve the synchronization in Figure 9. Figure 10 displays the error state time response between systems (4.1) and (4.2).

6. Conclusion

In this paper, based on the stability theory of fractional-order systems and Routh-Hurwitz stability condition, some sufficient conditions for control and synchronization of the fractional-order chaotic financial system by linear feedback control have been derived. Finally, numerical simulations are provided to verify the effectiveness of the results obtained. The results

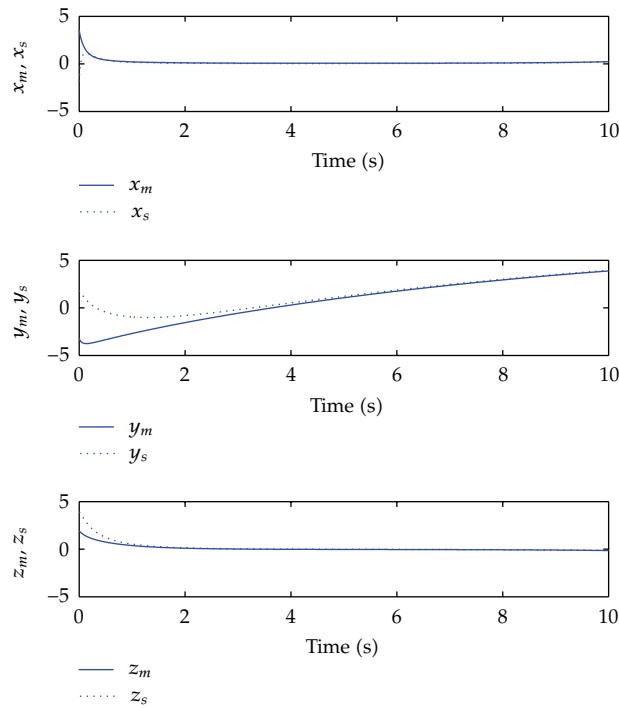


Figure 7: Synchronization states of systems (4.1) and (4.2) ($k_1 = 12, k_2 = 9, k_3 = -2$).

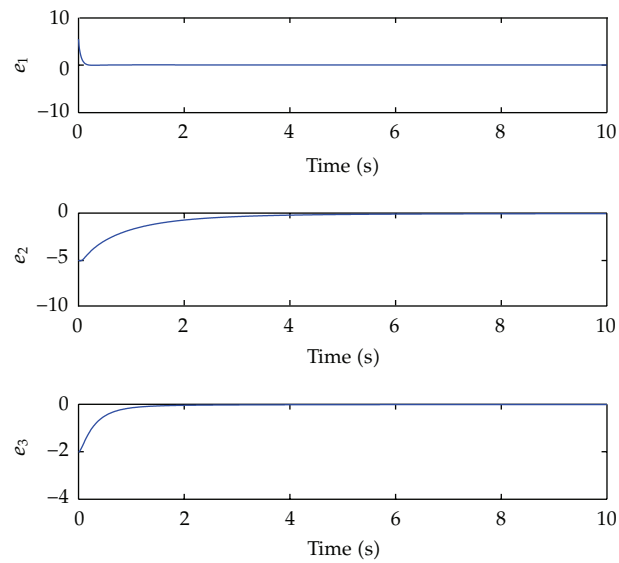


Figure 8: Synchronization error states of systems (4.1) and (4.2) ($k_1 = 12, k_2 = 9, k_3 = -2$).

