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Research Article

Unique Existence Theorem of Solution of Almost Periodic Differential Equations on Time Scales

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By using the theory of calculus on time scales and M-matrix theory, the unique existence theorem of solution of almost periodic differential equations on almost periodic time scales is established. The result can be used to a large of dynamic systems.

1. Introduction

It is well known that in Celestial mechanics, almost periodic solutions and stable solutions to differential equations or difference equations are intimately related. In the same way, stable electronic circuits, ecological systems, neural networks, and so on exhibit almost periodic behavior. A vast amount of researches have been directed toward studying these phenomena, we refer the readers to [1–5] and the references therein.

Also, the theory of calculus on time scales (see [6] and references cited therein) was initiated by Stefan Hilger in his Ph.D. thesis in 1988 [7] in order to unify continuous and discrete analysis, and it has a tremendous potential for applications and has received much attention since his foundational work (see, e.g., [8–12]). Therefore, it is practicable to study that on time scales which can unify the continuous and discrete situations.

Recently, the conceptions of almost periodic time scales and almost periodic functions on almost periodic time scales have been established, one can see [8]. Consider the following almost periodic system:

$$x^{\Delta}(t) = A(t)x(t) + f(t), \qquad t \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is an almost periodic time scale, $A(t)$ is an almost periodic matrix function, $f(t)$ is an almost periodic vector function. The authors in [8] only proved the existence of almost periodic solution for system (1.1) (see Lemma 2.13 in [8]), but the uniqueness has not been considered. However, the unique existence theorem of solution usually plays an important role in applications, so, the theories need to be explored.

The main purpose of this paper is by using the theory of calculus on time scales and M-matrix theory to establish the unique existence theorem of solution of system (1.1).

2. Preliminaries

The basic theories of calculus on time scales, one can see [6]. In order to obtain the unique existence theorem of solution of system (1.1), we first make the following preparations.

Lemma 2.1 (see [6]). *Let $g : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be a function with*

$$g(t, x_1) \leq g(t, x_2), \quad \forall t \in \mathbb{T}, \; x_1 \leq x_2. \tag{2.1}$$

Let $\upsilon, \omega : \mathbb{T} \to \mathbb{R}$ be differentiable with

$$\upsilon^{\Delta}(t) \leq g(t, \upsilon(t)), \qquad \omega^{\Delta}(t) \geq g(t, \omega(t)), \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}. \tag{2.2}$$

Then,

$$\upsilon(t_0) < \omega(t_0), \quad t_0 \in \mathbb{T}, \tag{2.3}$$

implies

$$\upsilon(t) < \omega(t), \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}. \tag{2.4}$$

Theorem 2.2. *If the following conditions satisfy:*

(1)

$$D^{+} x_i^{\Delta}(t) \leq \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} \overline{x}_j(t), \quad t \in [t_0, +\infty)_{\mathbb{T}}, \; i, j = 1, 2, \ldots, n, \tag{2.5}$$

where $a_{ij} \geq 0 \; (i \neq j)$, $b_{ij} \geq 0$, $\sum_{i=1}^{n} \overline{x}_i(t_0) > 0$, $\overline{x}_i(t) = \sup_{s \in [t-\tau_0, t]_{\mathbb{T}}} x_i(s)$, and $\tau_0 > 0$ is a constant;

(2) $\widetilde{M} := -(a_{ij} + b_{ij})_{n \times n}$ *is an M-matrix;*

then there exists constants $\gamma_i > 0$ and $a > 0$, such that the solutions of inequality (1) satisfy

$$x_i(t) \leq \gamma_i \left(\sum_{j=1}^{n} \overline{x}_j(t_0) \right) e_{\ominus a}(t, t_0), \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}, \; i = 1, 2, \ldots, n. \tag{2.6}$$

Proof. Assume that

$$G(t, x(t), \overline{x}(t)) = \left(g_1(t, x(t), \overline{x}(t)), g_2(t, x(t), \overline{x}(t)), \dots, g_n(t, x(t), \overline{x}(t))\right), \tag{2.7}$$

where

$$g_i(t, x(t), \overline{x}(t)) = \left(\sum_{j=1}^{n} a_{ij} x_i(t) + \sum_{j=1}^{n} b_{ij} \overline{x}_i(t)\right), \quad i = 1, 2, \dots, n. \tag{2.8}$$

