Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 208167, 8 pages
doi:10.1155/2012/208167

Research Article

Global Asymptotic Stability of a Predator-Prey Model with Modified Leslie-Gower and Holling-Type II Schemes

Shengbin Yu

Sunshine College, Fuzhou University, Fuzhou, Fujian 350015, China

Correspondence should be addressed to Shengbin Yu, yushengbin.8@163.com

Received 3 April 2012; Accepted 22 May 2012

Academic Editor: Zhen Jin

We study the predator-prey model proposed by Aziz-Alaoui and Okiye (Appl. Math. Lett. 16 (2003) 1069–1075) First, the structure of equilibria and their linearized stability is investigated. Then, we provide two sufficient conditions on the global asymptotic stability of a positive equilibrium by employing the Fluctuation Lemma and Lyapunov direct method, respectively. The obtained results not only improve but also supplement existing ones.

1. Introduction

One of the important interactions among species is the predator-prey relationship and it has been extensively studied because of its universal existence. There are many factors affecting the dynamics of predator-prey models. One of the familiar factors is the functional response, referring to the change in the density of prey attached per unit time per predator as the prey density changes. In the classical Lotka-Volterra model, the functional response is linear, which is valid first-order approximations of more general interaction. To build more realistic models, Holling [1] suggested three different kinds of functional responses, and Leslie and Gower [2] introduced the so-called Leslie-Gower functional response.

Recently, Aziz-Alaoui and Daher Okiye [3] proposed and studied the following predator-prey model with modified Leslie-Gower and Holling-type II schemes,

$$\begin{aligned} \frac{dx}{dt} &= \left(r_1 - b_1 x - \frac{a_1 y}{x + k_1}\right)x, \\ \frac{dy}{dt} &= \left(r_2 - \frac{a_2 y}{x + k_2}\right)y. \end{aligned} \tag{1.1}$$

Here, all the parameters are positive, and we refer to Aziz-Alaoui and Daher Okiye [3] for their biological meanings. System (1.1) can be considered as a representation of an insect pest-spider food chain, nature abounds in systems which exemplify this model; see [3].

Since then, system (1.1) and its nonautonomous versions have been studied by incorporating delay, impulses, harvesting, and so on (see, e.g., [4–11]). In spite of this extensive study, the dynamics of (1.1) is not fully understood and some existing results are not true. For example, the main result (Theorem 6 on global stability of a positive equilibrium) of Aziz-Alaoui and Daher Okiye [3] is not true as the condition (i) and condition (iii) cannot hold simultaneously. In fact, it follows from condition (i), $(1/4a_2b_1)(a_2r_1(r_1+4)+(r_2+1)^2(r_1+b_1k_2)) < r_1k_1/2a_1$, that $2a_1a_2r_1 < a_2b_1r_1k_1$. On the other hand, condition (iii), $4(r_1+b_1k_1) < a_1$, implies that $a_1 > 4b_1k_1$. Then, one can have $8a_2b_1r_1k_1 < a_2b_1r_1k_1$, which is impossible. One purpose of this paper is to establish several sufficient conditions on the global asymptotic stability of a positive equilibrium.

Let $\Omega_0 = \{(x,y) : x \geq 0,\ y \geq 0\}$. As a result of biological meaning, we only consider solutions $(x(t),y(t))$ of (1.1) with $(x(0),y(0)) \in \Omega_0$. Moreover, solutions $(x(t),y(t))$ of (1.1) with $(x(0),y(0)) \in \Omega_0$ are called positive solutions. An equilibrium $E^* = (x^*,y^*)$ of (1.1) is called *globally asymptotically stable* if $x(t) \to x^*$ and $y(t) \to y^*$ as $t \to \infty$ for any positive solution $(x(t),y(t))$ of (1.1). System (1.1) is *permanent* if there exists $0 < \alpha < \beta$ such that, for any positive solution $(x(t),y(t))$ of (1.1),

$$\alpha \leq \min\left\{\liminf_{t\to\infty} x(t), \liminf_{t\to\infty} y(t)\right\} \leq \max\left\{\limsup_{t\to\infty} x(t), \limsup_{t\to\infty} y(t)\right\} \leq \beta. \tag{1.2}$$

The remaining part of this paper is organized as follows. In Section 2, we discuss the structure of nonnegative equilibria to (1.1) and their linearized stability. This has not been done yet, and the results will motivate us to study global asymptotic stability of (1.1) in Section 3. The obtained results not only improve but also supplement existing ones.

