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Research Article

New Inequalities of Opial's Type on Time Scales and Some of Their Applications

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We will prove some new dynamic inequalities of Opial's type on time scales. The results not only extend some results in the literature but also improve some of them. Some continuous and discrete inequalities are derived from the main results as special cases. The results will be applied on second-order half-linear dynamic equations on time scales to prove several results related to the spacing between consecutive zeros of solutions and the spacing between zeros of a solution and/or its derivative. The results also yield conditions for disfocality of these equations.

1. Introduction

In 1960 Opial [1] proved that if y is an absolutely continuous function on $[a,b]$ with $y(a) = y(b) = 0$, then

$$\int_a^b |y(t)||y'(t)|dt \le \frac{(b-a)}{4}\int_a^b |y'(t)|^2 dt. \tag{1.1}$$

In further simplifying the proof of the Opial inequality which had already been simplified by Olech [2], Beesack [3], Levinson [4], Mallows [5], and Pederson [6], it is proved that if y is real absolutely continuous on $(0,b)$ and with $y(0) = 0$, then

$$\int_0^b |y(t)||y'(t)|dt \le \frac{b}{2}\int_0^b |y'(t)|^2 dt. \tag{1.2}$$

Since the discovery of Opial's inequality much work has been done, and many papers which deal with new proofs, various generalizations, extensions, and their discrete analogues have appeared in the literature. The discrete analogy of the (1.1) has been proved in [7] and the discrete analogy of (1.2) has been proved in [8, Theorem 5.2.2]. It is worth to mention here that many results concerning differential inequalities carry over quite easily to corresponding results for difference inequalities, while other results seem to be completely different from their continuous counterparts.

In recent years, there has been much research activity concerning the qualitative theory of dynamic equations on time scales. It has been created in [9] in order to unify the study of differential and difference equations, and it also extends these classical cases to cases “in between,” for example, to the so-called q-difference equations. The general idea is to prove a result for a dynamic equation or a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. A cover story article in New Scientist [10] discusses several possible applications of time scales. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [11]), that is, when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$ where $q>1$.

One of the main subjects of the qualitative analysis on time scales is to prove some new dynamic inequalities. These on the one hand generalize and on the other hand furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of dynamic equations on time scales. In the following, we recall some results obtained for dynamic inequalities on time scales that serve and motivate the contents of this paper.

Bohner and Kaymakçalan in [12] established the time scale analogy of (1.2) and proved that if $y:[0,b]\cap\mathbb{T}\to\mathbb{R}$ is delta differentiable with $y(0)=0$, then

$$\int_0^h \left|y(t)+y^\sigma(t)\right|\left|y^\Delta(t)\right|\Delta t \le h\int_0^h \left|y^\Delta(t)\right|^2\Delta t. \tag{1.3}$$

Also in [12] the authors proved that if r and q are positive rd-continuous functions on $[0,b]_{\mathbb{T}}$, $\int_0^b(\Delta t/r(t))<\infty$, q nonincreasing and $y:[0,b]\cap\mathbb{T}\to\mathbb{R}$ is delta differentiable with $y(0)=0$, then

$$\int_0^b q^\sigma(t)\left|\left(y(t)+y^\sigma(t)\right)y^\Delta(t)\right|\Delta t \le \int_0^b\frac{\Delta t}{r(t)}\int_0^b r(t)q(t)\left|y^\Delta(t)\right|^2\Delta t. \tag{1.4}$$

Since the discovery of the inequalities (1.3) and (1.4) some papers which deal with new proofs, various generalizations, and extensions of (1.3) and (1.4) have appeared in the literature, we refer to the results in [13–15] and the references cited therein. Karpuz et al. [13] proved an inequality similar to the inequality (1.4) of the form

$$\int_a^b q(t)\left|\left(y(t)+y^\sigma(t)\right)y^\Delta(t)\right|\Delta t \le K_q(a,b)\int_a^b\left|y^\Delta(t)\right|^2\Delta t, \tag{1.5}$$

where q is a positive rd-continuous function on $[a,b]$, and $y:[a,b]\cap\mathbb{T}\to\mathbb{R}$ is delta differentiable with $y(a)=0$, and

$$K_q(a,b)=\left(2\int_a^b q^2(u)(\sigma(u)-a)\Delta u\right)^{1/2}. \tag{1.6}$$

Wong et al. [14] and Sirvastava et al. [15] proved that if r is a positive rd-continuous function on $[a,b]$, we have

$$\int_a^b r(t)\left|y(t)\right|^p\left|y^{\Delta}(t)\right|^q\Delta t \le \frac{q}{p+q}(b-a)^p\int_a^b r(t)\left|y^{\Delta}(t)\right|^{p+q}\Delta t, \tag{1.7}$$

where $y : [a,b] \cap \mathbb{T} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$.

In [16] the author proved that if $y : [a,X]\cap\mathbb{T} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$, then

$$\int_a^X s(x)\left|y(x)\right|^p\left|y^{\Delta}(x)\right|^q\Delta x \le K_1(a,X,p,q)\int_a^X r(x)\left|y^{\Delta}(x)\right|^{p+q}\Delta x, \tag{1.8}$$

where p, q are positive real numbers such that $p+q > 1$, and let r, s be nonnegative rd-continuous functions on $(a,X)_{\mathbb{T}}$ such that $\int_a^X r^{-1/(p+q-1)}(t)\Delta t < \infty$ and

$$\begin{aligned} K_1(a,X,p,q) = {}& \left(\frac{q}{p+q}\right)^{q/(p+q)} \\ & \times\left(\int_a^X (s(x))^{(p+q)/p}(r(x))^{-q/p}\left(\int_a^x (r(t))^{-1/(p+q-1)}\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)}. \end{aligned} \tag{1.9}$$

If $y(a)$ is replaced by $y(b)$, then

$$\int_X^b s(x)\left|y(x)\right|^p\left|y^{\Delta}(x)\right|^q\Delta x \le K_2(X,b,p,q)\int_X^b r(x)\left|y^{\Delta}(x)\right|^{p+q}\Delta x, \tag{1.10}$$

where

$$\begin{aligned} K_2(X,b,p,q) = {}& \left(\frac{q}{p+q}\right)^{q/(p+q)} \\ & \times\left(\int_X^b (s(x))^{(p+q)/p}(r(x))^{-q/p}\left(\int_x^b (r(t))^{-1/(p+q-1)}\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)}. \end{aligned} \tag{1.11}$$

For contributions of different types of inequalities on time scales, we also refer the reader to the papers [16–24] and the references cited therein.

The paper is organized as follows: in Section 2, we will prove some new dynamic inequalities of Opial's type on time scales of the form

$$\int_a^X s(x)\left|y(x)+y^{\sigma}(x)\right|^p\left|y^{\Delta}(x)\right|^q\Delta x \le K(a,X,p,q)\int_a^X r(x)\left|y^{\Delta}(x)\right|^{p+q}\Delta x, \tag{1.12}$$

where p, q are positive real numbers such that $p \leq 1$, $p+q > 1$ and $K(a,X,p,q)$ is the coefficient of the inequality. As special cases, we derive some differential and discrete inequalities on continuous and discrete time scales. In Section 3, we will apply the obtained inequalities in Section 2 on the second-order half-linear dynamic equation

$$\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + q(t)\left(y^{\sigma}(t)\right)^{\gamma} = 0, \quad \text{on } [a,b]_{\mathbb{T}}, \tag{1.13}$$

where $\mathbb{T}$ is an arbitrary time scale, $0 < \gamma \leq 1$ is a quotient of odd positive integers, r and q are real rd-continuous functions defined on $\mathbb{T}$ with $r(t) > 0$. In particular, we will prove several results related to the problems:

(i) obtain lower bounds for the spacing $\beta - \alpha$ where y is a solution of (1.13) and satisfies $y(\alpha) = y^{\Delta}(\beta) = 0$, or $y^{\Delta}(\alpha) = y(\beta) = 0$,

(ii) obtain lower bounds for the spacing between consecutive zeros of solutions of (1.13).