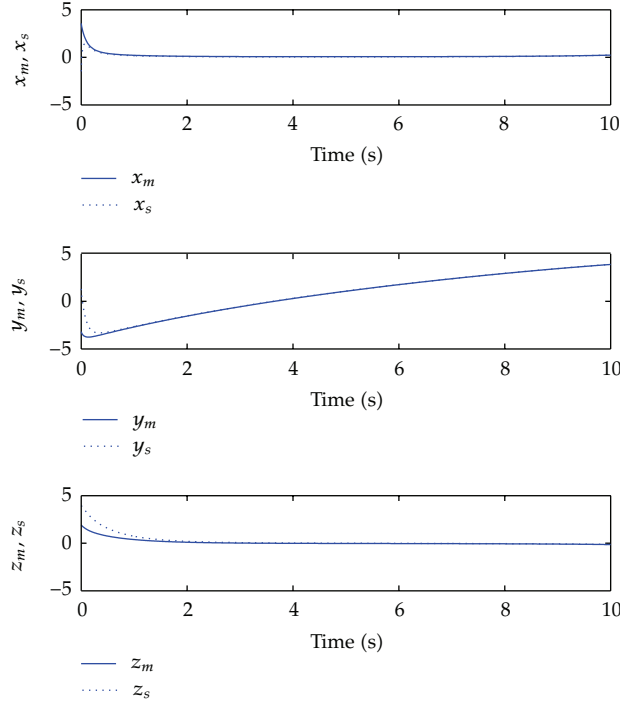


Figure 9: Synchronization states of systems (4.1) and (4.2) ($k_1 = 8$, $k_2 = 0$, $k_3 = -2$).

are economical, reliable and efficient. It is noted that the method applied in the paper can also be extended to other fractional-order chaotic systems.

Appendix

An improved predictor-corrector algorithm [31] for fractional-order differential equations is presented in brief. In comparison with the classical one-step Adams-Bashforth-Moulton algorithm, the numerical approximation of the improved algorithm is more accurate and the computational cost is lower.

The following differential equation:

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= f(t, x), \quad 0 \leq t \leq T, \\ x^{(k)} &= x_0^{(k)}, \quad k = 0, 1, 2, \dots, [\alpha] - 1, \end{aligned} \tag{A.1}$$

is equivalent to the Volterra integral equation [32]

$$x = \sum_{k=0}^{[\alpha]-1} x_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, x)}{(t-\tau)^{1-\alpha}} d\tau. \tag{A.2}$$

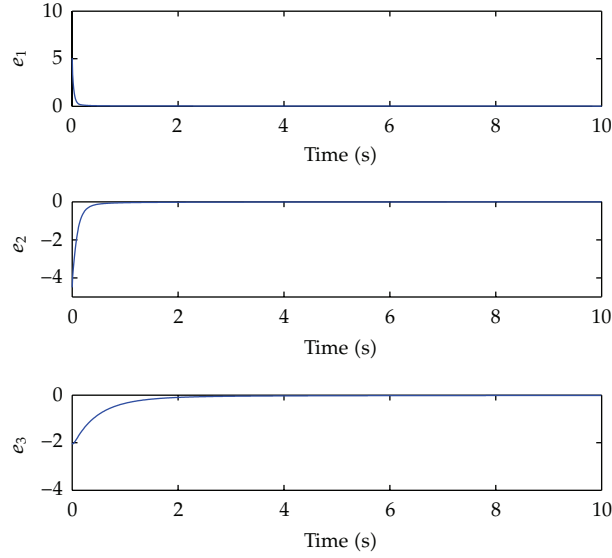


Figure 10: Synchronization error states of systems (4.1) and (4.2) ($k_1 = 8$, $k_2 = 0$, $k_3 = -2$).

Set $h = T/N$, $t_n = nh$ ($n = 0, 1, 2, \dots, N$). Then the above equation can be discretized as follows:

$$x_h(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} x_0^{(k)} \frac{t^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, x_h^\rho(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^n a_{j,n+1} f(t_j, x_h(t_j)), \quad (\text{A.3})$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha+1}, & j=0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \leq j \leq n, \\ 1, & j=n+1, \end{cases} \quad (\text{A.4})$$

$$x_h^\rho(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} f(t_j, x_h(t_j)),$$

in which, $b_{j,n+1} = h^\alpha / \alpha((n-j+1)^\alpha - (n-j)^\alpha)$.

The error estimate e is $\text{Max} |x(t_j) - x_h(t_j)| = O(h^\rho)$ ($j = 0, 1, \dots, N$), where $\rho = \text{Min}(2, 1+\alpha)$.

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