2. Nonnegative Equilibria and Their Linearized Stability

The Jacobian matrix of (1.1) is

$$J(x,y) = \begin{pmatrix} r_1 - 2b_1x - \dfrac{a_1k_1y}{(x+k_1)^2} & -\dfrac{a_1x}{x+k_1} \\ \dfrac{a_2y^2}{(x+k_2)^2} & r_2 - \dfrac{2a_2y}{x+k_2} \end{pmatrix}. \tag{2.1}$$

An equilibrium E of (1.1) is (linearly) stable if the real parts of both eigenvalues of $J(E)$ are negative and therefore a sufficient condition for stability is

$$\operatorname{tr}(J(E)) < 0, \qquad \det(J(E)) > 0. \tag{2.2}$$

Obviously, (1.1) has three boundary equilibria, $E_0 = (0,0)$, $E_1 = (r_1/b_1, 0)$, and $E_2 = (0, r_2k_2/a_2)$, whose Jacobian matrices are

$$\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \quad \begin{pmatrix} -r_1 & -\dfrac{a_1r_1}{r_1+b_1k_1} \\ 0 & r_2 \end{pmatrix}, \quad \begin{pmatrix} r_1 - \dfrac{a_1r_2k_2}{a_2k_1} & 0 \\ \dfrac{r_2^2}{a_2} & -r_2 \end{pmatrix}, \tag{2.3}$$

respectively. As a direct consequence of (2.2), we have the following result.

Proposition 2.1. (i) *Both E_0 and E_1 are unstable.*

(ii) *E_2 is stable if $a_1r_2k_2 > a_2r_1k_1$, while it is unstable if $a_1r_2k_2 < a_2r_1k_1$.*

Besides the three boundary equilibria, (1.1) may have (componentwise) positive equilibria. Suppose that $\widehat{E} = (\widehat{x}, \widehat{y})$ is such an equilibrium. Then,

$$\begin{aligned} r_1 - b_1\widehat{x} - \frac{a_1\widehat{y}}{\widehat{x}+k_1} &= 0, \\ r_2 - \frac{a_2\widehat{y}}{\widehat{x}+k_2} &= 0. \end{aligned} \tag{2.4}$$

One can easily see that $\widehat{x}$ satisfies

$$a_2b_1\widehat{x}^2 + B\widehat{x} + (a_1r_2k_2 - a_2r_1k_1) = 0, \tag{2.5}$$

where $B \triangleq a_1r_2 - a_2r_1 + a_2b_1k_1$. Moreover, for convenience, we denote $\Delta \triangleq B^2 - 4a_2b_1(a_1r_2k_2 - a_2r_1k_1)$. Equation (2.5) can have at most two positive solutions, and hence (1.1) can have at most two positive equilibria. Precisely, we have the following three cases.

Case 1. Suppose one of the following conditions holds.

(i) $a_1r_2k_2 < a_2r_1k_1$.

(ii) $a_1r_2k_2 = a_2r_1k_1$ and $B < 0$.

(iii) $a_1r_2k_2 > a_2r_1k_1$, $B < 0$, and $\Delta = 0$.

Then, (1.1) has a unique positive equilibrium $E_{3,1} = (x_{3,1}, y_{3,1})$ with $x_{3,1} = (-B + \sqrt{\Delta})/2a_2b_1$ and $y_{3,1} = r_2(x_{3,1} + k_2)/a_2$.

Case 2. If $a_1r_2k_2 > a_2r_1k_1$, $B < 0$, and $\Delta > 0$, then (1.1) has two positive equilibria $E_{3,\pm} = (x_{3,\pm}, y_{3,\pm})$, where $x_{3,\pm} = (-B \pm \sqrt{\Delta})/2a_2b_1$ and $y_{3,\pm} = r_2(x_{3,\pm} + k_2)/a_2$.