Our motivation comes from that fact that the inequalities obtained in the literature cannot be applied on the half-linear dynamic equation (1.13) to prove results related to the problems (i)-(ii).

2. Main Results

The main inequalities will be proved in this section by making use of the Hölder inequality (see [25, Theorem 6.13])

$$\int_a^h |f(t)g(t)|\Delta t \leq \left[\int_a^h |f(t)|^{\gamma}\Delta t\right]^{1/\gamma}\left[\int_a^h |g(t)|^{\nu}\Delta t\right]^{1/\nu}, \tag{2.1}$$

where $a,\ h \in \mathbb{T}$ and $f;\ g \in C_{\mathrm{rd}}(\mathbb{I},\mathbb{R})$, $\gamma > 1$ and $1/\nu + 1/\gamma = 1$, and the inequality (see [8, page 51])

$$2^{r-1}(a^r + b^r) \leq (a+b)^r \leq (a^r + b^r), \quad \text{where } a,b \geq 0,\ 0 \leq r \leq 1. \tag{2.2}$$

For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} :\ s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} :\ s < t\}, \tag{2.3}$$

where $\sup\emptyset = \inf\mathbb{T}$. A point $t \in \mathbb{T}$ is said to be left dense if $\rho(t) = t$ and $t > \inf\mathbb{T}$, is right dense if $\sigma(t) = t$, is left scattered if $\rho(t) < t$ and right scattered if $\sigma(t) > t$.

A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right dense continuous (rd-continuous) provided g is continuous at right dense points and at left dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T})$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We will assume that $\sup \mathbb{T} = \infty$ and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$.

Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t with

$$\left|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]\right| \leq \epsilon|\sigma(t) - s|, \quad \forall s \in U. \tag{2.4}$$

In this case, we say $x^{\Delta}(t)$ is the (delta) derivative of x at t and that x is (delta) differentiable at t.

We will frequently use the results in the following theorem which is due to Hilger [9]. Assume that $g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.

(i) If g is differentiable at t, then g is continuous at t.

(ii) If g is continuous at t and t is right scattered, then g is differentiable at t with

$$g^{\Delta}(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)}. \tag{2.5}$$

(iii) If g is differentiable and t is right-dense, then

$$g^{\Delta}(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}. \tag{2.6}$$

(iv) If g is differentiable at t, then $g(\sigma(t)) = g(t) + \mu(t) g^{\Delta}(t)$.

In this paper, we will refer to the (delta) integral which we can define as follows: if $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by

$$\int_a^t g(s)\Delta s := G(t) - G(a). \tag{2.7}$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$, here $g^{\sigma} = g \circ \sigma$) of two differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}, \qquad \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}. \tag{2.8}$$

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0,\ t \in \mathbb{T}$. The integration by parts formula is given by

$$\int_a^b f(t) g^{\Delta}(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t) g^{\sigma}(t)\Delta t. \tag{2.9}$$

It can be shown (see [25]) that if $g \in C_{\mathrm{rd}}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^{t} g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t),\ t \in \mathbb{T}$. The integration on discrete time scales is defined by

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} \mu(t) f(t). \tag{2.10}$$

To prove the main results, we need the formula

$$\left(x^{\gamma}(t)\right)^{\Delta} = \gamma \int_0^1 \left[hx^{\sigma} + (1-h)x\right]^{\gamma-1} dh x^{\Delta}(t), \qquad \gamma > 0, \tag{2.11}$$

which is a simple consequence of Keller's chain rule [25, Theorem 1.90]. The books on the subject of time scales by Bohner and Peterson [25, 26] summarize and organize much of time scale calculus and contain some results for dynamic equations on time scales. To admit functions such that y and y^{Δ} may change sign on $(a,b)_{\mathbb{T}}$, we note that if

$$y(x) = \int_a^x y^{\Delta}(x)\Delta x, \quad \text{ then } v(x) = \int_a^x \left|y^{\Delta}(x)\right|\Delta x \geq \left|y(x)\right|, \tag{2.12}$$

with equality if and only if y^{Δ} does not change sign. (The same result holds if $y = -\int_x^b y^{\Delta}(x)\Delta x$ and $v(x) = \int_x^b |y^{\Delta}(x)|\Delta x$.) Now, we are ready to state and prove the main results.

Theorem 2.1. *Let $\mathbb{T}$ be a time scale with $a, X \in \mathbb{T}$ and p, q be positive real numbers such that $p \leq 1$, $p+q > 1$ and let r, s be nonnegative rd-continuous functions on $(a,X)_{\mathbb{T}}$ such that $\int_a^X r^{-1/(p+q-1)}(t)\Delta t < \infty$. If $y : [a,X]\cap\mathbb{T} \to \mathbb{R}^{+}$ is delta differentiable with $y(a) = 0$, then one has*

$$\int_a^X s(x)\left|y(x) + y^{\sigma}(x)\right|^p \left|y^{\Delta}(x)\right|^q \Delta x \leq K_1(a,X,p,q) \int_a^X r(x)\left|y^{\Delta}(x)\right|^{p+q}\Delta x, \tag{2.13}$$

where

$$\begin{aligned} K_1(a,X,p,q) = \sup_{a\leq x\leq X}&\left(\mu^p(x)\frac{s(x)}{r(x)}\right) + 2^p\left(\frac{q}{p+q}\right)^{q/(p+q)} \\ &\times\left(\int_a^X \frac{(s(x))^{(p+q)/p}}{(r(x))^{q/p}}\left(\int_a^x r^{-1/(p+q-1)}(t)\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)}. \end{aligned} \tag{2.14}$$

Proof. Since

$$\left|y(x)\right| \leq \int_a^x \left|y^{\Delta}(t)\right|\Delta t, \quad \text{ for } x \in [a,X]_{\mathbb{T}}, \tag{2.15}$$

and r is nonnegative on $(a, X)_{\mathbb{T}}$, then it follows from the Hölder inequality (2.1) with

$$f(t) = \frac{1}{(r(t))^{1/(p+q)}}, \qquad g(t) = (r(t))^{1/(p+q)}\left|y^{\Delta}(t)\right|, \qquad \gamma = \frac{p+q}{p+q-1}, \qquad \nu = p+q, \tag{2.16}$$

that

$$\int_a^x \left|y^{\Delta}(t)\right| \Delta t \le \left(\int_a^x \frac{1}{(r(t))^{1/(p+q-1)}} \Delta t\right)^{(p+q-1)/(p+q)} \left(\int_a^x r(t)\left|y^{\Delta}(t)\right|^{p+q} \Delta t\right)^{1/(p+q)}. \tag{2.17}$$

Then, for $a \le x \le X$, we get (noting that $y(a) = 0$) that

$$|y(x)|^p \le \left(\int_a^x \frac{1}{(r(t))^{1/(p+q-1)}} \Delta t\right)^{p((p+q-1)/(p+q))} \left(\int_a^x r(t)\left|y^{\Delta}(t)\right|^{p+q} \Delta t\right)^{p/(p+q)}. \tag{2.18}$$

Since $y^{\sigma} = y + \mu y^{\Delta}$, we have

$$y(x) + y^{\sigma}(x) = 2y(x) + \mu y^{\Delta}(x). \tag{2.19}$$

Applying the inequality (2.2), we get (where $p \le 1$) that

$$|y + y^{\sigma}|^p = \left|2y(x) + \mu y^{\Delta}(x)\right|^p \le 2^p|y|^p + \mu^p\left|y^{\Delta}\right|^p. \tag{2.20}$$

