Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 293734, 19 pages
doi:10.1155/2012/293734

Research Article

Positive Solutions for p-Laplacian Fourth-Order Differential System with Integral Boundary Conditions

Jiqiang Jiang,[1] Lishan Liu,[1,2] and Yonghong Wu[2]

[1] *School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China*
[2] *Department of Mathematics and Statistics, Curtin University of Technology, Perth, WA 6845, Australia*

Correspondence should be addressed to Jiqiang Jiang, qfjjq@163.com
and Lishan Liu, lls@mail.qfnu.edu.cn

Received 25 March 2012; Accepted 15 May 2012

Academic Editor: Cengiz Çinar

This paper investigates the existence of positive solutions for a class of singular p-Laplacian fourth-order differential equations with integral boundary conditions. By using the fixed point theory in cones, explicit range for λ and μ is derived such that for any λ and μ lie in their respective interval, the existence of at least one positive solution to the boundary value system is guaranteed.

1. Introduction

Boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics and so on. Fourth-order differential equations boundary value problems, including those with the p-Laplacian operator, have their origin in beam theory [1, 2], ice formation [3, 4], fluids on lungs [5], brain warping [6, 7], designing special curves on surfaces [8], and so forth. In beam theory, more specifically, a beam with a small deformation, a beam of a material that satisfies a nonlinear power-like stress and strain law, and a beam with two-sided links that satisfies a nonlinear powerlike elasticity law can be described by fourth order differential equations along with their boundary value conditions. For more background and applications, we refer the reader to the work by Timoshenko [9] on elasticity, the monograph by Soedel [10] on deformation of structure, and the work by Dulcska [11] on the effects of soil settlement. Due to their wide applications, the existence and multiplicity of positive solutions for fourth-order (including p-Laplacian operator) boundary value problems has also attracted increasing attention over the last decades; see [12–33] and references therein. In [28], Zhang and Liu studied the following singular fourth-order four-point boundary value problem

$$\begin{gathered}
\left(\phi_p\left(u''(t)\right)\right)'' = f(t,u(t)), \quad 0 < t < 1, \\
u(0) = u(1) - au(\xi) = u''(0) = u''(1) - bu''(\eta) = 0,
\end{gathered} \tag{1.1}$$

where $\phi_p(x) = |x|^{p-2}x$, $p > 1$, $0 < \xi, \eta < 1$, $0 \leq a,\ b < 1$, $f \in C((0,1) \times (0,\infty), (0,\infty))$, $f(t,x)$ may be singular at $t = 0$ and/or $t = 1$ and $x = 0$. The authors gave sufficient conditions for the existence of one positive solution by using the upper and lower solution method, fixed point theorems, and the properties of the Green function.

In [32], Zhang et al. discussed the existence and nonexistence of symmetric positive solutions of the following fourth-order boundary value problem with integral boundary conditions:

$$\begin{gathered}
\left(\phi_p\left(u''(t)\right)\right)'' = w(t)f(t,u(t)), \quad 0 < t < 1, \\
u(0) = u(1) = \int_0^1 g(s)u(s)ds, \\
\phi_p\left(u''(0)\right) = \phi_p\left(u''(1)\right) = \int_0^1 h(s)\phi_p\left(u''(s)\right)ds,
\end{gathered} \tag{1.2}$$

where $\phi_p(x) = |x|^{p-2}x$, $p > 1$, $w \in L^1[0,1]$ is nonnegative, symmetric on the interval $[0,1]$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, $f(1-t,x) = f(t,x)$ for all $(t,x) \in [0,1] \times [0,+\infty)$, and $g, h \in L^1[0,1]$ are nonnegative, symmetric on $[0,1]$.

Motivated by the work of the above papers, in this paper, we study the existence of positive solutions of the following singular fourth-order boundary value system with integral boundary conditions:

$$\begin{gathered}
\left(\phi_{p_1}\left(u''(t)\right)\right)'' = \lambda^{p_1-1}a_1(t)f_1(t,u(t),v(t)), \quad 0 < t < 1, \\
\left(\phi_{p_2}\left(v''(t)\right)\right)'' = \mu^{p_2-1}a_2(t)f_2(t,u(t),v(t)), \\
u(0) = u(1) = \int_0^1 u(s)d\xi_1(s), \\
\phi_{p_1}\left(u''(0)\right) = \phi_{p_1}\left(u''(1)\right) = \int_0^1 \phi_{p_1}\left(u''(s)\right)d\eta_1(s), \\
v(0) = v(1) = \int_0^1 v(s)d\xi_2(s), \\
\phi_{p_2}\left(v''(0)\right) = \phi_{p_2}\left(v''(1)\right) = \int_0^1 \phi_{p_2}\left(v''(s)\right)d\eta_2(s),
\end{gathered} \tag{1.3}$$

where λ and μ are positive parameters, $\phi_{p_i}(x) = |x|^{p_i-2}x$, $p_i > 1$, $\phi_{q_i} = \phi_{p_i}^{-1}$, $1/p_i + 1/q_i = 1$, $\xi_i, \eta_i : [0,1] \to \mathbb{R}^+$ $(i = 1,2)$ are nondecreasing functions of bounded variation, and the integrals in (1.3) are Riemann-Stieltjes integrals, $f_1 : [0,1] \times \mathbb{R}_0^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and $f_2 : [0,1] \times \mathbb{R}^+ \times \mathbb{R}_0^+ \to \mathbb{R}^+$ are two continuous functions, and $f_1(t,x,y)$ may be singular at $x = 0$ while $f_2(t,x,y)$ may be singular at $y = 0$; $a_1, a_2 : (0,1) \to \mathbb{R}^+$ are continuous and may be singular at $t = 0$ and/or $t = 1$, in which $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{R}_0^+ = (0,+\infty)$.

Compared to previous results, our work presented in this paper has the following new features. Firstly, our study is on singular nonlinear differential systems, that is, a_1 and a_2 in (1.3) are allowed to be singular at $t = 0$ and/or $t = 1$, meanwhile $f_1(t,x,y)$ is allowed to be singular at $x = 0$ while $f_2(t,x,y)$ is allowed to be singular at $y = 0$, which bring about many difficulties. Secondly, the main tools used in this paper is a fixed-point theorem in cones, and the results obtained are the conditions for the existence of solutions to the more general system (1.3). Thirdly, the techniques used in this paper are approximation methods, and a special cone has been developed to overcome the difficulties due to the singularity and to apply the fixed-point theorem. Finally, we discuss the boundary value problem with integral boundary conditions, that is, system (1.3) including fourth-order three-point, multipoint and nonlocal boundary value problems as special cases. To our knowledge, very few authors studied the existence of positive solutions for p-Laplacian fourth-order differential equation with boundary conditions involving Riemann-Stieltjes integrals. Hence we improve and generalize the results of previous papers to some degree, and so it is interesting and important to study the existence of positive solutions for system (1.3).

The rest of this paper is organized as follows. In Section 2, we present some lemmas that are used to prove our main results. In Section 3, the existence of positive solution for system (1.3) is established by using the fixed point theory in cone. Finally, in Section 4, one example is also included to illustrate the main results.

Definition 1.1. A vector $(u,v) \in (C^2[0,1] \cap C^4(0,1)) \times (C^2[0,1] \cap C^4(0,1))$ is said to be a positive solution of system (1.3) if and only if (u,v) satisfies (1.3) and $u(t) > 0$, $v(t) \geq 0$ or $u(t) \geq 0$, $v(t) > 0$ for any $t \in (0,1)$.

Let K be a cone in a Banach space E. For $0 < r < R < +\infty$, let $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$, and $\overline{K}_{r,R} = \{x \in K : r \leq \|x\| \leq R\}$. The proof of the main theorem of this paper is based on the fixed point theory in cone. We list one lemma [34, 35] which is needed in our following argument.

Lemma 1.2. *Let K be a positive cone in real Banach space E and $T : \overline{K}_{r,R} \to K$ a completely continuous operator. If the following conditions hold*

(i) $\|Tx\| \leq \|x\|$ *for* $x \in \partial K_R$;

(ii) *there exists* $e \in \partial K_1$ *such that* $x \neq Tx + me$ *for any* $x \in \partial K_r$ *and* $m > 0$. *Then T has a fixed point in* $\overline{K}_{r,R}$.