Case 3. If no condition in Case 1 or Case 2 holds, then (1.1) has no positive equilibrium.

For a positive equilibrium $\hat{E} = (\hat{x}, \hat{y})$, $J(\hat{E})$ can be simplified to

$$J\left(\hat{E}\right) = \begin{pmatrix} \dfrac{\hat{x}(r_1 - b_1k_1 - 2b_1\hat{x})}{\hat{x} + k_1} & -\dfrac{a_1\hat{x}}{\hat{x} + k_1} \\ \dfrac{r_2^2}{a_2} & -r_2 \end{pmatrix} \tag{2.6}$$

by using (2.4). By simple computation, $\operatorname{tr}(J(\hat{E})) = (-2b_1\hat{x}^2 + (r_1 - r_2 - b_1k_1)\hat{x} - k_1r_2)/(\hat{x} + k_1)$, $\det(J(\hat{E})) = (r_2\hat{x}(2a_2b_1\hat{x} + B))/a_2(\hat{x} + k_1)$.

Then, one can easily see that $\det(J(E_{3,1})) > 0$ for Case 1(i)-(ii), $\det(J(E_{3,1})) = 0$ for Case 1(iii), $\det(J(E_{3,+})) > 0$, and $\det(J(E_{3,-})) < 0$. Therefore, we obtain the following.

Proposition 2.2. (i) *The positive equilibrium $E_{3,1}$ in Case 1(i)(ii) is stable if $2b_1x_{3,1}^2 - (r_1 - r_2 - b_1k_1)x_{3,1} + k_1r_2 > 0$.*

(ii) *The positive equilibrium $E_{3,-}$ is unstable, while the positive equilibrium $E_{3,+} = (x_{3,+}, y_{3,+})$ is stable if $2b_1x_{3,+}^2 - (r_1 - r_2 - b_1k_1)x_{3,+} + k_1r_2 > 0$.*

Remark 2.3. In [3, 7, 8], only existence of the positive equilibrium of (1.1) for Case 1(i) was considered, which is stable if either (a) $r_1 \le r_2$ and $k_1 \ge k_2$ [3] or (b) $a_1r_2k_2 < a_2r_1k_1$ and $r_1 < b_1k_1$ [7, 8]. Obviously, Proposition 2.2 greatly improves these results.

Propositions 2.1 and 2.2 naturally motivate us to seek sufficient conditions on global asymptotic stability of equilibrium to (1.1) and permanence of (1.1).

Nindjin et al. [5] showed that if

$$a_1r_1r_2 + a_1b_1r_2k_2 < a_2b_1r_1k_1, \tag{H1}$$

then

$$\limsup_{t\to\infty} x(t) \le K \triangleq \frac{r_1}{b_1}, \qquad \limsup_{t\to\infty} y(t) \le L \triangleq \frac{r_1r_2 + b_1r_2k_2}{a_2b_1}, \tag{2.7}$$

$$\liminf_{t\to\infty} x(t) \ge M \triangleq \frac{r_1k_1 - a_1L}{b_1k_1}, \qquad \liminf_{t\to\infty} y(t) \ge N \triangleq \frac{r_2(M + k_2)}{a_2} \tag{2.8}$$

for a positive solution $(x(t), y(t))$ of (1.1). Therefore, system (1.1) is permanent if (H1) holds. With the help of these bounds, it was shown that E_2 is globally asymptotically stable if $r_1(k_1 + K) < a_1N$ holds (see [5]).

In the coming section, we present two results on the global asymptotic stability of a positive equilibrium, which not only supplement Theorem 7 of Nindjin et al. [5] but also improve it by including more situations.

3. Global Asymptotic Stability of a Positive Equilibrium

The first result is established by employing the Fluctuation Lemma, and we refer to [12–16] for details.