Setting

$$z(x) := \int_a^x r(t)\left|y^{\Delta}(t)\right|^{p+q} \Delta t, \tag{2.21}$$

we see that $z(a) = 0$, and

$$z^{\Delta}(x) = r(x)\left|y^{\Delta}(x)\right|^{p+q} > 0. \tag{2.22}$$

From this, we get that

$$\left|y^{\Delta}(x)\right|^{p+q} = \frac{z^{\Delta}(x)}{r(x)}, \qquad \left|y^{\Delta}(x)\right|^{q} = \left(\frac{z^{\Delta}(x)}{r(x)}\right)^{q/(p+q)}. \tag{2.23}$$

Thus, since s is nonnegative on $(a, X)_{\mathbb{T}}$, we have from (2.20) and (2.23) that

$$\begin{aligned} s(x)\left|y(x)+y^{\sigma}(x)\right|^{p}\left|y^{\Delta}(x)\right|^{q} &\leq 2^{p} s(x)|y(x)|^{p}\left|y^{\Delta}(x)\right|^{q}+\mu^{p}(x) s(x)\left|y^{\Delta}\right|^{p+q} \\ &\leq 2^{p} s(x)\left(\frac{1}{r(x)}\right)^{q/(p+q)} \times\left(\int_{a}^{x} \frac{1}{r^{1/(p+q-1)}(t)} \Delta t\right)^{p((p+q-1)/(p+q))} \\ &\quad \times(z(x))^{p/(p+q)}\left(z^{\Delta}(x)\right)^{q/(p+q)}+\mu^{p}(x) s(x)\left(\frac{z^{\Delta}(x)}{r(x)}\right). \end{aligned} \tag{2.24}$$

This implies that

$$\begin{aligned} &\int_{a}^{X} s(x)\left|y(x)+y^{\sigma}(x)\right|^{p}\left|y^{\Delta}(x)\right|^{q} \Delta x \\ &\quad\leq 2^{p} \int_{a}^{X} s(x)\left(\frac{1}{r(x)}\right)^{q/(p+q)} \times\left(\int_{a}^{x} \frac{1}{r^{1/(p+q-1)}(t)} \Delta t\right)^{p((p+q-1)/(p+q))} \\ &\quad\quad \times(z(x))^{p/(p+q)}\left(z^{\Delta}(x)\right)^{q/(p+q)} \Delta x+\int_{a}^{X}\left(\mu^{p} \frac{s(x)}{r(x)}\right) z^{\Delta}(x) \Delta x \\ &\quad\leq 2^{p} \int_{a}^{X} s(x)\left(\frac{1}{r(x)}\right)^{q/(p+q)} \times\left(\int_{a}^{x} \frac{1}{(r(t))^{1/(p+q-1)}} \Delta t\right)^{p((p+q-1)/(p+q))} \\ &\quad\quad \times(z(x))^{p/(p+q)}\left(z^{\Delta}(x)\right)^{q/(p+q)} \Delta x+\sup_{a \leq x \leq X}\left(\mu^{p} \frac{s(x)}{r(x)}\right) \int_{a}^{X} z^{\Delta}(x) \Delta x. \end{aligned} \tag{2.25}$$

Supposing that the integrals in (2.25) exist and again applying the Hölder inequality (2.1) with indices $p+q/p$ and $p+q/q$ on the first integral on the right-hand side of (2.25), we have

$$\begin{aligned} &\int_{a}^{X} s(x)\left|y(x)+y^{\sigma}(x)\right|^{p}\left|y^{\Delta}(x)\right|^{q} \Delta x \\ &\quad\leq 2^{p}\left(\int_{a}^{X}(s(x))^{(p+q)/p}\left(\frac{1}{r(x)}\right)^{q/p}\left(\int_{a}^{x} \frac{1}{(r(t))^{1/(p+q-1)}} \Delta t\right)^{(p+q-1)} \Delta x\right)^{p/(p+q)} \\ &\quad\quad \times\left(\int_{a}^{X}(z(x))^{p/q} z^{\Delta}(x) \Delta x\right)^{q/(p+q)}+\sup_{a \leq x \leq X}\left(\mu^{p} \frac{s(x)}{r(x)}\right) \int_{a}^{X} z^{\Delta}(x) \Delta x. \end{aligned} \tag{2.26}$$

From (2.22), and the chain rule (2.11), we obtain

$$z^{p/q}(x) z^{\Delta}(x) \leq \frac{q}{p+q}\left(z^{(p+q)/q}(x)\right)^{\Delta}. \tag{2.27}$$

Substituting (2.27) into (2.26) and using the fact that $z(a) = 0$, we have that

$$\begin{aligned}
&\int_a^X s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x \\
&\quad\le 2^p\left(\int_a^X (s(x))^{(p+q)/p}\left(\frac{1}{r(x)}\right)^{q/p}\left(\int_a^x \frac{1}{(r(t))^{1/(p+q-1)}}\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)} \\
&\qquad\times\left(\frac{p}{p+q}\right)^{q/(p+q)}\left(\int_a^X \left((z(t))^{(p+q)/q}\right)^\Delta\Delta t\right)^{q/(p+q)}+\left(\mu^p\frac{s(x)}{r(x)}\right)\int_a^X z^\Delta(x)\Delta x \\
&\quad=\left(\int_a^X (s(x))^{(p+q)/p}\left(\frac{1}{r(x)}\right)^{q/p}\left(\int_a^x \frac{1}{(r(t))^{1/(p+q-1)}}\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)} \\
&\qquad\times 2^p\left(\frac{q}{p+q}\right)^{q/(p+q)}z(X)+\sup_{a\le x\le X}\left(\mu^p\frac{s(x)}{r(x)}\right)z(X).
\end{aligned} \tag{2.28}$$

Using (2.21), we have from the last inequality that

$$\int_a^X s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\le K_1(a,X,p,q)\int_a^X r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x, \tag{2.29}$$

which is the desired inequality (2.13). The proof is complete. □

Here, we only state the following theorem, since its proof is the same as that of Theorem 2.1, with $[a, X]$ replaced by $[b, X]$ and $|y(x)| = \int_x^b |y^\Delta(t)|\Delta t$.

Theorem 2.2. *Let $\mathbb{T}$ be a time scale with $X, b \in \mathbb{T}$ and let p, q be positive real numbers such that $p \le 1$, $p + q > 1$ and let r, s be nonnegative rd-continuous functions on $(X, b)_{\mathbb{T}}$ such that $\int_X^b r^{-1/(p+q-1)}(t)\Delta t < \infty$. If $y : [X, b] \cap \mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(b) = 0$, then one has*

$$\int_X^b s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\le K_2(X,b,p,q)\int_X^b r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x, \tag{2.30}$$

where

$$\begin{aligned}
K_2(X,b,p,q) = \sup_{X\le x\le b}\left(\mu^p(x)\frac{s(x)}{r(x)}\right)+2^p\left(\frac{q}{p+q}\right)^{q/(p+q)} \\
\times\left(\int_X^b \frac{(s(x))^{(p+q)/p}}{(r(x))^{q/p}}\left(\int_x^b (r(t))^{-1/(p+q-1)}\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)}.
\end{aligned} \tag{2.31}$$

Note that when $\mathbb{T} = \mathbb{R}$, we have $y^\sigma = y$ and $\mu(x) = 0$. Then from Theorems 2.1 and 2.2 we have the following differential inequalities.