Remark 1.3. If (i) and (ii) are satisfied for $x \in \partial K_r$ and $x \in \partial K_R$, respectively. Then Lemma 1.2 is still true.

2. Preliminaries and Lemmas

The basic space used in this paper is $E = C[0,1] \times C[0,1]$. Obviously, the space E is a Banach space if it is endowed with the norm as follows:

$$\|(u,v)\| := \|u\| + \|v\|, \quad \|u\| = \max_{0 \leq t \leq 1} |u(t)|, \quad \|v\| = \max_{0 \leq t \leq 1} |v(t)| \tag{2.1}$$

for any $(u,v) \in E$. Denote $C^+[0,1] = \{u \in C[0,1] : u(t) \geq 0,\ 0 \leq t \leq 1\}$. For convenience, we list the following assumptions:

(H_1) $a_1, a_2 : (0,1) \to \mathbb{R}^+$ are continuous and

$$0 < L_i := \left(\int_0^1 e(s)a_i(s)ds\right)^{q_i-1} < +\infty, \quad i = 1,2, \tag{2.2}$$

where $e(s) = s(1-s)$, $s \in [0,1]$.

(H_2) $\xi_i, \eta_i : [0,1] \to \mathbb{R}^+$ $(i = 1,2)$ are nondecreasing functions of bounded variation, and $\alpha_i \in [0,1)$, $\beta_i \in [0,1)$, where

$$\alpha_i = \int_0^1 d\xi_i(s), \quad \beta_i = \int_0^1 d\eta_i(s), \quad i = 1,2. \tag{2.3}$$

(H_3) $f_1 : [0,1] \times \mathbb{R}_0^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $f_2 : [0,1] \times \mathbb{R}^+ \times \mathbb{R}_0^+ \to \mathbb{R}^+$ are continuous and satisfy

$$\begin{aligned} f_1(t,x,y) &\leq g_1(t,x) + h_1(t,y), \quad \forall (t,x,y) \in [0,1] \times \mathbb{R}_0^+ \times \mathbb{R}^+, \\ f_2(t,x,y) &\leq g_2(t,x) + h_2(t,y), \quad \forall (t,x,y) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}_0^+, \end{aligned} \tag{2.4}$$

where $g_1, h_2 : [0,1] \times R_0^+ \to R^+$ are continuous and nonincreasing in the second variable, and $g_2, h_1 : [0,1] \times R^+ \to R^+$ are continuous and for any constant $r > 0$,

$$0 < \int_0^1 e(s)a_1(s)g_1(s,r)ds < +\infty, \quad 0 < \int_0^1 e(s)a_2(s)h_2(s,r)ds < +\infty. \tag{2.5}$$

Similar to the proof of Lemmas 2.1 and 2.2 in [32], the following two lemmas are valid.

Lemma 2.1. *If (H_2) holds, then for any $y \in L(0,1)$, the boundary value problem*

$$\begin{aligned} -x''(t) &= \phi_{q_i}(y(t)), \quad 0 < t < 1, \\ x(0) = x(1) &= \int_0^1 x(s)d\xi_i(s) \end{aligned} \tag{2.6}$$

has a unique solution

$$x(t) = \int_0^1 H_i(t,s)\phi_{q_i}(y(s))\,ds, \tag{2.7}$$

where

$$\begin{aligned} H_i(t,s) &= G(t,s) + \frac{1}{1-\alpha_i}\int_0^1 G(\tau,s)d\xi_i(\tau), \quad i = 1,2, \\ G(t,s) &= \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases} \end{aligned} \tag{2.8}$$

Lemma 2.2. *If (H_2) holds, then for any $z \in L(0,1)$, the boundary value problem*

$$\begin{gathered}-y''(t) = z(t), \quad 0 < t < 1, \\ y(0) = y(1) = \int_0^1 y(s)d\eta_i(s)\end{gathered} \tag{2.9}$$

has a unique solution

$$y(t) = \int_0^1 K_i(t,s)z(s)ds, \tag{2.10}$$

where

$$K_i(t,s) = G(t,s) + \frac{1}{1-\beta_i}\int_0^1 G(\tau,s)d\eta_i(\tau), \quad i = 1,2. \tag{2.11}$$

Remark 2.3. For t, $s \in [0,1]$, we have

$$e(t)e(s) \le G(t,s) \le e(s) \quad \text{or} \quad e(t) \le \max_{t\in[0,1]} e(t) = \frac{1}{4}. \tag{2.12}$$

Remark 2.4. If (H_2) holds, it is easy to testify $H_i(t,s)$ defined by (2.8) that:

$$\rho_i e(s) \le H_i(t,s) \le \gamma_i e(s) \le \frac{1}{4}\gamma_i < \gamma_i, \quad t,s \in [0,1], \ i = 1,2, \tag{2.13}$$

where

$$\gamma_i = \frac{1}{1-\alpha_i}, \quad \rho_i = \frac{\int_0^1 e(\tau)d\xi_i(\tau)}{1-\alpha_i}, \quad i = 1,2. \tag{2.14}$$

Remark 2.5. From (2.11), we can prove that the properties of $K_i(t,s)$ $(i = 1,2)$ are similar to those of $H_i(t,s)$ $(i = 1,2)$.

Lemma 2.6. *For $x > 0$, $y > 0$, we have*

$$\begin{aligned}\phi_{q_i}(x+y) &\le \begin{cases} 2^{q_i-1}\left[\phi_{q_i}(x) + \phi_{q_i}(y)\right], & q_i \ge 2 \\ \phi_{q_i}(x) + \phi_{q_i}(y), & 1 < q_i < 2 \end{cases}, \\ &\le 2^{q_i-1}\left[\phi_{q_i}(x) + \phi_{q_i}(y)\right], \quad q_i > 1, \ i = 1,2,\end{aligned} \tag{2.15}$$

$$\phi_{q_i}(x) > \phi_{q_i}(y) > \phi_{q_i}(0) = 0, \quad x > y > 0, \ q_i > 1, \ i = 1,2. \tag{2.16}$$

Proof. The proof of this lemma is easy, and we omit it. □

Let

$$K = \Big\{ (u,v) \in C^{+}[0,1] \times C^{+}[0,1] : u, v \text{ are concave on } [0,1], \\ \min_{t\in[0,1]} u(t) \geq \Lambda \|u\|, \min_{t\in[0,1]} v(t) \geq \Lambda \|v\| \Big\}, \tag{2.17}$$

where

$$\Lambda = \min\left\{ \frac{\rho_1 \sigma_1^{q_1-1}}{\gamma_1 \nu_1^{q_1-1}}, \frac{\rho_2 \sigma_2^{q_2-1}}{\gamma_2 \nu_2^{q_2-1}} \right\}, \quad \sigma_i = \frac{\int_0^1 e(s)d\eta_i(s)}{1-\beta_i}, \ \nu_i = \frac{1}{1-\beta_i}, \ i = 1,2. \tag{2.18}$$

It is easy to see that K is a cone of E. For any $0 < r < R$, let $K_{r,R} = \{(u,v) \in K : r < \|u\| < R, \ r < \|v\| < R\}$.

Remark 2.7. By the definition of ρ_i, σ_i, γ_i, ν_i $(i = 1,2)$, we have $0 < \Lambda < 1$.

To overcome singularity, we consider the following approximate problem of (1.3):

$$\begin{gathered} \left(\phi_{p_1}(u''(t))\right)'' = \lambda^{p_1-1} a_1(t) f_{1n}(t,u(t),v(t)), \quad 0 < t < 1, \\ \left(\phi_{p_2}(v''(t))\right)'' = \mu^{p_2-1} a_2(t) f_{2n}(t,u(t),v(t)), \\ u(0) = u(1) = \int_0^1 u(s)d\xi_1(s), \\ \phi_{p_1}(u''(0)) = \phi_{p_1}(u''(1)) = \int_0^1 \phi_{p_1}(u''(s))d\eta_1(s), \\ v(0) = v(1) = \int_0^1 v(s)d\xi_2(s), \\ \phi_{p_2}(v''(0)) = \phi_{p_2}(v''(1)) = \int_0^1 \phi_{p_2}(v''(s))d\eta_2(s), \end{gathered} \tag{2.19}$$

where n is a positive integer and

$$f_{1n}(t,u,v) = f_1\left(t, \max\left\{u, n^{-1}\right\}, v\right), \qquad f_{2n}(t,u,v) = f_2\left(t, u, \max\left\{v, n^{-1}\right\}\right). \tag{2.20}$$

Clearly, $f_{in} \in C([0,1] \times \mathbb{R}^{+} \times \mathbb{R}^{+}, \mathbb{R}^{+})$ $(i = 1,2)$.