Theorem 3.1. *In addition to* (H1) *, further suppose that*

$$2a_2b_1M + (a_2b_1k_1 - a_2r_1 - a_1r_2) > 0, \tag{H2}$$

where M *is defined in* (2.8)*. Then, system* (1.1) *has a unique positive equilibrium which is globally asymptotically stable.*

Proof. Obviously, (H1) implies $a_1r_2k_2 < a_2r_1k_1$, that is, condition (i) of Case 1 holds. Thus, (1.1) has a unique positive equilibrium. Let $(x(t), y(t))$ be any positive solution of (1.1). By the results at the end of Section 2, $\overline{x} \triangleq \limsup_{t\to\infty} x(t) \geq \underline{x} \triangleq \liminf_{t\to\infty} x(t) \geq M$, $\overline{y} \triangleq \limsup_{t\to\infty} y(t) \geq \underline{y} \triangleq \liminf_{t\to\infty} y(t) > 0$.

We claim $\overline{x} = \underline{x}$. Otherwise, $\overline{x} > \underline{x}$. According to the Fluctuation lemma, there exist sequences $\xi_n \to \infty$, $\eta_n \to \infty$, $\tau_n \to \infty$, and $\sigma_n \to \infty$ as $n \to \infty$ such that $\dot{x}(\xi_n) \to 0$, $\dot{x}(\eta_n) \to 0$, $x(\xi_n) \to \overline{x}$, $x(\eta_n) \to \underline{x}$, $\dot{y}(\tau_n) \to 0$, $\dot{y}(\sigma_n) \to 0$, $y(\tau_n) \to \overline{y}$, and $y(\sigma_n) \to \underline{y}$ as $n \to \infty$. First, from the second equation of (1.1),

$$\dot{y}(\tau_n) \leq \left(r_2 - \frac{a_2 y(\tau_n)}{\sup_{t\geq\tau_n} x(t) + k_2}\right) y(\tau_n), \qquad \dot{y}(\tau_n) \geq \left(r_2 - \frac{a_2 y(\tau_n)}{\inf_{t\geq\tau_n} x(t) + k_2}\right) y(\tau_n). \tag{3.1}$$

Letting $n \to \infty$, we obtain that $0 \leq (r_2 - a_2\overline{y}/(\overline{x} + k_2))\overline{y}$ and $0 \geq (r_2 - a_2\overline{y}/(\underline{x} + k_2))\overline{y}$. Hence,

$$\frac{r_2(\underline{x} + k_2)}{a_2} \leq \overline{y} \leq \frac{r_2(\overline{x} + k_2)}{a_2}. \tag{3.2}$$

Similar arguments as above also produce

$$\frac{r_2(\underline{x} + k_2)}{a_2} \leq \underline{y} \leq \frac{r_2(\overline{x} + k_2)}{a_2}. \tag{3.3}$$

Second, from the first equation of (1.1),

$$\dot{x}(\xi_n) = \left(r_1 - b_1 x(\xi_n) - \frac{a_1 y(\xi_n)}{x(\xi_n) + k_1}\right) x(\xi_n). \tag{3.4}$$

Equation (3.4) implies $\dot{x}(\xi_n) \leq (r_1 - b_1 x(\xi_n) - (a_1 \inf_{t\geq\xi_n} y(t)/(x(\xi_n) + k_1)))x(\xi_n)$.

Taking limit as $n \to \infty$, one obtains $0 \leq (r_1 - b_1\overline{x} - a_1\underline{y}/(\overline{x} + k_1))\overline{x}$. This, combined with (3.3), gives us $0 \leq (r_1 - b_1\overline{x} - a_1r_2(\underline{x} + k_2)/a_2(\overline{x} + k_1))\overline{x}$. It follows that

$$(a_2r_1 - a_2b_1k_1)\overline{x} - a_2b_1\overline{x}^2 + a_2r_1k_1 \geq a_1r_2(\underline{x} + k_2). \tag{3.5}$$

Similarly, one can show that

$$(a_2r_1 - a_2b_1k_1)\underline{x} - a_2b_1\underline{x}^2 + a_2r_1k_1 \leq a_1r_2(\overline{x} + k_2). \tag{3.6}$$