Corollary 2.3. *Assume that $p,\ q$ are positive real numbers such that $p \leq 1,\ p+q>1$ and let $r,\ s$ be nonnegative continuous functions on $(a,X)_{\mathbb{R}}$ such that $\int_a^X (r(t))^{-1/(p+q-1)}dt < \infty$. If $y:[a,X]\cap\mathbb{R} \to \mathbb{R}^+$ is differentiable with $y(a)=0$, then one has*

$$\int_a^X s(x)|y(x)|^p |y'(x)|^q dx \leq C_1(a,X,p,q)\int_a^X r(x)|y'(x)|^{p+q}dx, \tag{2.32}$$

where

$$C_1(a,X,p,q) = 2^p\left(\frac{q}{p+q}\right)^{q/(p+q)} \times \left(\int_a^X \frac{(s(x))^{(p+q)/p}}{(r(x))^{q/p}}\left(\int_a^x (r(t))^{-1/(p+q-1)}dt\right)^{(p+q-1)}dx\right)^{p/(p+q)}. \tag{2.33}$$

Corollary 2.4. *Assume that $p,\ q$ are positive real numbers such that $p \leq 1,\ p+q>1$ and let $r,\ s$ be nonnegative continuous functions on$(X,b)_{\mathbb{R}}$ such that $\int_X^b (r(t))^{-1/(p+q-1)}dt < \infty$. If $y:[X,b]\cap\mathbb{R} \to \mathbb{R}^+$ is delta differentiable with $y(b)=0$, then one has*

$$\int_X^b s(x)|y(x)|^p |y'(x)|^q dx \leq C_2(X,b,p,q)\int_X^b r(x)|y'(x)|^{p+q}dx, \tag{2.34}$$

where

$$C_2(X,b,p,q) = 2^p\left(\frac{q}{p+q}\right)^{q/(p+q)} \times \left(\int_X^b \frac{(s(x))^{(p+q)/p}}{(r(x))^{q/p}}\left(\int_x^b (r(t))^{-1/(p+q-1)}dt\right)^{(p+q-1)}dx\right)^{p/(p+q)}. \tag{2.35}$$

In the following, we assume that there exists $h \in (a,b)$ which is the unique solution of the equation

$$K(p,q) = K_1(a,h,p,q) = K_2(h,b,p,q) < \infty, \tag{2.36}$$

where $K_1(a,h,p,q)$ and $K_2(h,b,p,q)$ are defined as in Theorems 2.1 and 2.2.

Theorem 2.5. *Let $\mathbb{T}$ be a time scale with $a,b \in \mathbb{T}$ and let $p,\ q$ be positive real numbers such that $p \leq 1,\ p+q>1$ and let $r,\ s$ be nonnegative rd-continuous functions on $(a,b)_{\mathbb{T}}$ such that $\int_a^b (r(t))^{-1/(p+q-1)}\Delta t < \infty$. If $y:[a,b]\cap\mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(a)=0=y(b)$, then one has*

$$\int_a^b s(x)|y(x)+y^{\sigma}(x)|^p \left|y^{\Delta}(x)\right|^q \Delta x \leq K(p,q)\int_a^b r(x)\left|y^{\Delta}(x)\right|^{p+q}\Delta x. \tag{2.37}$$

Proof. Since

$$\begin{aligned}\int_a^b s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x &= \int_a^X s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\\&\quad+\int_X^b s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x,\end{aligned}\tag{2.38}$$

then the rest of the proof will be a combination of Theorems 2.1 and 2.2 and hence is omitted. The proof is complete. □

As a special case if $r=s$ in Theorem 2.1, then we obtain the following result.

Corollary 2.6. *Let $\mathbb{T}$ be a time scale with $a,X\in\mathbb{T}$ and let p, q be positive real numbers such that $p\le 1$, $p+q>1$ and let r be a nonnegative rd-continuous function on $(a,X)_{\mathbb{T}}$ such that $\int_a^X(r(t))^{-1/(p+q-1)}\Delta t<\infty$. If $y:[a,X]\cap\mathbb{T}\to\mathbb{R}^+$ is delta differentiable with $y(a)=0$, then one has*

$$\int_a^X r(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\le K_1^*(a,X,p,q)\int_a^X r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x,\tag{2.39}$$

where

$$\begin{aligned}K_1^*(a,X,p,q)&=\sup_{a\le x\le X}\left(\mu^p(x)\right)+2^p\left(\frac{q}{p+q}\right)^{q/(p+q)}\\&\quad\times\left(\int_a^X r(x)\left(\int_a^x r^{-1/(p+q-1)}(t)\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)}.\end{aligned}\tag{2.40}$$

From Theorems 2.2 and 2.5 one can derive similar results by setting $r=s$. The details are left to the reader.

On a time scale $\mathbb{T}$, we note from the chain rule (2.11) that

$$\begin{aligned}\left((t-a)^{p+q}\right)^\Delta&=(p+q)\int_0^1\left[h(\sigma(t)-a)+(1-h)(t-a)\right]^{p+q-1}dh\\&\ge(p+q)\int_0^1\left[h(t-a)+(1-h)(t-a)\right]^{p+q-1}dh\\&=(p+q)(t-a)^{p+q-1}.\end{aligned}\tag{2.41}$$

This implies that

$$\int_a^X(x-a)^{(p+q-1)}\Delta x\le\int_a^X\frac{1}{(p+q)}\left((x-a)^{p+q}\right)^\Delta\Delta x=\frac{(X-a)^{p+q}}{(p+q)}.\tag{2.42}$$

From this and (2.40) (by putting $r(t) = 1$), we get that

$$\begin{aligned} K_1^*(a,X,p,q) &= 2^p\left(\frac{q}{p+q}\right)^{q/(p+q)} \times \left(\int_a^X (x-a)^{(p+q-1)}\Delta x\right)^{p/(p+q)} \\ &\leq 2^p\left(\frac{q}{p+q}\right)^{q/(p+q)}\left(\frac{(X-a)^{p+q}}{(p+q)}\right)^{p/(p+q)} + \max_{a\leq x\leq X}(\mu^p(x)) \\ &= \max_{a\leq x\leq X}(\mu^p(x)) + 2^p\frac{q^{q/(p+q)}}{p+q}(X-a)^p. \end{aligned} \tag{2.43}$$

So setting $r = 1$ in (2.39) and using (2.43), we have the following result.

Corollary 2.7. *Let $\mathbb{T}$ be a time scale with $a, X \in \mathbb{T}$ and let $p,\ q$ be positive real numbers such that $p \leq 1$ and $p+q > 1$. If $y : [a,X]\cap\mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(a) = 0$, then one has*

$$\int_a^X \left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x \leq L(a,X,p,q)\int_a^X \left|y^\Delta(x)\right|^{p+q}\Delta x, \tag{2.44}$$

where

$$L(a,X,p,q) := \left(2^p\frac{q^{q/(p+q)}}{p+q}\times(X-a)^p + \sup_{a\leq x\leq X}\mu^p(x)\right). \tag{2.45}$$

Remark 2.8. Note that when $\mathbb{T} = \mathbb{R}$, we have $y^\sigma = y$, $\mu(x) = 0$ and then the inequality (2.44) becomes

$$\int_a^X |y(x)|^p|y'(x)|^q dx \leq \frac{q^{q/(p+q)}}{(p+q)}\times(X-a)^p\int_a^X |y'(x)|^{p+q}dx. \tag{2.46}$$

Note also that when $p = 1$ and $q = 1$, then the inequality (2.46) becomes

$$\int_a^X |y(x)||y'(x)|dx \leq \frac{(X-a)}{2}\int_a^X |y'(x)|^2dx, \tag{2.47}$$

which is the Opial inequality (1.2).

When $\mathbb{T} = \mathbb{N}$, we have form (2.44) the following discrete Opial's type inequality.

Corollary 2.9. *Assume that $p,\ q$ are positive real numbers such that $p \leq 1,\ p+q > 1$ and $\{r_i\}_{0\leq i\leq N}$ are a nonnegative real sequence. If $\{y_i\}_{0\leq i\leq N}$ is a sequence of positive real numbers with $y(0) = 0$, then*

$$\sum_{n=1}^{N-1} r(n)|y(n)+y(n+1)|^p|\Delta y(n)|^q \leq \left(2^p\frac{q^{q/(p+q)}(N-a)^p}{(p+q)}+1\right)\sum_{n=0}^{N-1} r(n)|\Delta y(n)|^{p+q}. \tag{2.48}$$

The inequality (2.44) has an immediate application to the case where $y(a) = y(b) = 0$. Choose $X = (a+b)/2$ and apply (2.40) to $[a, X]$ and $[X, b]$ and adding we obtain the following inequality.

