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Research Article

Stochastically Perturbed Epidemic Model with Time Delays

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We investigate a stochastic epidemic model with time delays. By using Liapunov functionals, we obtain stability conditions for the stochastic stability of endemic equilibrium.

1. Introduction

In [1], Zhen et al. introduced a deterministic SIRS model

$$\begin{aligned}\dot{S}(t) &= b - \mu S(t) - \beta S(t)\int_0^h f(s)I(t-s)ds + \alpha R(t),\\ \dot{I}(t) &= \beta S(t)\int_0^h f(s)I(t-s)ds - (\mu + c + \lambda)I(t),\\ \dot{R}(t) &= \lambda I(t) - (\mu+\alpha)R(t),\end{aligned} \tag{1.1}$$

where $S(t)$ is the number of susceptible population, $I(t)$ is the number of infective members and $R(t)$ is the number of recovered members. b is the rate at which population is recruited, μ is the death rate for classes $S(t)$, $I(t)$, and $R(t)$, c is the disease-induced death rate, β is the transmission rate, λ is the recovery rate, and α is the loss of immunity rate. Equation (1.1) represents an SIRS model with epidemics spreading via a vector, whose incubation time period is a distributed parameter over the interval $[0, h]$. $h \in \mathbb{R}^+$ is the limit superior of incubation time periods in the vector population. The $f(s)$ is usually nonnegative and continuous and is the distribution function of incubation time periods among the vectors and $\int_0^h f(s)ds = 1$.

To be more general, the following model is formulated:

$$\dot{S}(t) = b - \mu_1 S(t) - \beta S(t) \int_0^h f(s) I(t-s) ds + \alpha R(t),$$
$$\dot{I}(t) = \beta S(t) \int_0^h f(s) I(t-s) ds - (\mu_2 + \lambda) I(t), \tag{1.2}$$
$$\dot{R}(t) = \lambda I(t) - (\mu_3 + \alpha) R(t).$$

The positive constants μ_1, μ_2, and μ_3 represent the death rates of susceptibles, infectives, and recovered, respectively. It is natural biologically to assume that $\mu_1 < \min\{\mu_2, \mu_3\}$. If $\alpha = 0$, model (1.2) was considered in [2–5]. For $\alpha = 0$ and fixed delay, the global asymptotic stability of (1.2) was considered in [6].

The basic reproduction number for (1.2) is

$$R_0 = \frac{\beta b}{\mu_1(\mu_2 + \lambda)}. \tag{1.3}$$

If $R_0 \leq 1$, the system (1.2) has just one disease-free equilibrium $E_0 = (b/\mu_1, 0, 0)$; otherwise, if $R_0 > 1$, the disease-free equilibrium E_0 is still present, but there is also a unique positive endemic equilibrium $E^* = (S^*, I^*, R^*)$, given by $S^* = (\mu_2 + \lambda)/\beta$, $I^* = (b(\mu_3 + \alpha)(R_0 - 1))/(R_0[\mu_2(\mu_3 + \alpha) + \mu_3\lambda])$, $R^* = (\lambda/(\mu_3 + \alpha))I^*$.

2. Stability Analysis of the Atochastic Delay Model

Since environmental fluctuations have great influence on all aspects of real life, then it is natural to study how these fluctuations affect the epidemiological model (1.2). We assume that stochastic perturbations are of white noise type and that they are proportional to the distances of S, I, R from S^*, I^*, R^*, respectively. Then the system (1.2) will be reduced to the following form:

$$\dot{S}(t) = b - \mu_1 S(t) - \beta S(t) \int_0^h f(s) I(t-s) ds + \alpha R(t) + \sigma_1 (S(t) - S^*) \dot{w}_1(t),$$
$$\dot{I}(t) = \beta S(t) \int_0^h f(s) I(t-s) ds - (\mu_2 + \lambda) I(t) + \sigma_2 (I(t) - I^*) \dot{w}_2(t), \tag{2.1}$$
$$\dot{R}(t) = \lambda I(t) - (\mu_3 + \alpha) R(t) + \sigma_3 (R(t) - R^*) \dot{w}_3(t).$$

Here, σ_1, σ_2, and σ_3 are constants, and $w(t) = (w_1(t), w_2(t), w_3(t))$ represents a three-dimensional standard Wiener processes.