By condition (1), then

$$D^+ x_i^{\Delta}(t) \le g_i(t, x(t), \overline{x}(t)), \quad \forall t \in [t_0, +\infty)_{\mathbb{T}},\ i = 1, 2, \dots, n. \tag{2.9}$$

By condition (2), there exist constants $\xi > 0$ and $d_i > 0$ $(i = 1, 2, \dots, n)$ such that

$$\sum_{j=1}^{n} (a_{ij} + b_{ij}) d_i < -\xi, \quad i = 1, 2, \dots, n. \tag{2.10}$$

Choose $0 < a \ll 1$, such that

$$a d_i + \sum_{j=1}^{n} \left(a_{ij} d_i + b_{ij} d_i e_a(t, t - \tau_0)\right) < 0, \quad \forall t \in [t_0, +\infty)_{\mathbb{T}},\ i = 1, 2, \dots, n. \tag{2.11}$$

If $t \in [t_0 - \tau_0, t_0]_{\mathbb{T}}$, choose $F \gg 1$, such that

$$F d_i e_{\ominus a}(t, t_0) > 1, \quad i = 1, 2, \dots, n. \tag{2.12}$$

For any $\varepsilon > 0$, let

$$q_i(t) = F d_i \left(\sum_{j=1}^{n} \overline{x}_j(t_0) + \varepsilon\right) e_{\ominus a}(t, t_0), \quad i = 1, 2, \dots, n. \tag{2.13}$$

From (2.11), for any $t \in [t_0, +\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
D^+ q_i^{\Delta}(t) &= (\ominus a) F d_i \left(\sum_{j=1}^{n} \overline{x}_j(t_0) + \varepsilon\right) e_{\ominus a}(t, t_0) \\
&\ge -a F d_i \left(\sum_{j=1}^{n} \overline{x}_j(t_0) + \varepsilon\right) e_{\ominus a}(t, t_0) \\
&> \sum_{j=1}^{n} \left(a_{ij} d_i + b_{ij} d_i e_a(t, t - \tau_0)\right) F \left(\sum_{j=1}^{n} \overline{x}_j(t_0) + \varepsilon\right) e_{\ominus a}(t, t_0)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{n} a_{ij} d_i F\left(\sum_{j=1}^{n} \overline{x}_j(t_0) + \varepsilon\right) e_{\ominus a}(t, t_0) \\
&\quad + \sum_{j=1}^{n} b_{ij} d_i F\left(\sum_{j=1}^{n} \overline{x}_j(t_0) + \varepsilon\right) e_{\ominus a}(t - \tau_0, t_0) \\
&\geq \sum_{j=1}^{n} a_{ij} q_i(t) + \sum_{j=1}^{n} b_{ij} \overline{q}_i(t) \\
&= g_i(t, q(t), \overline{q}(t)), \quad i = 1, 2, \ldots, n,
\end{aligned} \tag{2.14}$$

that is,

$$D^+ q_i^{\Delta}(t) > g_i(t, q(t), \overline{q}(t)), \quad \forall t \in [t_0, +\infty)_{\mathbb{T}},\ i = 1, 2, \ldots, n. \tag{2.15}$$

For $t \in [t_0 - \tau_0, t_0]_{\mathbb{T}}$, by (2.12), we can get

$$q_i(t) = F d_i \left(\sum_{j=1}^{n} \overline{x}_j(t_0) + \varepsilon\right) e_{\ominus a}(t, t_0) > \sum_{j=1}^{n} \overline{x}_j(t_0) + \varepsilon, \quad i = 1, 2, \ldots, n. \tag{2.16}$$

Let $x_i(t) \leq \sum_{j=1}^{n} \overline{x}_j(t_0) + \varepsilon$, $t \in [t_0 - \tau_0, t_0]_{\mathbb{T}}$, then

$$q_i(t_0) > x_i(t_0), \quad i = 1, 2, \ldots, n. \tag{2.17}$$

Together with (2.9), (2.15), and (2.17), by Lemma 2.1, we can get

$$x_i(t) < q_i(t) = F d_i \left(\sum_{j=1}^{n} \overline{x}_j(t_0) + \varepsilon\right) e_{\ominus a}(t, t_0), \quad \forall t \in (t_0, +\infty)_{\mathbb{T}},\ i = 1, 2, \ldots, n. \tag{2.18}$$