Corollary 2.10. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$ and let p, q be positive real numbers such that $p \leq 1$ and $p + q > 1$. If $y : [a, b] \cap \mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(a) = 0 = y(b)$, then one has*

$$\int_a^b |y(x) + y^\sigma(x)|^p \left|y^\Delta(x)\right|^q \Delta x \leq F(a, b, p, q) \int_a^b \left|y^\Delta(x)\right|^{p+q} \Delta x, \tag{2.49}$$

where

$$F(a, b, p, q) := \frac{q^{q/(p+q)}}{p+q}(b-a)^p + \sup_{a \leq x \leq b} (\mu^p(x)). \tag{2.50}$$

From this inequality, we have the following discrete Opial type inequality.

Corollary 2.11. *Assume that p, q are positive real numbers such that $p \leq 1$ and $p + q > 1$. If $\{y_i\}_{0 \leq i \leq N}$ is a sequence of real numbers with $y(0) = 0 = y(N)$, then*

$$\sum_{n=1}^{N-1} r(n)|y(n) + y(n+1)|^p |\Delta y(n)|^q \leq \left(\frac{q^{q/(p+q)}}{p+q}(N-a)^p + 1\right) \sum_{n=0}^{N-1} r(n)|\Delta y(n)|^{p+q}. \tag{2.51}$$

By setting $p = q = 1$ in (2.49) we have the following Opial type inequality on a time scale.

Corollary 2.12. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$. If $y : [a, X] \cap \mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(a) = 0 = y(b)$, then one has*

$$\int_a^b |y(x) + y^\sigma(x)| \left|y^\Delta(x)\right| \Delta x \leq \left(\frac{(b-a)}{2} + \sup_{a \leq x \leq b} \mu(x)\right) \int_a^b \left|y^\Delta(x)\right|^2 \Delta x. \tag{2.52}$$

As special cases from (2.52) on the continuous and discrete spaces, that is, when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$, one has the following inequalities.

Corollary 2.13. *If $y : [a, b] \cap \mathbb{T} \to \mathbb{R}$ is differentiable with $y(a) = 0 = y(b)$, then one has the Opial inequality*

$$\int_a^b |y(x)||y'(x)| dx \leq \frac{(b-a)}{4} \int_a^b |y'(x)|^2 dx. \tag{2.53}$$

Corollary 2.14. *If $\{y_i\}_{0 \leq i \leq N}$ is a sequence of real numbers with $y(0) = 0 = y(N)$, then*

$$\sum_{n=1}^{N-1} |y(n) + y(n+1)||\Delta y(n)| \leq \left(\frac{N}{2} + 1\right) \sum_{n=0}^{N-1} |\Delta y(n)|^2. \tag{2.54}$$

3. Applications

Our aim in this section, is to apply the dynamic inequalities of Opial's type proved in Section 2 to prove several results related to the problems (i)-(ii) for the second-order half-linear dynamic equation

$$\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+q(t)\left(y^{\sigma}(t)\right)^{\gamma}=0, \quad \text{on } [a,b]_{\mathbb{T}}, \tag{3.1}$$

on an arbitrary time scale $\mathbb{T}$, where $0<\gamma\leq 1$ is a quotient of odd positive integers, r and q are real rd-continuous functions defined on $\mathbb{T}$ with $r(t)>0$. The terminology half-linear arises because of the fact that the space of all solutions of (3.1) is homogeneous, but not generally additive. Thus, it has just "half" of the properties of a linear space. It is easily seen that if $y(t)$ is a solution of (3.1), then so also is $cy(t)$.

By a solution of (3.1) on an interval $\mathbb{I}$, we mean a nontrivial real-valued function $y\in C_{\mathrm{rd}}(\mathbb{I})$, which has the property that $r(t)y^{\Delta}(t)\in C^{1}_{\mathrm{rd}}(\mathbb{I})$ and satisfies (3.1) on $\mathbb{I}$. We say that a solution y of (3.1) has a generalized zero at t if $y(t)=0$ and has a generalized zero in $(t,\sigma(t))$ in case $y(t)y^{\sigma}(t)<0$ and $\mu(t)>0$. Equation (3.1) is disconjugate on the interval $[t_0,b]_{\mathbb{T}}$, if there is no nontrivial solution of (3.1) with two (or more) generalized zeros in $[t_0,b]_{\mathbb{T}}$.

Equation (3.1) is said to be nonoscillatory on $[t_0,\infty)_{\mathbb{T}}$ if there exists $c\in[t_0,\infty)_{\mathbb{T}}$ such that this equation is disconjugate on $[c,d]_{\mathbb{T}}$ for every $d>c$. In the opposite case (3.1) is said to be oscillatory on $[t_0,\infty)_{\mathbb{T}}$. The oscillation of solutions of (3.1) may equivalently be defined as follows: a nontrivial solution $y(t)$ of (3.1) is called oscillatory if it has infinitely many (isolated) generalized zeros in $[t_0,\infty)_{\mathbb{T}}$; otherwise it is called nonoscillatory. So that the solution $y(t)$ of (3.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. This means that the property of oscillation or nonoscillation is the behavior in the neighborhood of the infinite points.

We say that (3.1) is right disfocal (left disfocal) on $[\alpha,\beta]_{\mathbb{T}}$ if the solutions of (3.1) such that $y^{\Delta}(\alpha)=0(y^{\Delta}(\beta)=0)$ have no generalized zeros in $[\alpha,\beta]_{\mathbb{T}}$.

Note that (3.1) in its general form involves some different types of differential and difference equations depending on the choice of the time scale $\mathbb{T}$. For example when $\mathbb{T}=\mathbb{R}$, (3.1) becomes a second-order half-linear differential equation and $\sigma(t)=t$. When $\mathbb{T}=\mathbb{Z}$, (3.1) becomes a second-order half-linear difference equation and $\sigma(t)=t+1$. When $\mathbb{T}=h\mathbb{N}$, (3.1) becomes a generalized difference equation and $\sigma(t)=t+h$. When $\mathbb{T}=\{t: t=\varrho^{k},\ k\in\mathbb{N}_0,\ \varrho>1\}$, (3.1) becomes a quantum difference equation (see [11]) and $\sigma(t)=\varrho t$. Note also that the results in this paper can be applied on the time scales $\mathbb{T}=\mathbb{N}^2=\{t^2: t\in\mathbb{N}\}$, $\mathbb{T}_2=\{\sqrt{n}: n\in\mathbb{N}_0\}$, $\mathbb{T}_3=\{\sqrt[3]{n}: n\in\mathbb{N}_0\}$, and when $\mathbb{T}=\mathbb{T}_n=\{t_n: n\in\mathbb{N}_0\}$ where $\{t_n\}$ is the set of harmonic numbers. In these cases we see that when $\mathbb{T}=\mathbb{N}_0^2=\{t^2: t\in\mathbb{N}_0\}$, we have $\sigma(t)=(\sqrt{t}+1)^2$ and when $\mathbb{T}=\mathbb{T}_n=\{t_n: n\in\mathbb{N}\}$ where $(t_n\}$ is the harmonic numbers that are defined by $t_0=0$ and $t_n=\sum_{k=1}^{n}1/k,\ n\in\mathbb{N}_0$, we have $\sigma(t_n)=t_{n+1}$. When $\mathbb{T}_2=\{\sqrt{n}: n\in\mathbb{N}_0\}$, we have $\sigma(t)=\sqrt{t^2+1}$, and when $\mathbb{T}_3=\{\sqrt[3]{n}: n\in\mathbb{N}_0\}$, we have $\sigma(t)=\sqrt[3]{t^3+1}$.