Multiplying (3.5) by −1 and adding it to (3.6), we have

$$a_2 b_1\left(\overline{x}^2 - \underline{x}^2\right) + (a_2 b_1 k_1 - a_2 r_1 - a_1 r_2)(\overline{x} - \underline{x}) \leq 0. \tag{3.7}$$

Due to $\overline{x} > \underline{x}$, one gets $a_2 b_1(\overline{x}+\underline{x})+(a_2 b_1 k_1 - a_2 r_1 - a_1 r_2) \leq 0$ which contradicts (H2) . Therefore, $\overline{x} = \underline{x}$, and the claim is proved.

The claim implies that $\lim_{t\to\infty} x(t)$ exists and we denote it by x^*. Then, it follows from (3.2) and (3.3) that $\lim_{t\to\infty} y(t)$ exists and $\lim_{t\to\infty} y(t) \triangleq y^* = r_2(x^* + k_2)/a_2 > 0$. Letting $n \to \infty$ in (3.4) gives us $r_1 - b_1 x^* - a_1 r_2(x^* + k_2)/a_2(x^* + k_1) = 0$. Then, one can see that (x^*, y^*) satisfies (2.4), that is, (x^*, y^*) is a positive equilibrium of (1.1). This completes the proof as the positive equilibrium is unique. □

Theorem 3.2. *Suppose that* (1.1) *has a unique positive equilibrium* $E^* = (x^*, y^*)$. *Further assume that*

$$\left(\frac{2k_2 + k_1}{2k_1 k_2}\right) a_1 L + \frac{a_1}{2} < b_1(x^* + k_1), \quad L < k_2, \tag{H3}$$

where L is defined in (2.7). *Then,* E^* *is globally asymptotically stable.*

Proof. Let $(x(t), y(t))$ be any positive solution of (1.1). From (H3) , we can choose an $\varepsilon > 0$ such that

$$\left(\frac{2k_2 + k_1}{2k_1 k_2}\right) a_1 (L + \varepsilon) + \frac{a_1}{2} < b_1(x^* + k_1), \quad L + \varepsilon < k_2. \tag{3.8}$$

Moreover, it follows from (2.7) that there exists $T > 0$ such that

$$0 < y(t) \leq L + \varepsilon \quad \text{for } t \geq T. \tag{3.9}$$

According to the proof of Theorem 6 in [3], let

$$V(x, y) = (x^* + k_1)\left(x - x^* - x^* \ln\left(\frac{x}{x^*}\right)\right) + \frac{a_1(x^* + k_2)}{a_2}\left(y - y^* - y^* \ln\left(\frac{y}{y^*}\right)\right). \tag{3.10}$$

Then, by the positivity of x, (3.8), and (3.9),

$$\begin{aligned}\frac{dV}{dt} &= \left(-b_1(x^* + k_1) + \frac{a_1 y}{x + k_1}\right)(x - x^*)^2 - a_1(y - y^*)^2 + \left(-a_1 + \frac{a_1 y}{x + k_2}\right)(x - x^*)(y - y^*) \\ &\le \left(-b_1(x^* + k_1) + \frac{a_1 y}{k_1}\right)(x - x^*)^2 - a_1(y - y^*)^2 + \left(a_1 + \frac{a_1 y}{x + k_2}\right)\frac{(x - x^*)^2 + (y - y^*)^2}{2} \\ &\le \left(-b_1(x^* + k_1) + \frac{a_1 y}{k_1}\right)(x - x^*)^2 - a_1(y - y^*)^2 + \left(a_1 + \frac{a_1 y}{k_2}\right)\frac{(x - x^*)^2 + (y - y^*)^2}{2} \\ &= \left(-b_1(x^* + k_1) + \frac{a_1 y}{k_1} + \frac{a_1 y}{2k_2} + \frac{a_1}{2}\right)(x - x^*)^2 + \left(\frac{a_1 y}{2k_2} - \frac{a_1}{2}\right)(y - y^*)^2 \\ &\le \left(-b_1(x^* + k_1) + \left(\frac{2k_2 + k_1}{2k_1 k_2}\right)a_1(L + \varepsilon) + \frac{a_1}{2}\right)(x - x^*)^2 + \frac{a_1}{2}\left(\frac{L + \varepsilon}{k_2} - 1\right)(y - y^*)^2 \\ &< 0.\end{aligned} \tag{3.11}$$