Let $\varepsilon \to 0^+$, $F d_i = \gamma_i$, then

$$x_i(t) \leq \gamma_i \left(\sum_{j=1}^{n} \overline{x}_j(t_0)\right) e_{\ominus a}(t, t_0), \quad \forall t \in (t_0, +\infty)_{\mathbb{T}},\ i = 1, 2, \ldots, n. \tag{2.19}$$

The proof is completed. □

Definition 2.3. The almost periodic solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ of (1.1) is said to be exponentially stable, if there exists a positive α such that for any $\delta \in [t_0 - \tau_0, t_0]_{\mathbb{T}}$, $\tau_0 > 0$, there exists $N = N(\delta) \geq 1$ such that for any solution $x = (x_1, x_2, \ldots, x_n)^T$ satisfying

$$\|x - x^*\| \leq N \|\phi - x^*\| e_{\ominus \alpha}(t, \delta), \quad t \in [t_0, +\infty)_{\mathbb{T}}, \tag{2.20}$$

where $\phi(s)$, $s \in [t_0 - \tau_0, t_0]_{\mathbb{T}}$, is the initial condition.

3. Unique Existence Theorem

In this section, we will establish the unique existence theorem of solution of system (1.1) based on the theory of calculus on time scales and M-matrix theory. The conceptions of almost periodic time scales and almost periodic functions on almost periodic time scales, one can see [8].

Definition 3.1 (see [8]). Let $x \in \mathbb{R}^n$, and let $A(t)$ be an $n \times n$ rd-continuous matrix on $\mathbb{T}$, the linear system

$$x^{\Delta}(t) = A(t)x(t), \quad t \in \mathbb{T}, \tag{3.1}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constants k, α, projection P, and the fundamental solution matrix $X(t)$ of (3.1), satisfying

$$\begin{aligned} \left|X(t)PX^{-1}(\sigma(s))\right|_0 &\le ke_{\ominus\alpha}(t,\sigma(s)), \quad s,t \in \mathbb{T},\ t \ge \sigma(s), \\ \left|X(t)(I-P)X^{-1}(\sigma(s))\right|_0 &\le ke_{\ominus\alpha}(\sigma(s),t), \quad s,t \in \mathbb{T},\ t \le \sigma(s), \end{aligned} \tag{3.2}$$

where $|\cdot|_0$ is a matrix norm on $\mathbb{T}$.

Lemma 3.2 (see [8]). *If the linear system* (3.1) *admits exponential dichotomy, then system* (1.1) *has a bounded solution $x(t)$ as follows:*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s, \tag{3.3}$$

where $X(t)$ is the fundamental solution matrix of (3.1).

Let $A(t) = (a_{ij}(t))_{n\times n}$, $\overline{A} = (\sup(a_{ij}(t)))_{n\times n}$, $1 \le i, j \le n$, $t \in \mathbb{T}$.

Lemma 3.3. *Assume that the conditions of Lemma 3.2 hold, if $-\overline{A}$ is an M-matrix, then the almost periodic solution of system* (1.1) *is globally exponentially stable and unique.*

Proof. According to Lemma 3.2, we know that (1.1) has an almost periodic solution $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$. Suppose that $x = (x_1, x_2, \dots, x_n)^T$ be an arbitrary solution of (1.1). Then, system (1.1) can be written as

$$(x(t) - x^*(t))^{\Delta} = A(t)x(t) - A(t)x^*(t). \tag{3.4}$$

Assume that the initial condition of (1.1) is $\phi(s) = (\phi_1(s), \dots, \phi_n(s))^T$, $s \in [t_0 - \tau_0, t_0]_{\mathbb{T}}$, $\tau_0 > 0$, then the initial condition of (3.4) is $\widehat{\phi}(s) = \phi(s) - x^*(s)$, $s \in [t_0 - \tau_0, t_0]_{\mathbb{T}}$.

Let $V(t) = |x(t) - x^*(t)|$, the upper right derivative $D^+V^{\Delta}(t)$ along the solutions of system (3.4) is as follows:

$$D^+V^{\Delta}(t) = \operatorname{sgn}(x(t) - x^*(t))(x(t) - x^*(t))^{\Delta} \le \overline{A}V(t) + \mathbf{0}\overline{V}(t), \tag{3.5}$$

where $\mathbf{0}$ is an $n \times n$-matrix with all its elements are zeros.