Perhaps the best known existence results of types (i)-(ii) for a special case of (3.1) (when $\gamma=1$ and $r(t)=1$) is due to Bohner et al. [27], where they extended the Lyapunov inequality obtained for differential equations in [28]. In particular the authors in [27] considered the dynamic equation

$$y^{\Delta\Delta}(t)+q(t)y^{\sigma}(t)=0, \tag{3.2}$$

where $q(t)$ is a positive rd-continuous function defined on $\mathbb{T}$ and proved that if $y(t)$ is a solution of (3.2) with $y(a) = y(b) = 0$ $(a < b)$, then

$$\int_a^b p(t)\Delta t > \frac{4}{b-a}. \tag{3.3}$$

Karpuz et al. [13] proved that if $q \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$ and $y(t)$ is a solution of (3.2) with $y(a) = y^{\Delta_\sigma}(b) = 0$, then

$$\left(2\int_a^{\sigma(b)} Q^2(u)(\sigma(u)-a)\Delta u\right)^{1/2} \geq 1, \quad \text{where } Q(u) = \int_u^{\sigma(b)} q(t)\Delta t. \tag{3.4}$$

Saker [22] considered the second-order half-linear dynamic equation

$$\left(r(t)\varphi\left(x^{\Delta}\right)\right)^{\Delta} + q(t)\varphi(x^{\sigma}(t)) = 0, \tag{3.5}$$

on an arbitrary time scale $\mathbb{T}$, where $\varphi(u) = |u|^{\gamma-1}u$, $\gamma \geq 1$ is a positive constant, r and q are real rd-continuous positive functions defined on $\mathbb{T}$ and proved that if $x(t)$ is a positive solution of (3.1) which satisfies $x(a) = x(b) = 0$, $x(t) \neq 0$ for $t \in (a,b)$ and $x(t)$ has a maximum at a point $c \in (a,b)$, then

$$\left(\int_a^b \frac{\Delta t}{r^{\gamma}(t)}\right)^{\gamma} \int_a^b q(t)\Delta t \geq 2^{\gamma+1}. \tag{3.6}$$

Of particular interest in this paper is when q is oscillatory which is different from the conditions imposed on q in [12, 22, 27]. The results also yield conditions for disfocality for (3.1) on time scales. As special cases, the results include some results obtained for differential equations and give new results for difference equations on discrete time scales.

Now, we are ready to state and prove the main results in this section. To simplify the presentation of the results, we define

$$\begin{aligned} M(\beta) &:= \sup_{\alpha \leq t \leq \beta} \mu^{\gamma}(t)\frac{|Q(t)|}{r(t)}, \quad \text{where } Q(t) = \int_t^{\beta} q(s)\Delta s, \\ M(\alpha) &:= \sup_{\alpha \leq t \leq \beta} \mu^{\gamma}(t)\frac{|Q(t)|}{r(t)}, \quad \text{where } Q(t) = \int_{\alpha}^{t} q(s)\Delta s. \end{aligned} \tag{3.7}$$

Note that when $\mathbb{T} = \mathbb{R}$, we have $M(\alpha) = 0 = M(\beta)$, and when $\mathbb{T} = \mathbb{Z}$, we have

$$M(\beta) = \sup_{\alpha \leq t \leq \beta} \frac{\left|\sum_{s=t}^{\beta-1} q(s)\right|}{r(t)}, \qquad M(\alpha) = \sup_{\alpha \leq t \leq \beta} \frac{\left|\sum_{s=\alpha}^{t-1} q(s)\right|}{r(t)}. \tag{3.8}$$

Theorem 3.1. *Suppose that y is a nontrivial solution of* (3.1). *If $y(\alpha) = y^{\Delta}(\beta) = 0$, then*

$$\frac{2}{(\gamma+1)^{1/(\gamma+1)}} \times \left(\int_{\alpha}^{\beta} \frac{|Q(x)|^{(\gamma+1)/\gamma}}{r^{1/\gamma}(x)} \left(\int_{\alpha}^{x} \frac{\Delta t}{r^{1/\gamma}(t)}\right)^{\gamma} \Delta x\right)^{\gamma/(\gamma+1)} + 2^{1-\gamma} M(\beta) \geq 1, \tag{3.9}$$

where $Q(t) = \int_t^{\beta} q(s)\Delta s$. If $y^{\Delta}(\alpha) = y(\beta) = 0$, then

$$\frac{2}{(\gamma+1)^{1/(\gamma+1)}} \left(\int_{\alpha}^{\beta} \frac{|Q(x)|^{(\gamma+1)/\gamma}}{r^{1/\gamma}(x)} \left(\int_{x}^{\beta} \frac{\Delta t}{r^{1/\gamma}(t)}\right)^{\gamma} \Delta x\right)^{\gamma/(\gamma+1)} + 2^{1-\gamma} M(\alpha) \geq 1, \tag{3.10}$$

where $Q(t) = \int_{\alpha}^{t} q(s)\Delta s$.

Proof. We prove (3.9). Without loss of generality we may assume that $y(t) > 0$ in $[\alpha,\beta]_{\mathbb{T}}$. Multiplying (3.1) by y^{σ} and integrating by parts, we have

$$\begin{aligned} \int_{\alpha}^{\beta} \left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} y^{\sigma}(t)\Delta t &= r(t)\left(y^{\Delta}(t)\right)^{\gamma} y(t)\Big|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t \\ &= -\int_{\alpha}^{\beta} q(t)\left(y^{\sigma}(t)\right)^{\gamma+1}\Delta t. \end{aligned} \tag{3.11}$$

Using the assumptions that $y(\alpha) = y^{\Delta}(\beta) = 0$ and $Q(t) = \int_t^{\beta} q(s)\Delta s$, we have

$$\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t = \int_{\alpha}^{\beta} q(t)\left(y^{\sigma}(t)\right)^{\gamma+1}\Delta t = -\int_{\alpha}^{\beta} Q^{\Delta}(t)\left(y^{\sigma}(t)\right)^{\gamma+1}\Delta t. \tag{3.12}$$

Integrating by parts the right-hand side (see (2.9)), we see that

$$\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t = -Q(t)\left(y(t)\right)^{\gamma+1}\Big|_{\alpha}^{\beta} + \int_{\alpha}^{\beta} Q(t)\left(y^{\gamma+1}(t)\right)^{\Delta}\Delta t. \tag{3.13}$$

Again using the facts that $y(\alpha) = 0 = Q(\beta)$, we obtain

$$\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^{\gamma+1} dt = \int_{\alpha}^{\beta} Q(t)\left(y^{\gamma+1}(t)\right)^{\Delta} dt. \tag{3.14}$$

Applying the chain rule formula (2.11) and the inequality (2.2), we see that

$$\begin{aligned}\left|\left(y^{\gamma+1}(t)\right)^{\Delta}\right| &\leq (\gamma+1)\int_0^1 \left|hy^{\sigma}(t)+(1-h)y(t)\right|^{\gamma}dh\left|y^{\Delta}(t)\right| \\ &\leq (\gamma+1)\left|y^{\Delta}(t)\right|\int_0^1 \left|hy^{\sigma}(t)\right|^{\gamma}dh+(\gamma+1)\left|y^{\Delta}(t)\right|\int_0^1 \left|(1-h)y(t)\right|^{\gamma}dh \\ &= \left|y^{\Delta}(t)\right|\left|y^{\sigma}(t)\right|^{\gamma}+\left|y^{\Delta}(t)\right|\left|y(t)\right|^{\gamma} \\ &\leq 2^{1-\gamma}\left|y^{\sigma}(t)+y(t)\right|^{\gamma}\left|y^{\Delta}(t)\right|.\end{aligned} \tag{3.15}$$

This and (3.14) imply that

$$\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t \leq 2^{1-\gamma}\int_{\alpha}^{\beta}|Q(t)|\left|y(t)+y^{\sigma}(t)\right|^{\gamma}\left|y^{\Delta}(t)\right|\Delta t. \tag{3.16}$$