By Lemmas 2.1 and 2.2, for each $n \in \mathbb{N}$, $\lambda > 0$, $\mu > 0$, let us define operators $A_n^\lambda : K \to C[0,1]$, $B_n^\mu : K \to C[0,1]$, and $T_n : K \to E$ by

$$A_n^\lambda(u,v)(t) = \lambda \int_0^1 H_1(t,s)\phi_{q_1}\left(\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau,u(\tau),v(\tau))d\tau\right)ds, \tag{2.21}$$

$$B_n^\mu(u,v)(t) = \mu \int_0^1 H_2(t,s)\phi_{q_2}\left(\int_0^1 K_2(s,\tau)a_2(\tau)f_{2n}(\tau,u(\tau),v(\tau))d\tau\right)ds, \tag{2.22}$$

and $T_n(u,v) = (A_n^\lambda(u,v), B_n^\mu(u,v))$, respectively.

Lemma 2.8. *Assume that* (H_1)–(H_3) *hold, then for each* $\lambda > 0$, $\mu > 0$, $n \in \mathbb{N}$, $T_n : \overline{K}_{r,R} \to K$ *is a completely continuous operator.*

Proof. Let $\lambda > 0$, $\mu > 0$, and $n \in \mathbb{N}$ be fixed. For any $(u,v) \in K$, by (2.21), we have

$$\begin{aligned}\left(A_n^\lambda(u,v)\right)''(t) &= -\lambda\phi_{q_1}\left(\int_0^1 K_1(t,\tau)a_1(\tau)f_{1n}(\tau,u(\tau),v(\tau))d\tau\right) \le 0,\\ A_n^\lambda(u,v)(0) &= A_n^\lambda(u,v)(1)\\ &= \lambda\int_0^1 H_1(0,s)\phi_{q_1}\left(\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau,u(\tau),v(\tau))d\tau\right)ds \ge 0,\end{aligned} \tag{2.23}$$

which implies that A_n^λ is nonnegative and concave on $[0,1]$. Similarly, by (2.22) we can obtain that B_n^μ is nonnegative and concave on $[0,1]$. For any $(u,v) \in K$ and $t \in [0,1]$, it follows from (2.13) that

$$\begin{aligned}A_n^\lambda(u,v)(t) &= \lambda\int_0^1 H_1(t,s)\phi_{q_1}\left(\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau,u(\tau),v(\tau))d\tau\right)ds\\ &\le \lambda\gamma_1\nu_1^{q_1-1}\int_0^1 e(s)\phi_{q_1}\left(\int_0^1 e(\tau)a_1(\tau)f_{1n}(\tau,u(\tau),v(\tau))d\tau\right)ds.\end{aligned} \tag{2.24}$$

Thus

$$\left\|A_n^\lambda(u,v)\right\| \le \lambda\gamma_1\nu_1^{q_1-1}\int_0^1 e(s)\phi_{q_1}\left(\int_0^1 e(\tau)a_1(\tau)f_{1n}(\tau,u(\tau),v(\tau))d\tau\right)ds. \tag{2.25}$$

On the other hand, by (2.13) and (2.18), we have

$$
\begin{aligned}
A_n^{\lambda}(u,v)(t) &= \lambda \int_0^1 H_1(t,s)\phi_{q_1}\left(\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau,u(\tau),v(\tau))d\tau\right)ds \\
&\geq \lambda\rho_1\sigma_1^{q_1-1}\int_0^1 e(s)\phi_{q_1}\left(\int_0^1 e(\tau)a_1(\tau)f_{1n}(\tau,u(\tau),v(\tau))d\tau\right)ds \\
&\geq \frac{\rho_1\sigma_1^{q_1-1}}{\gamma_1\nu_1^{q_1-1}}\left\|A_n^{\lambda}(u,v)\right\| \geq \Lambda\left\|A_n^{\lambda}(u,v)\right\|.
\end{aligned}
\tag{2.26}
$$

This implies that

$$
\min_{t\in[0,1]} A_n^{\lambda}(u,v)(t) \geq \Lambda\left\|A_n^{\lambda}(u,v)\right\|. \tag{2.27}
$$

Similar to (2.27), we also have

$$
\min_{t\in[0,1]} B_n^{\mu}(u,v)(t) \geq \Lambda\left\|B_n^{\mu}(u,v)\right\|. \tag{2.28}
$$

Therefore, $T_n(K) \subset K$.

Next, we prove that $T_n : \overline{K}_{r,R} \to K$ is completely continuous. Suppose $(u_m, v_m) \in \overline{K}_{r,R}$ and $(u_0,v_0) \in \overline{K}_{r,R}$ with $\|(u_m,v_m)-(u_0,v_0)\| \to 0$ $(m \to \infty)$. We notice that $t \in [0,1]$ $f_{in}(t,u_m(t),v_m(t)) - f_{in}(t,u_0(t),v_0(t)) \to 0$ $(m \to \infty)$. Using the Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
&\Bigg|\phi_{q_1}^{p_1-1}\left(\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau,u_m(\tau),v_m(\tau))d\tau\right) \\
&\quad -\phi_{q_1}^{p_1-1}\left(\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau,u_0(\tau),v_0(\tau))d\tau\right)\Bigg| \\
&\quad\leq \nu_1\int_0^1 e(\tau)a_1(\tau)\left|f_{1n}(\tau,u_m(\tau),v_m(\tau)) - f_{1n}(\tau,u_0(\tau),v_0(\tau))\right|d\tau \longrightarrow 0, \quad m\longrightarrow\infty.
\end{aligned}
\tag{2.29}
$$

Therefore,

$$
\begin{aligned}
&\left\|A_n^{\lambda}(u_m,v_m) - A_n^{\lambda}(u_0,v_0)\right\| \\
&\quad\leq \lambda\gamma_1\int_0^1 e(s)\Bigg|\phi_{q_1}\left(\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau,u_m(\tau),v_m(\tau)d\tau)\right) \\
&\qquad -\phi_{q_1}\left(\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau,u_0(\tau),v_0(\tau))d\tau\right)\Bigg|ds \longrightarrow 0, \quad m\longrightarrow\infty.
\end{aligned}
\tag{2.30}
$$

Similarly, we also have

$$\left\| B_n^{\mu}(u_m, v_m) - B_n^{\mu}(u_0, v_0) \right\| \longrightarrow 0, \quad m \longrightarrow \infty. \tag{2.31}$$

So $A_n^{\lambda} : \overline{K}_{r,R} \to C[0,1]$ and $B_n^{\beta} : \overline{K}_{r,R} \to C[0,1]$ are continuous. Therefore, $T_n : \overline{K}_{r,R} \to K$ is also continuous.