Since $-(\overline{A}+\mathbf{0}) = -\overline{A}$ is an M-matrix, according to Theorem 2.2, then there exist constants $\alpha > 0$, $\gamma_0 > 0$, for any $\delta \in [t_0-\tau_0, t_0]_{\mathbb{T}}$,

$$\begin{aligned}\left|x_i(t)-x_i^*(t)\right| &\le \gamma_0\left[\sup_{\delta\in[t_0-\tau_0,t_0]_{\mathbb{T}}}\left|\phi_i(\delta)-x_i^*(\delta)\right|\right]e_{\ominus\alpha}(t,t_0)\\ &\le \frac{\gamma_0}{e_{\ominus\alpha}(t_0,\delta)}\left[\sup_{\delta\in[t_0-\tau_0,t_0]_{\mathbb{T}}}\left|\phi_i(\delta)-x_i^*(\delta)\right|\right]e_{\ominus\alpha}(t,\delta),\end{aligned} \tag{3.6}$$

where $t \in [t_0,+\infty)_{\mathbb{T}}$, $i = 1,2,\ldots,n$.

Then, there exists a positive number $\eta > e_{\ominus\alpha}(t_0,\delta)/\gamma_0$, such that

$$\|x-x^*\| \le N\left\|\phi-x^*\right\|e_{\ominus\alpha}(t,\delta), \quad t\in[t_0,+\infty)_{\mathbb{T}}, \tag{3.7}$$

where $N = N(\delta) = \eta\gamma_0/e_{\ominus\alpha}(t_0,\delta) > 1$, $\|x\| = \max_{1\le i\le n}\sup_{t\in[t_0,+\infty)_{\mathbb{T}}}|x_i(t)|$.

From Definition 2.3, the almost periodic solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ is globally exponentially stable. Thus, the almost periodic solution of system (1.1) is globally exponentially stable.

In (3.7), let $t \to +\infty$, then $e_{\ominus\alpha}(t,\delta) \to 0$, so, we can get $x = x^*$. Hence, the almost periodic system (1.1) has a unique almost periodic solution. The proof is completed. □

Together with Lemmas 3.2 and 3.3, we can get the following theorem.

Theorem 3.4. *If the linear system* (3.1) *admits an exponential dichotomy, $-\overline{A}$ is an M-matrix, then system* (1.1) *has a unique almost periodic solution $x(t)$, and*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s, \tag{3.8}$$

where $X(t)$ is the fundamental solution matrix of (3.1).

Lemma 3.5 (see [8]). *Let $c_i(t)$ be an almost periodic function on $\mathbb{T}$, where $c_i(t) > 0$, $-c_i(t) \in \mathcal{R}^+$, for all $t \in \mathbb{T}$, and*

$$\min_{1\le i\le n}\left\{\inf_{t\in\mathbb{T}} c_i(t)\right\} = \widetilde{m} > 0, \tag{3.9}$$

then the linear system

$$x^{\Delta}(t) = \operatorname{diag}(-c_1(t), -c_2(t), \ldots, -c_n(t))x(t) \tag{3.10}$$

admits an exponential dichotomy on $\mathbb{T}$.

Corollary 3.6. *In system* (1.1)*, if* $A(t) = \mathrm{diag}(-a_{11}(t), -a_{22}(t), \ldots, -a_{nn}(t))$, $t \in \mathbb{T}$*, and* $\min_{1\le i\le n}\{\inf_{t\in\mathbb{T}} a_{ii}(t)\} = \widehat{a} > 0$*, then system* (1.1) *has a unique almost periodic solution* $x(t)$*, and*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s, \tag{3.11}$$

where $X(t)$ *is the fundamental solution matrix of* (3.1).

Proof. Obviously, $-\overline{A}$ is an M-matrix, since $A(t) = \mathrm{diag}(-a_{11}(t), -a_{22}(t), \ldots, -a_{nn}(t))$, $t \in \mathbb{T}$, and $\min_{1\le i\le n}\{\inf_{t\in\mathbb{T}} a_{ii}(t)\} = \widehat{a} > 0$. By Lemma 3.5, the linear system (3.1) admits an exponential dichotomy. According to Theorem 3.4, it is easy to see that the almost periodic system (1.1) has exactly one almost periodic solution. The proof is completed. □

Remark 3.7. As an application, consider system (1.1) in paper [8], by using fixed-point theorem, the authors in [8] proved (1.1) has a unique almost periodic solution, one can see Theorem 3.2 in [8] for more detail. However, from the proof of Theorem 3.2 in paper [8], one can see that (3.5) is a solution of system (3.4), but the uniqueness cannot be determined, so, the proof of Theorem 3.2 in paper [8] is questionable. Our results obtained in this paper can solve the problem. By Corollary 3.6, one can get system (3.4) has exactly one solution as (3.5) in [8], then by the same method in [8], under the conditions of Theorem 3.2, system (1.1) has a unique almost periodic solution. Also, the results can be used to other neural networks.