Applying the inequality (2.13) with $s(t)=|Q(t)|,\ p=\gamma$ and $q=1$, we have

$$\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t \leq 2^{1-\gamma}K_1(\alpha,\beta,\gamma,1)\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t, \tag{3.17}$$

where

$$\begin{aligned}K_1(\alpha,\beta,\gamma,1) &= M(\beta)+2^{\gamma}\left(\frac{1}{\gamma+1}\right)^{1/(\gamma+1)} \\ &\quad\times\left(\int_{\alpha}^{\beta}|Q(x)|^{(\gamma+1)/\gamma}r^{-1/\gamma}(x)\left(\int_{\alpha}^{x} r^{-1/\gamma}(t)\Delta t\right)^{\gamma}\Delta x\right)^{\gamma/(\gamma+1)}.\end{aligned} \tag{3.18}$$

Then, we have from (3.17) after cancelling the term $\int_{\alpha}^{\beta} r(t)|y^{\Delta}(t)|^{\gamma+1}\Delta t$, that

$$2^{1-\gamma}M(\beta)+\frac{2}{(\gamma+1)^{1/(\gamma+1)}}\times\left(\int_{\alpha}^{\beta}\frac{|Q(x)|^{(\gamma+1)/\gamma}}{r^{1/\gamma}(x)}\left(\int_{\alpha}^{x}\frac{\Delta t}{r^{1/\gamma}(t)}\right)^{\gamma}\Delta x\right)^{\gamma/(\gamma+1)} \geq 1, \tag{3.19}$$

which is the desired inequality (3.9). The proof of (3.10) is similar to (3.9) by using the integration by parts and (2.30) of Theorem 2.2 and (2.31) instead of (2.14). The proof is complete. □

As a special case of Theorem 3.1, when $r(t)=1$, we have the following result.

Corollary 3.2. *Suppose that y is a nontrivial solution of*

$$\left(\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+q(t)\left(y^{\sigma}(t)\right)^{\gamma}=0,\qquad t\in[\alpha,\beta]_{\mathbb{T}}. \tag{3.20}$$

If $y(\alpha) = y^{\Delta}(\beta) = 0$, *then*

$$\frac{2}{(\gamma+1)^{1/(\gamma+1)}} \times \left[\int_{\alpha}^{\beta} |Q(t)|^{(1+\gamma)/\gamma} (t-\alpha)^{\gamma} \Delta t\right]^{\gamma/(\gamma+1)} + 2^{1-\gamma} \sup_{\alpha \le t \le \beta} \left(\mu^{\gamma}(t)|Q(t)|\right) \ge 1, \tag{3.21}$$

where $Q(t) = \int_t^{\beta} q(s)\Delta s$. *If* $y^{\Delta}(\alpha) = y(\beta) = 0$, *then*

$$\frac{2}{(\gamma+1)^{1/(\gamma+1)}} \left[\int_{\alpha}^{\beta} |Q(t)|^{(1+\gamma)/\gamma} (\beta-t)^{\gamma} \Delta t\right]^{\gamma/(\gamma+1)} + 2^{1-\gamma} \sup_{\alpha \le t \le \beta} \left(\mu^{\gamma}(t)|Q(t)|\right) \ge 1, \tag{3.22}$$

where $Q(t) = \int_{\alpha}^{t} q(s)\Delta s$.

As a special case of Theorem 3.1, when $\gamma = 1$, we have the following result.

Corollary 3.3. *Suppose that* y *is a nontrivial solution of*

$$\left(r(t)y^{\Delta}(t)\right)^{\Delta} + q(t)y^{\sigma}(t) = 0, \quad t \in [\alpha,\beta]_{\mathbb{T}}. \tag{3.23}$$

If $y(\alpha) = y^{\Delta}(\beta) = 0$, *then*

$$\sqrt{2}\left(\int_{\alpha}^{\beta} \frac{|Q(x)|^2}{r(x)} \left(\int_{\alpha}^{x} \frac{\Delta t}{r(t)}\right) \Delta x\right)^{1/2} + M(\beta) \ge 1, \tag{3.24}$$

where $Q(t) = \int_t^{\beta} q(s)\Delta s$. *If* $y^{\Delta}(\alpha) = y(\beta) = 0$, *then*

$$\sqrt{2}\left(\int_{\alpha}^{\beta} \frac{|Q(x)|^2}{r(x)} \left(\int_{x}^{\beta} \frac{\Delta t}{r(t)}\right) \Delta x\right)^{1/2} + M(\alpha) \ge 1, \tag{3.25}$$

where $Q(t) = \int_{\alpha}^{t} q(s)\Delta s$.

As a special case when $\mathbb{T} = \mathbb{R}$, we have $M(\alpha) = M(\beta) = 0$ and then the results in Corollary 3.3 reduce to the following results obtained by Brown and Hinton [29] for the second-order differential equation

$$y'' + q(t)y(t) = 0, \quad \alpha \le t \le \beta. \tag{3.26}$$

Corollary 3.4 (see [29]). *If* y *is a solution of* (3.26) *such that* $y(\alpha) = y'(\beta) = 0$, *then*

$$2\int_{\alpha}^{\beta} Q^2(s)(s-\alpha)ds > 1, \tag{3.27}$$

where $Q(t) = \int_t^\beta q(s)ds$. *If instead* $y'(\alpha) = y(\beta) = 0$, *then*

$$2\int_\alpha^\beta Q^2(s)(\beta - s)ds > 1, \tag{3.28}$$

where $Q(t) = \int_\alpha^t q(s)ds$.

Remark 3.5. Note that if $\mathbb{T} = \mathbb{N}$, then $\mu(t) = 1$ and (3.1) (when $r(t) = 1$) becomes

$$\Delta^2 y(n) + q(n)y(n+1) = 0, \tag{3.29}$$

and as a special case of Corollary 3.3, we have the following result.

Corollary 3.6. *If y is a solution of* (3.29) *such that* $y(\alpha) = \Delta y(\beta) = 0$, *then*

$$\sqrt{2}\left(\sum_{n=\alpha}^{\beta-1}(Q(n))^2(n-\alpha)\right)^{1/2} + \sup_{\alpha\le n\le\beta}|Q(n)| > 1, \tag{3.30}$$

where $Q(n) = \sum_{s=n}^{\beta-1} q(s)$. *If instead* $\Delta y(\alpha) = y(\beta) = 0$, *then*

$$\sqrt{2}\left(\sum_{n=\alpha}^{\beta-1}(Q(n))^2(\beta-n)\right)^{1/2} + \sup_{\alpha\le n\le\beta}|Q(n)| > 1, \tag{3.31}$$

where $Q(n) = \sum_{s=\alpha}^{n-1} q(s)$.

Remark 3.7. By using the maximum of $|Q|$ on $[\alpha,\beta]_{\mathbb{T}}$ and

$$\int_a^X (x-a)^{(p+q-1)}\Delta x \le \int_a^X \frac{1}{(p+q)}\left((x-a)^{p+q}\right)^{\Delta}\Delta x = \frac{(X-a)^{p+q}}{(p+q)}, \tag{3.32}$$

in (3.21) and (3.22) with $p = \gamma$ and $q = 1$, we have the following results.