Let $D \subset \overline{K}_{r,R}$ be any bounded set, then for any $(u,v) \in D$, we have $(u,v) \in K$, $r \leq \|u\| \leq R$, $r \leq \|v\| \leq R$, and then $0 < \Lambda r \leq u(\tau) \leq R$, $0 < \Lambda r \leq v(\tau) \leq R$ for any $\tau \in [0,1]$. By (H_3), we have

$$L_r := \left(\int_0^1 e(\tau) a_1(\tau) g_1(s, r\Lambda) d\tau \right)^{q_1 - 1} < +\infty. \tag{2.32}$$

It is easy to show that $A_n^{\lambda}(D)$ is uniformly bounded. In order to show that T_n is a compact operator, we only need to show that $A_n^{\lambda}(D)$ is equicontinuous. By the uniformly continuity of $H_1(t,s)$ on $[0,1] \times [0,1]$, for all $\varepsilon > 0$, there is $\delta > 0$ such that for any t_1, t_2, $s \in [0,1]$ and $|t_1 - t_2| < \delta$, we have

$$|H_1(t_1, s) - H_1(t_2, s)| < \varepsilon. \tag{2.33}$$

This together with (2.15) and (2.32) implies

$$\begin{aligned}
&\left| A_n^{\lambda}(u,v)(t_1) - A_n^{\lambda}(u,v)(t_2) \right| \\
&\quad \leq \lambda \int_0^1 |H_1(t_1,s) - H_1(t_2,s)| \phi_{q_1}\left(\int_0^1 K_1(s,\tau) a_1(\tau) f_{1n}(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
&\quad < \varepsilon \lambda \nu_1^{q_1-1} \phi_{q_1}\left(\int_0^1 e(\tau) a_1(\tau) \left[g_1\left(\tau, \max\left\{u(\tau), n^{-1}\right\}\right) + h_1(\tau, v(\tau)) \right] d\tau \right) \\
&\quad \leq \varepsilon \lambda \nu_1^{q_1-1} \phi_{q_1}\left(\int_0^1 e(\tau) a_1(\tau) \left[g_1(\tau, r\Lambda) + h_1(\tau, v(\tau)) \right] d\tau \right) \\
&\quad \leq \varepsilon \lambda \nu_1^{q_1-1} 2^{q_1-1} \left[\phi_{q_1}\left(\int_0^1 e(\tau) a_1(\tau) g_1(\tau, r\Lambda) d\tau \right) + \phi_{q_1}\left(\int_0^1 e(\tau) a_1(\tau) h_1(\tau, v(\tau)) d\tau \right) \right] \\
&\quad \leq \varepsilon \lambda \nu_1^{q_1-1} 2^{q_1-1} \left[L_r + L_1 \left(\max_{\substack{\tau \in [0,1] \\ y \in [r\Lambda, R]}} h_1(\tau, y) \right)^{q_1-1} \right], \quad |t_1 - t_2| < \delta, \ (u,v) \in D.
\end{aligned} \tag{2.34}$$

This means that $A_n^{\lambda}(D)$ is equicontinuous. By the Arzela-Ascoli theorem, $A_n^{\lambda}(D)$ is a relatively compact set and that $A_n^{\lambda} : \overline{K}_{r,R} \to C[0,1]$ is a completely continuous operator.

In the same way, we can show that $B_n^{\mu}:\overline{K}_{r,R}\to C[0,1]$ is also completely continuous, and so $T_n:\overline{K}_{r,R}\to K$ is completely continuous. Now since λ, μ, and n are given arbitrarily, the conclusion of this lemma is valid. □

3. Main Results

For notational convenience, we denote by

$$M_i=6\left(\rho_i\sigma^{q_i-1}\Lambda L_i\right)^{-1},\qquad N_i=\left(\gamma_i\nu_i^{q_i-1}L_i\right)^{-1},\quad i=1,2,$$

$$f_1^{\alpha}=\left(\limsup_{x\to\alpha}\sup_{\substack{t\in[0,1]\\ y\in\mathbb{R}^+}}\frac{f_1(t,x,y)}{\phi_{p_1}(x)}\right)^{q_1-1},\qquad f_2^{\alpha}=\left(\limsup_{y\to\alpha}\sup_{\substack{t\in[0,1]\\ x\in\mathbb{R}^+}}\frac{f_2(t,x,y)}{\phi_{p_2}(y)}\right)^{q_2-1},$$
$$f_{1\alpha}=\left(\liminf_{x\to\alpha}\inf_{\substack{t\in[0,1]\\ y\in\mathbb{R}^+}}\frac{f_1(t,x,y)}{\phi_{p_1}(x)}\right)^{q_1-1},\qquad f_{2\alpha}=\left(\liminf_{y\to\alpha}\inf_{\substack{t\in[0,1]\\ x\in\mathbb{R}^+}}\frac{f_2(t,x,y)}{\phi_{p_2}(y)}\right)^{q_2-1}, \tag{3.1}$$

where α denotes 0 or ∞. The main results of this paper are the following.

Theorem 3.1. *Assume that (H_1)–(H_3) hold. Then we have:*

(C_1) *If f_1^0, $f_{1\infty}$, $f_2^0\in(0,\infty)$ and $M_1/f_{1\infty}<N_1/f_1^0$, then for each $\lambda\in(M_1/f_{1\infty},N_1/f_1^0)$, $\mu\in(0,N_2/f_2^0)$, the system* (1.3) *has at least one positive solution.*

(C_2) *If f_1^0, f_2^0, $f_{2\infty}\in(0,\infty)$ and $M_2/f_{2\infty}<N_2/f_2^0$, then for each $\lambda\in(0,N_1/f_1^0)$, $\mu\in(M_2/f_{2\infty},N_2/f_2^0)$, the system* (1.3) *has at least one positive solution.*

(C_3) *If $f_1^0=0$, $f_{1\infty}=\infty$, $0<f_2^0<\infty$, then for each $\lambda\in(0,\infty)$, $\mu\in(0,N_2/f_2^0)$, the system* (1.3) *has at least one positive solution.*

(C_4) *If $0<f_1^0<\infty$, $f_2^0=0$, $f_{2\infty}=\infty$, then for each $\lambda\in(0,N_1/f_1^0\ )$, $\mu\in(0,\infty)$, the system* (1.3) *has at least one positive solution.*

(C_5) *If $f_i^0=0$, $f_{i\infty}=\infty$ $(i=1,2)$, then for each $\lambda\in(0,\infty)$, $\mu\in(0,\infty)$, the system* (1.3) *has at least one positive solution.*

(C_6) *If $0<f_1^0<\infty$, $f_{1\infty}=\infty$ or $f_{2\infty}=\infty$, $0<f_2^0<\infty$, then for each $\lambda\in(0,N_1/f_1^0)$, $\mu\in(0,N_2/f_2^0)$, the system* (1.3) *has at least one positive solution.*

(C_7) *If $f_1^0=0$, $0<f_{1\infty}<\infty$, and $f_1^0=0$, $0<f_{2\infty}<\infty$, then for each $\lambda\in(M_1/f_{1\infty},\infty)$, $\mu\in(0,\infty)$ or $\lambda\in(0,\infty)$, $\mu\in(M_2/f_{2\infty},\infty)$, the system* (1.3) *has at least one positive solution.*

Proof. We only prove the condition in which (C_1) holds. The other cases can be proved similarly.

Let $\lambda\in(M_1/f_{1\infty},N_1/f_1^0)$, $\mu\in(0,N_2/(f_2^0))$, choose $\varepsilon_1>0$ such that $f_{1\infty}-\varepsilon_1>0$ and

$$\frac{M_1}{f_{1\infty}-\varepsilon_1}\le\lambda\le\frac{N_1}{f_1^0+\varepsilon_1},\qquad 0<\mu\le\frac{N_2}{f_2^0+\varepsilon_1}. \tag{3.2}$$

It follows from $f_i^0 \in (0,\infty)$ of (C_1) that there exists $r_1 > 0$ such that for any $t \in [0,1]$,

$$f_1(t,x,y) \le \left(f_1^0+\varepsilon_1\right)^{p_1-1}\phi_{p_1}(x) \le \left[r_1\left(f_1^0+\varepsilon_1\right)\right]^{p_1-1}, \quad 0 < x \le r_1, y \ge 0, \tag{3.3}$$

$$f_2(t,x,y) \le \left(f_2^0+\varepsilon_1\right)^{p_2-1}\phi_{p_2}(y) \le \left[r_1\left(f_2^0+\varepsilon_1\right)\right]^{p_2-1}, \quad x \ge 0,\ 0 < y \le r_1. \tag{3.4}$$