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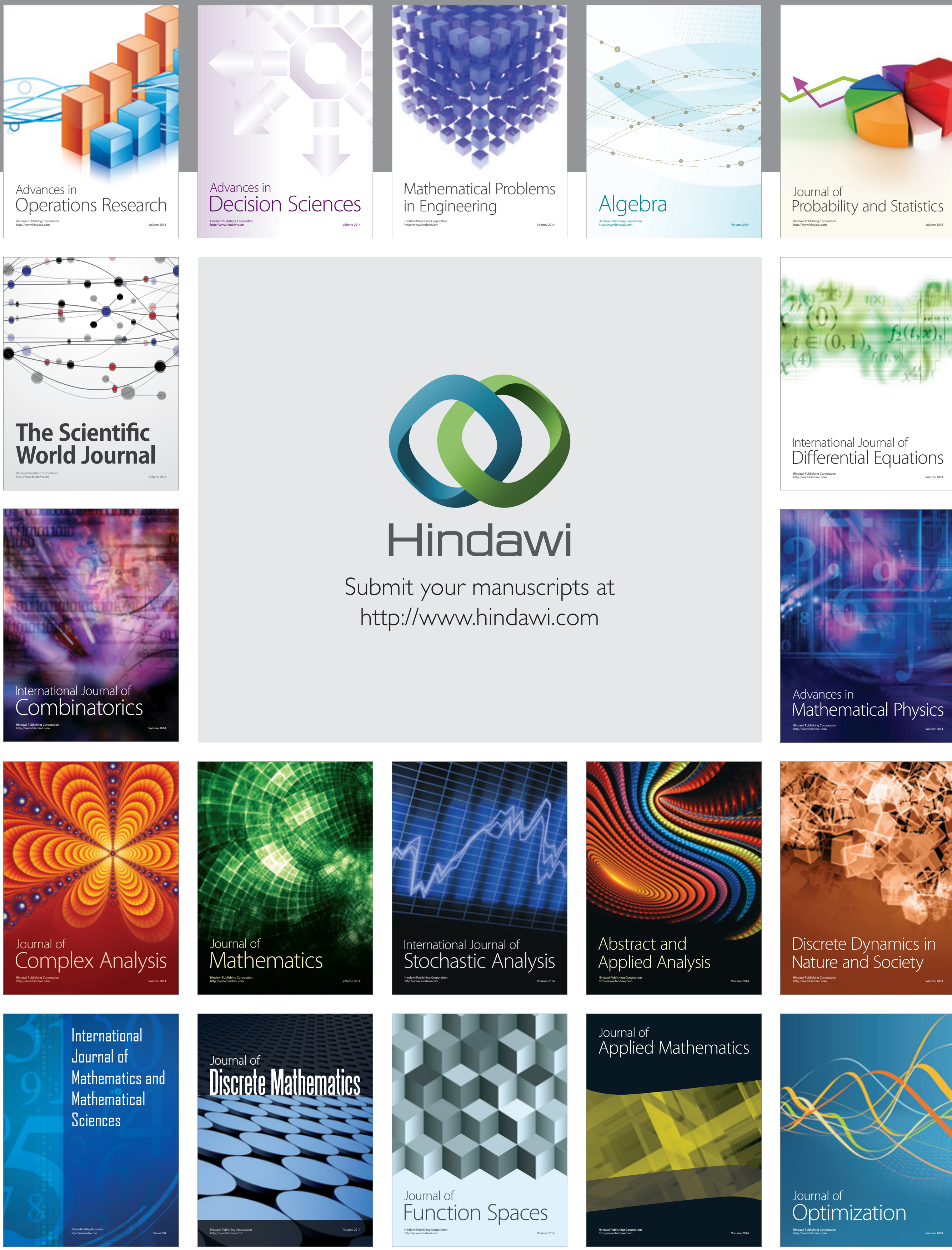

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