Corollary 3.8. *Suppose that y is a nontrivial solution of* (3.20), *where $\gamma \le 1$ is a quotient of odd positive integers. If* $y(\alpha) = y^\Delta(\beta) = 0$, *then*

$$\frac{2(\beta-\alpha)^\gamma}{(\gamma+1)}\max_{\alpha\le t\le\beta}\left|\int_t^\beta q(s)\Delta s\right| + 2^{1-\gamma}\sup_{\alpha\le t\le\beta}\left(\mu^\gamma(t)\left|\int_t^\beta q(s)\Delta s\right|\right) \ge 1, \tag{3.33}$$

and if $y^\Delta(\alpha) = y(\beta) = 0$, *then*

$$\frac{2(\beta-\alpha)^\gamma}{(\gamma+1)}\max_{\alpha\le t\le\beta}\left|\int_\alpha^t q(s)\Delta s\right| + 2^{1-\gamma}\sup_{\alpha\le t\le\beta}\left(\mu^\gamma(t)\left|\int_\alpha^t q(s)\Delta s\right|\right) \ge 1. \tag{3.34}$$

As a special when $\mathbb{T} = \mathbb{R}$, we have $M(\alpha) = M(\beta) = 0$ and then the results in Corollary 3.8 reduce to the following results for the second-order half-linear differential equation:

$$((y'(t))^{\gamma})' + q(t)(y(t))^{\gamma} = 0, \quad \alpha \leq t \leq \beta, \tag{3.35}$$

where $\gamma \leq 1$ is a quotient of odd positive integers.

Corollary 3.9. *Assume that $\gamma \leq 1$ is a quotient of odd positive integers. Suppose that y is a nontrivial solution of* (3.35)*. If $y(\alpha) = y'(\beta) = 0$, then*

$$\frac{2}{(\gamma+1)}(\beta-\alpha)^{\gamma} \sup_{\alpha \leq t \leq \beta} \left| \int_t^{\beta} q(s)ds \right| \geq 1. \tag{3.36}$$

If instead $y'(\alpha) = y(\beta) = 0$, then

$$\frac{2}{(\gamma+1)}(\beta-\alpha)^{\gamma} \sup_{\alpha \leq t \leq \beta} \left| \int_{\alpha}^{t} q(s)ds \right| \geq 1. \tag{3.37}$$

As a special case of Corollary 3.9 when $\gamma = 1$, we have the following results that has been established by Harris and Kong [30].

Corollary 3.10 (see [30]). *If y is a solution of the equation*

$$y'' + q(t)y(t) = 0, \quad a \leq t \leq b, \tag{3.38}$$

with no zeros in (a, b) and such that $y(a) = y'(b) = 0$, then

$$(b-a) \sup_{a \leq t \leq b} \left| \int_t^{b} q(s)ds \right| > 1. \tag{3.39}$$

If instead $y'(a) = y(b) = 0$, then

$$(b-a) \sup_{a \leq t \leq b} \left| \int_a^{t} q(s)ds \right| > 1. \tag{3.40}$$

As a special when $\mathbb{T} = \mathbb{Z}$, we see that $M(\alpha)$ and $M(\beta)$ are defined as in (3.8) and then the results in Corollary 3.8 reduce to the following results for the second-order half-linear difference equation

$$\Delta((\Delta y(n))^{\gamma}) + q(n)(y(n+1))^{\gamma} = 0, \quad \alpha \leq n \leq \beta, \tag{3.41}$$

where $\gamma \leq 1$ is a quotient of odd positive integers.

Corollary 3.11. *Suppose that y is a nontrivial solution of* (3.41)*, where $\gamma \le 1$ is a quotient of odd positive integers. If $y(\alpha) = \Delta y(\beta) = 0$, then*

$$\frac{2(\beta-\alpha)^{\gamma}}{(\gamma+1)} \max_{\alpha\le n\le\beta}\left|\sum_{s=n}^{\beta-1} q(s)\right| + 2^{1-\gamma}\sup_{\alpha\le n\le\beta}\left(\left|\sum_{s=n}^{\beta-1} q(s)\right|\right) \ge 1, \tag{3.42}$$

and if $\Delta y(\alpha) = y(\beta) = 0$, then

$$\frac{2(\beta-\alpha)^{\gamma}}{(\gamma+1)} \max_{\alpha\le n\le\beta}\left|\sum_{s=\alpha}^{n-1} q(s)\right| + 2^{1-\gamma}\sup_{\alpha\le n\le\beta}\left(\left|\sum_{s=\alpha}^{n-1} q(s)\right|\right) \ge 1. \tag{3.43}$$

Remark 3.12. The above results yield sufficient conditions for disfocality of (3.1), that is, sufficient conditions so that there does not exist a nontrivial solution y satisfying either $y(\alpha) = y^{\Delta}(\beta) = 0$ or $y^{\Delta}(\alpha) = y(\beta) = 0$.

In the following, we employ Theorem 2.5, to determine the lower bound for the distance between consecutive zeros of solutions of (3.1). Note that the applications of the above results allow the use of arbitrary antiderivative Q in the above arguments. In the following, we assume that $Q^{\Delta}(t) = q(t)$ and there exists $h \in (\alpha,\beta)$ which is the unique solution of the equation

$$K_1(\alpha,\beta) = K_1(\alpha,\beta,h) = K_1(\alpha,h,\beta) < \infty, \tag{3.44}$$

where

$$\begin{aligned} K_1(\alpha,\beta,h) &= \frac{2^{\gamma}}{(\gamma+1)^{1/(\gamma+1)}} \times \left(\int_{\alpha}^{\beta}\frac{|Q(x)|^{(\gamma+1)/\gamma}}{r^{1/\gamma}(x)}\left(\int_{\alpha}^{h}\frac{\Delta t}{r^{1/\gamma}(t)}\right)^{\gamma}\Delta x\right)^{\gamma/(\gamma+1)}, \\ K_1(\alpha,h,\beta) &= \frac{2^{\gamma}}{(\gamma+1)^{1/(\gamma+1)}}\left(\int_{\alpha}^{\beta}\frac{|Q(x)|^{(\gamma+1)/\gamma}}{r^{1/\gamma}(x)}\left(\int_{h}^{\beta}\frac{\Delta t}{r^{1/\gamma}(t)}\right)^{\gamma}\Delta x\right)^{\gamma/(\gamma+1)}. \end{aligned} \tag{3.45}$$

Theorem 3.13. *Assume that $Q^{\Delta}(t) = q(t)$ and y is a nontrivial solution of* (3.1)*. If $y(\alpha) = y(\beta) = 0$, then*

$$K_1(\alpha,\beta) \ge 1, \tag{3.46}$$

where $K_1\ (\alpha,\beta)$ is defined as in (3.44)*.*

Proof. Multiplying (3.1) by $y^{\sigma}(t)$, proceed as in Theorem 3.1 and use $y(\alpha) = y(\beta) = 0$, to get

$$\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t = \int_{\alpha}^{\beta} q(t)(y(t))^{\gamma+1}\Delta t = \int_{\alpha}^{\beta} Q^{\Delta}(t)(y^{\sigma}(t))^{\gamma+1}\Delta t. \tag{3.47}$$

Integrating by parts the right hand side (see (2.9)), we see that

$$\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t = Q(t)(y(t))^{\gamma+1}\Big|_{\alpha}^{\beta} + \int_{\alpha}^{\beta}(-Q(t))\left(y^{\gamma+1}(t)\right)^{\Delta}\Delta t. \tag{3.48}$$

Again using the facts that $y(\alpha) = 0 = y(\beta)$, we obtain

$$\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t \leq \int_{\alpha}^{\beta}|Q(t)|\left|y(t) + y^{\sigma}(t)\right|^{\gamma}\left|y^{\Delta}(t)\right|\Delta t. \tag{3.49}$$

Applying the inequality (2.37) with $s(t) = |Q(t)|,\ p = \gamma$ and $q = 1$, we have

$$\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}dt \leq 2^{1-\gamma}K_1(\alpha,\beta)\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t. \tag{3.50}$$

From this inequality, after cancelling $\int_{\alpha}^{\beta}|y^{\Delta}(t)|^{\gamma+1}\Delta t$, we get the desired inequality (3.46). This completes the proof. □

Problem 1. It would be interesting to extend the above results to cover the delay equation with oscillatory coefficients

$$\left(r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta} + q(t)|x(\tau(t))|^{\gamma-1}x(\tau(t)) = 0, \tag{3.51}$$

where the delay function $\tau(t)$ satisfies $\tau(t) < t$ and $\lim_{t\to\infty}\tau(t) = \infty$.

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