Let $K_{r_1} = \{(u,v) \in K : \|u\| < r_1, \|v\| < r_1\}$. For any $(u,v) \in \partial K_{r_1}$, $n > 1/r_1$, by (2.13), (3.3), we have

$$\begin{aligned}
\left\|A_n^\lambda(u,v)\right\| &= \max_{t\in[0,1]}\lambda\int_0^1 H_1(t,s)\phi_{q_1}\left(\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau,u(\tau),v(\tau))d\tau\right)ds \\
&\le \lambda\gamma_1\nu_1^{q_1-1}\phi_{q_1}\int_0^1\left(e(\tau)a_1(\tau)(f_1^0+\varepsilon_1)^{p_1-1}\phi_{p_1}(u(\tau))d\tau\right) \\
&\le \lambda\gamma_1\nu_1^{q_1-1}\left(f_1^0+\varepsilon_1\right)L_1r_1 \\
&= \lambda N_1^{-1}\left(f_1^0+\varepsilon_1\right)r_1.
\end{aligned} \tag{3.5}$$

Similarly, we also have

$$\left\|B_n^\mu(u,v)\right\| \le \mu N_2^{-1}\left(f_2^0+\varepsilon_1\right)r_1. \tag{3.6}$$

Therefore, we have

$$\begin{aligned}
\|T_n(u,v)\| &= \left\|A_n^\lambda(u,v)\right\| + \left\|B_n^\mu(u,v)\right\| \\
&\le \left[\lambda N_1^{-1}\left(f_1^0+\varepsilon_1\right) + \mu N_2^{-1}\left(f_2^0+\varepsilon_1\right)\right]r_1 \\
&\le 2r_1 = \|(u,v)\|.
\end{aligned} \tag{3.7}$$

On the other hand, by $f_{1\infty} > f_{1\infty} - \varepsilon_1 > 0$, there exists $R_0 > 0$ such that

$$f_1(t,x,y) \ge (f_{1\infty}-\varepsilon_1)^{p_1-1}\phi_{p_1}(x), \quad t \in [0,1],\ x \ge R_0,\ y \ge 0. \tag{3.8}$$

Let $R_1 > \max\{2r_1, \Lambda^{-1}R_0\}$, $K_{R_1} = \{(u,v) \in K : \|u\| < R_1, \|v\| < R_1\}$. Next, we take $(\varphi_1,\varphi_2) = (1,1) \in \partial K_1$, and for any $(u,v) \in \partial K_{R_1}$, $m > 0$, $n \in \mathbb{N}$, we will show

$$(u,v) \ne A_n^\lambda(u,v) + m(\varphi_1,\varphi_2). \tag{3.9}$$

Otherwise, there exist $(u_0,v_0) \in \partial K_{R_1}$ and $m_0 > 0$ such that

$$(u_0,v_0) = A_n^\lambda(u_0,v_0) + m_0(\varphi_1,\varphi_2). \tag{3.10}$$

From $(u_0, v_0) \in \partial K_{R_1}$, we know that $\|u_0\| = R_1$ or $\|v_0\| = R_1$. Without loss of generality, we may suppose that $\|u_0\| = R_1$, then $u_0(\tau) \geq \Lambda\|u_0\| = \Lambda R_1 > R_0$ for any $\tau \in [0,1]$. So, by (2.13), (3.8), we have

$$\begin{aligned}
u_0(t) &= \lambda \int_0^1 H_1(t,s)\phi_{q_1}\left(\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau, u_0(\tau), v_0(\tau))d\tau\right)ds + m_0 \\
&\geq \lambda\rho_1\sigma_1^{q_1-1}\int_0^1 e(s)\phi_{q_1}\left(\int_0^1 e(\tau)a_1(\tau)f_{1n}(\tau, u_0(\tau), v_0(\tau))d\tau\right)ds + m_0 \\
&\geq \lambda\rho_1\sigma_1^{q_1-1}\int_0^1 e(s)\phi_{q_1}\left(\int_0^1 e(\tau)a_1(\tau)(f_{1\infty}-\varepsilon_1)^{p_1-1}\phi_{p_1}(u_0(\tau))\ d\tau\right)ds + m_0 \\
&\geq \lambda\rho_1\sigma_1^{q_1-1}\int_0^1 e(s)\phi_{q_1}\left(\int_0^1 e(\tau)a_1(\tau)(f_{1\infty}-\varepsilon_1)^{p_1-1}(\Lambda R_1)^{p_1-1}d\tau\right)ds + m_0 \\
&= \frac{1}{6}\lambda\rho_1\sigma_1^{q_1-1}(f_{1\infty}-\varepsilon_1)\Lambda R_1 L_1 + m_0 \\
&= \lambda M_1^{-1}(f_{1\infty}-\varepsilon_1)R_1 + m_0 > R_1.
\end{aligned} \tag{3.11}$$

This implies that $R_1 > R_1$, which is a contradiction. This yields that (3.9) holds. By (3.7), (3.9), and Lemma 1.2, for any $n > 1/r_1$ and $\lambda \in (M_1/f_{1\infty}, N_1/f_1^0)$, $\mu \in (0, N_2/f_2^0)$, we obtain that T_n has a fixed point (u_n, v_n) in $\overline{K}_{r_1,R_1}$ satisfying $r_1 < \|u_n\| < R_1, r_1 < \|v_n\| < R_1$.

Let $\{(u_n, v_n)\}_{n\geq n_1}$ be the sequence of solutions of boundary value problems (2.19), where $n_1 > 1/r_1$ is a fixed integer. It is easy to see that they are uniformly bounded. Next we show that $\{u_n\}_{n\geq n_1}$ are equicontinuous on $[0,1]$. From $(u_n, v_n) \in \overline{K}_{r_1,R_1}$, we know that $R_1 \geq u_n(\tau) \geq \Lambda\|u_n\| \geq \Lambda r_1$, $R_1 \geq v_n(\tau) \geq \Lambda\|v_n\| \geq \Lambda r_1$, $\tau \in [0,1]$. For any $\varepsilon > 0$, by the continuous of $H_1(t,s)$ in $[0,1]\times[0,1]$, there exists $\delta_1 > 0$ such that for any t_1, t_2, $s \in [0,1]$ and $|t_1 - t_2| < \delta_1$, we have

$$|H_1(t_1,s) - H_1(t_2,s)| < \varepsilon. \tag{3.12}$$

This combining with (2.15), (2.32) implies that for any t_1, $t_2 \in [0,1]$ and $|t_1 - t_2| < \delta_1$, we have

$$\begin{aligned}
&|u_n(t_1) - u_n(t_2)| \\
&\quad \leq \lambda \int_0^1 |H_1(t_1,s) - H_1(t_2,s)|\phi_{q_1}\left(\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau, u_n(\tau), v_n(\tau))d\tau\right)ds \\
&\quad < \varepsilon\lambda\nu_1^{q_1-1}\phi_{q_1}\left(\int_0^1 e(\tau)a_1(\tau)\left[g_1\left(\tau, \max\left\{u_n(\tau), n^{-1}\right\}\right) + h_1(\tau, v_n(\tau))\right]d\tau\right)
\end{aligned}$$

$$\le \varepsilon\lambda v_1^{q_1-1}2^{q_1-1}\left[\phi_{q_1}\left(\int_0^1 e(\tau)a_1(\tau)g_1(\tau,r\Lambda)d\tau\right)+\phi_{q_1}\left(\int_0^1 e(\tau)a_1(\tau)h_1(\tau,v(\tau))d\tau\right)\right]$$
$$\le \varepsilon\lambda v_1^{q_1-1}2^{q_1-1}\left[L_r+L_1\left(\max_{\substack{\tau\in[0,1]\\ y\in[r_1\Lambda,R_1]}} h_1(\tau,y)\right)^{q_1-1}\right]. \tag{3.13}$$

Similarly, $\{v_n\}_{n\ge n_1}$ are also equicontinuous on $[0,1]$. By the Ascoli-Arzela theorem, the sequence $\{(u_n,v_n)\}_{n\ge n_1}$ has a subsequence being uniformly convergent on $[0,1]$. From Lemma 2.2, we know that

$$\begin{aligned} u_n''(s) &= \lambda^{p_1-1}\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau,u_n(\tau),v_n(\tau))d\tau,\\ v_n''(s) &= \mu^{p_2-1}\int_0^1 K_2(s,\tau)a_2(\tau)f_{2n}(\tau,u_n(\tau),v_n(\tau))d\tau. \end{aligned} \tag{3.14}$$