Therefore, $E^*(x^*, y^*)$ is globally asymptotically stable, and this completes the proof. □

References

[1] C. S. Holling, "The functional response of predator to prey density and its role in mimicry and population regulation, Men," *Memoirs of the Entomological Society of Canada*, vol. 45, pp. 1–60, 1965.

[2] P. H. Leslie and J. C. Gower, "The properties of a stochastic model for the predator-prey type of interaction between two species," *Biometrika*, vol. 47, pp. 219–234, 1960.

[3] M. A. Aziz-Alaoui and M. Daher Okiye, "Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes," *Applied Mathematics Letters*, vol. 16, no. 7, pp. 1069–1075, 2003.

[4] H. Guo and X. Song, "An impulsive predator-prey system with modified Leslie-Gower and Holling type II schemes," *Chaos, Solitons and Fractals*, vol. 36, no. 5, pp. 1320–1331, 2008.

[5] A. F. Nindjin, M. A. Aziz-Alaoui, and M. Cadivel, "Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay," *Nonlinear Analysis: Real World Applications*, vol. 7, no. 5, pp. 1104–1118, 2006.

[6] X. Song and Y. Li, "Dynamic behaviors of the periodic predator-prey model with modified Leslie-Gower Holling-type II schemes and impulsive effect," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 1, pp. 64–79, 2008.

[7] R. Yafia, F. El Adnani, and H. Talibi Alaoui, "Stability of limit cycle in a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay," *Applied Mathematical Sciences*, vol. 1, no. 1–4, pp. 119–131, 2007.

[8] R. Yafia, F. El Adnani, and H. T. Alaoui, "Limit cycle and numerical similations for small and large delays in a predator-prey model with modified Leslie-Gower and Holling-type II schemes," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 5, pp. 2055–2067, 2008.

[9] T. K. Kar and A. Ghorai, "Dynamic behaviour of a delayed predator-prey model with harvesting," *Applied Mathematics and Computation*, vol. 217, no. 22, pp. 9085–9104, 2011.

[10] W. Khellaf and N. Hamri, "Boundedness and global stability for a predator-prey system with the Beddington-DeAngelis functional response," *Differential Equations and Nonlinear Mechanics*, vol. 2010, Article ID 813289, 24 pages, 2010.

[11] H.-F. Huo, X. Wang, and C. Castillo-Chavez, "Dynamics of a stage-structured Leslie-Gower predator-prey model," *Mathematical Problems in Engineering*, vol. 2011, Article ID 149341, 22 pages, 2011.

[12] W. M. Hirsch, H. Hanisch, and J.-P. Gabriel, "Differential equation models of some parasitic infections: methods for the study of asymptotic behavior," *Communications on Pure and Applied Mathematics*, vol. 38, no. 6, pp. 733–753, 1985.

[13] A. Tineo, "Asymptotic behaviour of positive solutions of the nonautonomous Lotka-Volterra competition equations," *Differential and Integral Equations*, vol. 6, no. 2, pp. 449–457, 1993.

[14] F. Chen, Y. Wu, and Z. Ma, "Stability property for the predator-free equilibrium point of predator-prey systems with a class of functional response and prey refuges," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 148942, 5 pages, 2012.

[15] Z. Li and F. Chen, "Extinction in two dimensional nonautonomous Lotka-Volterra systems with the effect of toxic substances," *Applied Mathematics and Computation*, vol. 182, no. 1, pp. 684–690, 2006.

[16] F. Montes De Oca and M. Vivas, "Extinction in two dimensional Lotka-Volterra system with infinite delay," *Nonlinear Analysis: Real World Applications*, vol. 7, no. 5, pp. 1042–1047, 2006.