Since the properties of $K_i(t,s)$ $(i=1,2)$ are similar to those of $H_i(t,s)$ $(i=1,2)$, so (u_n'',v_n'') have the similar properties of (u_n,v_n), that is, (u_n'',v_n'') also has a subsequence being uniformly convergent on $[0,1]$. Without loss of generality, we still assume that $\{(u_n,v_n)\}_{n\ge n_1}$ itself uniformly converges to (u,v) on $[0,1]$ and $\{(u_n'',v_n'')\}_{n\ge n_1}$ itself uniformly converges to (u'',v'') on $[0,1]$, respectively. Since $\{(u_n,v_n)\}_{n\ge n_1}\in \overline{K}_{r_1,R_1}\subset K$, so we have $u_n\ge 0$, $v_n\ge 0$. By (2.19), we have

$$\begin{aligned} u_n(t) = {}& u_n\left(\frac{1}{2}\right)+\left(t-\frac{1}{2}\right)u_n'\left(\frac{1}{2}\right)\\ &-\int_{1/2}^t ds\int_{1/2}^s \phi_{q_1}\left(u_n''^{p_1-1}\left(\frac{1}{2}\right)+\left(s_2-\frac{1}{2}\right)u_n'''^{p_1-1}\left(\frac{1}{2}\right)\right.\\ &\left.-\int_{1/2}^{s_2} ds_1\int_{1/2}^{s_1}\lambda^{p_1-1}a_1(\tau)f_{1n}(\tau,u_n(\tau),v_n(\tau))d\tau\right)ds_2, \quad t\in(0,1), \end{aligned} \tag{3.15}$$

$$\begin{aligned} v_n(t) = {}& v_n\left(\frac{1}{2}\right)+\left(t-\frac{1}{2}\right)v_n'\left(\frac{1}{2}\right)\\ &-\int_{1/2}^t ds\int_{1/2}^s \phi_{q_2}\left(v_n''^{p_2-1}\left(\frac{1}{2}\right)+\left(s_2-\frac{1}{2}\right)v_n'''^{p_2-1}\left(\frac{1}{2}\right)\right.\\ &\left.-\int_{1/2}^{s_2} ds_1\int_{1/2}^{s_1}\mu^{p_2-1}a_2(\tau)f_{2n}(\tau,u_n(\tau),v_n(\tau))d\tau\right)ds_2, \quad t\in(0,1). \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we know that $\{u_n'(1/2)\}_{n\ge n_1}$, $\{v_n'(1/2)\}_{n\ge n_1}$, $\{u_n''(1/2)\}_{n\ge n_1}$, $\{v_n''(1/2)\}_{n\ge n_1}$, $\{u_n'''(1/2)\}_{n\ge n_1}$, $\{v_n'''(1/2)\}_{n\ge n_1}$ are bounded sets. Without loss of generality, we may assume $(u_n'(1/2),v_n'(1/2))\to(c_1,d_1),(u_n''(1/2),v_n''(1/2))\to(c_2,d_2)$,

$(u_n'''(1/2), v_n'''(1/2)) \to (c_3, d_3)$ as $n \to \infty$. Then by (3.15), (3.16), and the Lebesgue dominated convergence theorem, we have

$$u(t) = u\left(\frac{1}{2}\right) + c_1\left(t - \frac{1}{2}\right) - \int_{1/2}^{t} ds \int_{1/2}^{s} \phi_{q_1}\left(c_2^{p_1-1} + c_3^{p_1-1}\left(s_2 - \frac{1}{2}\right)\right. \\ \left. - \int_{1/2}^{s_2} ds_1 \int_{1/2}^{s_1} \lambda^{p_1-1} a_1(\tau) f_1(\tau, u(\tau), v(\tau)) d\tau\right) ds_2, \ t \in (0,1), \tag{3.17}$$

$$v(t) = v\left(\frac{1}{2}\right) + d_1\left(t - \frac{1}{2}\right) \\ - \int_{1/2}^{t} ds \int_{1/2}^{s} \phi_{q_2}(d_2^{p_2-1} + d_3^{p_2-1}\left(s_2 - \frac{1}{2}\right) \\ - \int_{1/2}^{s_2} ds_1 \int_{1/2}^{s_1} \mu^{p_2-1} a_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau) ds_2 \quad t \in (0,1). \tag{3.18}$$

By (3.17) and (3.18), direct computation shows that

$$(\phi_{p_1}(u''(t)))'' = \lambda^{p_1-1} a_1(t) f_1(t, u(t), v(t)), \\ (\phi_{p_2}(v''(t)))'' = \mu^{p_2-1} a_2(t) f_2(t, u(t), v(t)), \quad 0 < t < 1. \tag{3.19}$$

On the other hand, (u, v) satisfies the boundary condition of (1.3). In fact, $u_n(0) = u_n(1) = \int_0^1 u_n(s) d\xi_1(s)$, $v_n(0) = v_n(1) = \int_0^1 v_n(s) d\xi_2(s)$, $\phi_{p_1}(u_n''(0)) = \phi_{p_1}(u_n''(1)) = \int_0^1 \phi_{p_1}(u_n''(s)) d\eta_1(s)$, $\phi_{p_2}(v_n''(0)) = \phi_{p_2}(v_n''(1)) = \int_0^1 \phi_{p_2}(v_n''(s)) d\eta_2(s)$, and so the conclusion holds by letting $n \to \infty$. □

Theorem 3.2. *Assume that* (H_1)–(H_3) *hold. Then we have:*

(D_1) *If* f_{10}, f_1^∞, $f_2^\infty \in (0, \infty)$ *and* $M_1/f_{10} < N_1/f_1^\infty$, *then for each* $\lambda \in (M_1/f_{10}, N_1/f_1^\infty)$, $\mu \in (0, N_2/f_2^\infty)$, *the system* (1.3) *has at least one positive solution.*

(D_2) *If* f_1^∞, f_{20}, $f_2^\infty \in (0, \infty)$ *and* $M_2/f_{20} < N_2/f_2^\infty$, *then for each* $\lambda \in (0, N_1/f_1^\infty)$, $\mu \in (M_2/f_{20}, N_2/f_2^\infty)$, *the system* (1.3) *has at least one positive solution.*

(D_3) *If* $f_{10} = \infty$, $f_1^\infty = 0$, $0 < f_2^\infty < \infty$, *then for each* $\lambda \in (0, \infty)$, $\mu \in (0, N_2/(f_2^\infty))$, *the system* (1.3) *has at least one positive solution.*

(D_4) *If* $0 < f_1^\infty < \infty$, $f_{20} = \infty$, $f_2^\infty = 0$, *then for each* $\lambda \in (0, N_1/f_1^\infty)$, $\mu \in (0, \infty)$, *the system* (1.3) *has at least one positive solution.*

(D_5) *If* $f_{i0} = \infty$, $f_i^\infty = 0$ $(i = 1, 2)$, *then for each* $\lambda \in (0, \infty)$, $\mu \in (0, \infty)$, *the system* (1.3) *has at least one positive solution.*

(D_6) *If* $0 < f_1^\infty < \infty$, $f_{10} = \infty$ *or* $f_{20} = \infty$, $0 < f_2^\infty < \infty$, *then for each* $\lambda \in (0, N_1/f_1^\infty)$, $\mu \in (0, N_2/f_2^\infty)$, *the system* (1.3) *has at least one positive solution.*

(D_7) *If* $f_1^\infty = 0$, $0 < f_{10} < \infty$, *and* $f_2^\infty = 0$, $0 < f_{20} < \infty$, *then for each* $\lambda \in (M_1/f_{10}, \infty)$, $\mu \in (0, \infty)$ *or* $\lambda \in (0, \infty)$, $\mu \in (M_2/f_{20}, \infty)$, *the system* (1.3) *has at least one positive solution.*

Proof. We may suppose that condition (D_1) holds. Similarly, we can prove the other cases.

Let $\lambda \in (M_1/f_{10}, N_1/f_1^{\infty})$, $\mu \in (0, N_2/f_2^{\infty})$. We can choose $\varepsilon_2 > 0$ such that $N_1 - \varepsilon_2 > 0$, $N_2 - \varepsilon_2 > 0$ and

$$\lambda f_1^{\infty} < N_1 - \varepsilon_2, \qquad \mu f_2^{\infty} < N_2 - \varepsilon_2. \tag{3.20}$$

It follows from (D_1) and (2.16) that there exists $R_2^* > 0$ such that for any $t \in [0,1]$

$$f_1(t,x,y) \leq \left(\frac{1}{\lambda}(N_1 - \varepsilon_2)\right)^{p_1-1} \phi_{p_1}(x), \quad x \geq R_2^*, \ y \geq 0, \tag{3.21}$$

$$f_2(t,x,y) \leq \left(\frac{1}{\lambda}(N_2 - \varepsilon_2)\right)^{p_2-1} \phi_{p_2}(y), \quad x \geq 0, \ y \geq R_2^*. \tag{3.22}$$

Let $R_2 = \Lambda^{-1} R_2^*$, $K_{R_2} = \{(u,v) \in K : \|u\| < R_2, \ \|v\| < R_2\}$. For any $(u,v) \in \partial K_{R_2}$, $n \in \mathbb{N}$, by (2.13), (3.21), we have

$$\begin{aligned}
\left\| A_n^{\lambda}(u,v) \right\| &= \max_{t\in[0,1]} \lambda \int_0^1 H_1(t,s) \phi_{q_1}\left(\int_0^1 K_1(s,\tau) a_1(\tau) f_{1n}(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
&\leq \lambda \gamma_1 \nu_1^{q_1-1} \phi_{q_1}\left(\int_0^1 e(\tau) a_1(\tau) \left(\frac{1}{\lambda}(N_1 - \varepsilon_2)\right)^{p_1-1} \phi_{p_1}(u(\tau)) d\tau \right) \\
&\leq \lambda \gamma_1 \nu_1^{q_1-1} \frac{1}{\lambda}(N_1 - \varepsilon_2) L_1 R_2 < R_2.
\end{aligned} \tag{3.23}$$

Similarly, by (3.22) we have $\|B_n^{\mu}(u,v)\| < R_2$. Therefore,

$$\|T_n(u,v)\| = \left\| A_n^{\lambda}(u,v) \right\| + \left\| B_n^{\mu}(u,v) \right\| \leq 2R_2 = \|(u,v)\|, \quad (u,v) \in \partial K_{R_2}, \ n \in \mathbb{N}. \tag{3.24}$$

On the other hand, choose $\varepsilon_3 > 0$ such that $M_1 + \varepsilon_3 < \lambda f_{10}$. By the condition $f_{10} \in (0, \infty)$ of (D_1) and (2.16), there exists $r_2^* > 0$ such that

$$f_1(t,x,y) \geq \left(\frac{1}{\lambda}(M_1 + \varepsilon_3)\right)^{p_1-1} \phi_{p_1}(x), \quad t \in [0,1], \ 0 < x \leq r_2^*, \ y \geq 0. \tag{3.25}$$

Let $0 < r_2 < \min\{R_2, r_2^*\}$, $K_{r_2} = \{(u,v) \in K : \|u\| < r_2, \ \|v\| < r_2\}$. Next, we take $(\varphi_1, \varphi_2) = (1,1) \in \partial K_1$, $n > 1/r_2$, and for any $(u,v) \in \partial K_{r_2}$, $m > 0$, we will show

$$(u,v) \neq A_n^{\lambda}(u,v) + m(\varphi_1, \varphi_2). \tag{3.26}$$

Otherwise, there exist $(u_0, v_0) \in \partial K_{r_2}$ and $m_0 > 0$ such that

$$(u_0, v_0) = A_n^\lambda(u_0, v_0) + m_0(\varphi_1, \varphi_2). \tag{3.27}$$

From $(u_0, v_0) \in \partial K_{r_2}$, we know that $\|u_0\| = r_2$ or $\|v_0\| = r_2$. Without loss of generality, we may suppose that $\|u_0\| = r_2$, then $u_0(\tau) \geq \Lambda\|u_0\| \geq \Lambda r_2$ for any $\tau \in [0,1]$. So, we have

$$\begin{aligned}
u_0(t) &= \lambda \int_0^1 H_1(t,s)\phi_{q_1}\left(\int_0^1 K_1(s,\tau)a_1(\tau)f_{1n}(\tau,u_0(\tau),v_0(\tau))d\tau\right)ds + m_0 \\
&\geq \lambda\rho_1\sigma_1^{q_1-1}\int_0^1 e(s)\phi_{q_1}\left(\int_0^1 e(\tau)a_1(\tau)f_{1n}(\tau,u_0(\tau),v_0(\tau))d\tau\right)ds + m_0 \\
&\geq \lambda\rho_1\sigma_1^{q_1-1}\int_0^1 e(s)\phi_{q_1}\left(\int_0^1 e(\tau)a_1(\tau)\left(\frac{1}{\lambda}(M_1+\varepsilon_3)\right)^{p_1-1}\phi_{p_1}(u_0(\tau))d\tau\right)ds + m_0 \\
&\geq \lambda\rho_1\sigma_1^{q_1-1}\int_0^1 e(s)\phi_{q_1}\left(\int_0^1 e(\tau)a_1(\tau)(M_1+\varepsilon_3)^{p_1-1}(\Lambda r_2)^{p_1-1}d\tau\right)ds + m_0 \\
&= \frac{1}{6}\lambda\rho_1\sigma_1^{q_1-1}\frac{1}{\lambda}(M_1+\varepsilon_3)\Lambda r_2 L_1 + m_0 > r_2.
\end{aligned} \tag{3.28}$$

This implies that $r_2 > r_2$, which is a contradiction. This yields that (3.26) holds. By (3.24), (3.26), and Lemma 1.2, for any $n > 1/r_2$ and $\lambda \in (M_1/f_{10}, N_1/f_1^\infty)$, $\mu \in (0, N_2/f_2^\infty)$, we obtain that T_n has a fixed point (u_n, v_n) in $\overline{K}_{r_2,R_2}$ and $r_2 < \|u_n\| < R_2$, $r_2 < \|v_n\| < R_2$. The rest of proof is similar to Theorem 3.1. □

4. An Example

Example 4.1. We consider system (1.3) with $p_1 = 3/2$, $p_2 = 7/3$, $a_1(t) = 1/(t\sqrt{(1-t)}$, $a_2(t) = 1/((1-t)\ \sqrt{t})$,

$$\begin{aligned}
f_1(t,u,v) &= \frac{t^2+1}{\sqrt{u}} + 1 + \sin\left(v^2+v+t\right), \quad (t,u,v) \in [0,1]\times\mathbb{R}_0^+\times\mathbb{R}^+, \\
f_2(t,u,v) &= 2 + \sin(u+\ln(t+1)) + \frac{t^4+t+3}{\sqrt{v}}, \quad (t,u,v) \in [0,1]\times\mathbb{R}^+\times\mathbb{R}_0^+.
\end{aligned} \tag{4.1}$$

Obviously, a_1, a_2 are singular at $t = 0$ and $t = 1$, $f_1(t,u,v)$ is singular at $u = 0$ and $f_2(t,u,v)$ is singular at $v = 0$. Choose $g_1(t,u) = (t^2+1)/\sqrt{u}$, $h_1(t,v) = 1+\sin(v^2+v+t)$, $g_2(t,u) = 2+\sin(u+\ln(t+1))$, and $h_2(t,v) = (t^4+t+3)/\sqrt{v}$. Let

$$\xi_1(s) = \begin{cases} 0, & s \in \left[0, \dfrac{1}{3}\right), \\ \dfrac{1}{5}, & s \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right), \\ \dfrac{1}{4}, & s \in \left[\dfrac{2}{3}, 1\right], \end{cases} \qquad \xi_2(s) = \begin{cases} 0, & s \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{1}{7}, & s \in \left[\dfrac{1}{2}, \dfrac{3}{4}\right), \\ \dfrac{1}{3}, & s \in \left[\dfrac{3}{4}, 1\right], \end{cases} \tag{4.2}$$

$$\eta_1(s) = \begin{cases} 0, & s \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{3}{5}, & s \in \left[\dfrac{1}{2}, 1\right], \end{cases} \qquad \eta_2(s) = \begin{cases} 0, & s \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{4}{7}, & s \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

By direct calculation, we have $\alpha_1 = 1/4$, $\alpha_2 = 1/3$, $\beta_1 = 3/5$, $\beta_2 = 4/7$, $\int_0^1 e(s)a_i(s)ds = (2/3)$ $(i = 1,2)$. It is easy to check that $f_{10} = f_{20} = \infty$, $f_1^\infty = f_2^\infty = 0$, and the conditions (H_1)–(H_3) and (D_5) are satisfied. By Theorem 3.2, system (1.3) has at least one positive solution provided $\lambda,\ \mu \in (0,+\infty)$.

Remark 4.2. Example 4.1 not only implies that $f_1(t,u,v)$, $f_2(t,u,v)$ can be singular at $u = 0$ and $v = 0$, respectively, but also indicates that there is a large number of functions that satisfy the conditions of Theorem 3.2. In addition, the condition (D_5) is also easy to check.

Acknowledgments

The first and second authors were supported financially by the National Natural Science Foundation of China (11071141, 11126231) and the Natural Science Foundation of Shandong Province of China (ZR2009AL014, ZR2011AQ008). The third author was supported financially by the Australia Research Council through an ARC Discovery Project Grant.

References

[1] F. Bernis, “Compactness of the support in convex and nonconvex fourth order elasticity problems,” *Nonlinear Analysis*, vol. 6, no. 11, pp. 1221–1243, 1982.
[2] D. G. Zill and M. R. Cullen, *Differential Equations and Boundary Value Problems: Computing and Modeling*, Brooks Cole, 5th edition, 2001.
[3] T. G. Myers and J. P. F. Charpin, “A mathematical model for atmospheric ice accretion and water flow on a cold surface,” *International Journal of Heat and Mass Transfer*, vol. 47, no. 25, pp. 5483–5500, 2004.
[4] T. G. Myers, J. P. F. Charpin, and S. J. Chapman, “The flow and solidification of a thin fluid film on an arbitrary three-dimensional surface,” *Physics of Fluids*, vol. 14, no. 8, pp. 2788–2803, 2002.
[5] D. Halpern, O. E. Jensen, and J. B. Grotberg, “A theoretical study of surfactant and liquid delivery into the lung,” *Journal of Applied Physiology*, vol. 85, no. 1, pp. 333–352, 1998.
[6] F. Mémoli, G. Sapiro, and P. Thompson, “Implicit brain imaging,” *NeuroImage*, vol. 23, supplement 1, pp. S179–S188, 2004.
[7] A. Toga, *Brain Warping*, Academic Press, New York, NY, USA, 1998.

[8] M. Hofer and H. Pottmann, "Energy-minimizing splines in manifolds," in *Proceedings of the ACM Transactions on Graphics (SIGGRAPH '04)*, pp. 284–293, August 2004.

[9] S. P. Timoshenko, *Theory of Elastic Stability*, McGraw-Hill, New York, NY, USA, 1961.

[10] W. Soedel, *Vibrations of Shells and Plates*, Marcel Dekker, New York, NY, USA, 1993.

[11] E. Dulcska, *Soil Settlement Effects on Buildings*, vol. 69 of *Developments in Geotechnical Engineering*, Elsevier, Amsterdam, The Netherlands, 1992.

[12] Z. Bai, "Existence and multiplicity of positive for a fourth-order p-Laplacian equation," *Applied Mathematics and Mechanics*, vol. 22, no. 12, pp. 1476–1480, 2001.

[13] Z. Bai, B. Huang, and W. Ge, "The iterative solutions for some fourth-order p-Laplace equation boundary value problems," *Applied Mathematics Letters*, vol. 19, no. 1, pp. 8–14, 2006.

[14] Z. Bai and H. Wang, "On positive solutions of some nonlinear fourth-order beam equations," *Journal of Mathematical Analysis and Applications*, vol. 270, no. 2, pp. 357–368, 2002.

[15] Z. Cheng and J. Ren, "Periodic solutions for a fourth-order Rayleigh type p-Laplacian delay equation," *Nonlinear Analysis*, vol. 70, no. 1, pp. 516–523, 2009.

[16] M. Feng, "Multiple positive solutions of fourth-order impulsive differential equations with integral boundary conditions and one-dimensional p-laplacian," *Boundary Value Problems*, vol. 2011, Article ID 654871, 26 pages, 2011.

[17] J. R. Graef and L. Kong, "Necessary and sufficient conditions for the existence of symmetric positive solutions of singular boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 1467–1484, 2007.

[18] J. R. Graef, C. Qian, and B. Yang, "A three point boundary value problem for nonlinear fourth order differential equations," *Journal of Mathematical Analysis and Applications*, vol. 287, no. 1, pp. 217–233, 2003.

[19] Z. Guo, J. Yin, and Y. Ke, "Multiplicity of positive solutions for a fourth-order quasilinear singular differential equation," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 27, pp. 1–15, 2010.

[20] S. Lu and S. Jin, "Existence of periodic solutions for a fourth-order p-Laplacian equation with a deviating argument," *Journal of Computational and Applied Mathematics*, vol. 230, no. 2, pp. 513–520, 2009.

[21] Y. Luo and Z. Luo, "Symmetric positive solutions for nonlinear boundary value problems with p-Laplacian operator," *Applied Mathematics Letters*, vol. 23, no. 6, pp. 657–664, 2010.

[22] S. Jin and S. Lu, "Periodic solutions for a fourth-order p-Laplacian differential equation with a deviating argument," *Nonlinear Analysis*, vol. 69, no. 5-6, pp. 1710–1718, 2008.

[23] G. Shi and X. Meng, "Monotone iterative for fourth-order p-Laplacian boundary value problems with impulsive effects," *Applied Mathematics and Computation*, vol. 181, no. 2, pp. 1243–1248, 2006.

[24] Z. Wang, L. Qian, and S. Lu, "On the existence of periodic solutions to a fourth-order p-Laplacian differential equation with a deviating argument," *Nonlinear Analysis*, vol. 11, no. 3, pp. 1660–1669, 2010.

[25] J. Xu and Z. Yang, "Positive solutions for a fourth order p-Laplacian boundary value problem," *Nonlinear Analysis*, vol. 74, no. 7, pp. 2612–2623, 2011.

[26] J. Yang and Z. Wei, "Existence of positive solutions for fourth-order m-point boundary value problems with a one-dimensional p-Laplacian operator," *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 2985–2996, 2009.

[27] J. Zhang and G. Shi, "Positive solutions for fourth-order singular p-Laplacian boundary value problems," *Applicable Analysis*, vol. 85, no. 11, pp. 1373–1382, 2006.

[28] X. Zhang and L. Liu, "Positive solutions of fourth-order four-point boundary value problems with p-Laplacian operator," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 2, pp. 1414–1423, 2007.

[29] X. Zhang and L. Liu, "A necessary and sufficient condition for positive solutions for fourth-order multi-point boundary value problems with p-Laplacian," *Nonlinear Analysis*, vol. 68, no. 10, pp. 3127–3137, 2008.

[30] X. Zhang and Y. Cui, "Positive solutions for fourth-order singular p-Laplacian differential equations with integral boundary conditions," *Boundary Value Problems*, vol. 2010, Article ID 862079, 23 pages, 2010.

[31] X. Zhang, M. Feng, and W. Ge, "Existence results for nonlinear boundary-value problems with integral boundary conditions in Banach spaces," *Nonlinear Analysis*, vol. 69, no. 10, pp. 3310–3321, 2008.

[32] X. Zhang, M. Feng, and W. Ge, "Symmetric positive solutions for p-Laplacian fourth-order differential equations with integral boundary conditions," *Journal of Computational and Applied Mathematics*, vol. 222, no. 2, pp. 561–573, 2008.

[33] X. Zhang and W. Ge, "Positive solutions for a class of boundary-value problems with integral boundary conditions," *Computers & Mathematics with Applications*, vol. 58, no. 2, pp. 203–215, 2009.

[34] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cone*, Academic Press, New York, NY, USA, 1988.

[35] K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, NY, USA, 1985.