This system has the same equilibria as system (1.2). We assume that $R_0 > 1$; we discuss the stability of the endemic equilibrium E^* of (2.1). The stochastic system (2.1) can be centered at its endemic equilibrium E^* by the changes of variables $x_1 = S - S^*$, $x_2 = I - I^*$, $x_3 = R - R^*$. By this way, we obtain

$$\dot{x}_1 = -(\beta I^* + \mu_1)x_1 - \beta x_1 \int_0^h f(s)x_2(t-s)ds - \beta S^* \int_0^h f(s)x_2(t-s)ds + \alpha x_3 + \sigma_1 x_1 \dot{w}_1(t),$$
$$\dot{x}_2 = \beta I^* x_1 - \beta S^* x_2 + \beta x_1 \int_0^h f(s)x_2(t-s)ds + \beta S^* \int_0^h f(s)x_2(t-s)ds + \sigma_2 x_2 \dot{w}_2(t),$$
$$\dot{x}_3 = \lambda x_2 - (\mu_3 + \alpha)x_3 + \sigma_3 x_3 \dot{w}_3(t). \tag{2.2}$$

In order to investigate the stability of endemic equilibrium of system (2.1), we study the stability of the trivial solution of system (2.2).

First, consider the stochastic functional differential equation

$$dy(t) = h(t, y_t)dt + g(t, y_t)dw(t), \quad t \geq 0, \; y_0 = \varphi \in H. \tag{2.3}$$

Let $\{\Omega, \sigma, P\}$ be the probability space, $\{f_t, t \geq 0\}$ the family of σ-algebra, $f_t \in \sigma$, H the space of f_0-adapted functions $\varphi(s) \in R^n$, $s \leq 0$, $\|\varphi\| = \sup_{s\leq 0}|\varphi(s)|$, $w(t)$ the m-dimensional f_t-adapted Wiener process, $h(t, y_t)$ the n-dimensional vector, and $g(t, y_t)$ the $n \times m$-dimensional matrix, both defined for $t \geq 0$. We assume that (2.3) has a unique global solution $y(t;\varphi)$ and that $h(t,0) = g(t,0) \equiv 0$. Then, (2.3) has the trivial solution $y(t) \equiv 0$ corresponding to the initial condition $y_0 = 0$.

Definition 2.1. The trivial solution of (2.3) is said to be stochastically stable if, for every $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\delta > 0$ such that

$$P\{|y(t;\varphi)| > r, \, t \geq 0\} \leq \varepsilon \tag{2.4}$$

for any initial condition $\varphi \in H$ satisfying $P\{\|\varphi\| \leq \delta\} = 1$.

Definition 2.2. The trivial solution of (2.3) is said to be mean square stable if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $E|y(t;\varphi)|^2 < \varepsilon$ for any $t \geq 0$ provided that $\sup_{s\leq 0} E|\varphi(s)|^2 < \delta$.

Definition 2.3. The trivial solution of (2.3) is said to be asymptotically mean square stable if it is mean square stable and $\lim_{t\to\infty} E|y(t;\varphi)|^2 = 0$.

The differential operator associated to (2.3) is defined by the formula

$$LV(t,\varphi) = \limsup_{\Delta \to 0} \frac{E_{t,\varphi} V(t+\Delta, y_{t+\Delta}) - V(t,\varphi)}{\Delta}, \tag{2.5}$$

where $y(s)$, $s \geq t$ is the solution of (2.3) with initial condition $y_t = \varphi \in H$, and $V(t,\varphi)$ is a functional defined for $t \geq 0$.

If $V(t,\varphi) = V(t,\varphi(0),\varphi(s))$, $s < 0$, we can define the function $V_\varphi(t,y) = V(t,\varphi) = V(t,y_t) = V(t,y,y(t+s))$, $s < 0$, $\varphi = y_t$, $y = \varphi(0) = y(t)$. Let us define $C_{1,2}$ as a class of function $V(t,\varphi)$ so that for almost all $t \geq 0$, the first and second derivatives with respect to y of $V_\varphi(t,y)$ are continuous, and the first derivative with respect to t is continuous and bounded. Then the generating operator L of (2.3) is defined by

$$LV(t,y_t)=\frac{\partial V_\varphi(t,y)}{\partial t}+h^T(t,y_t)\frac{\partial V_\varphi(t,y)}{\partial y}+\frac{1}{2}\text{trace}\left[g^T(t,y_t)\frac{\partial^2 V_\varphi(t,y)}{\partial y^2}g(t,y_t)\right]. \tag{2.6}$$

The following theorems [7] contain conditions under which the trivial solution of (2.3) is asymptotically mean square stable and stochastically stable.

Theorem 2.4. *If there exist a functional $V(t,\varphi)\in C_{1,2}$ such that*

$$c_1E|y(t)|^2\le EV(t,y_t)\le c_2\sup_{s\le 0}E|y(t+s)|^2,\quad ELV(t,y_t)\le -c_3E|y(t)|^2 \tag{2.7}$$

for $c_i>0$, $i=1,2,3$. Then, the trivial solution of (2.3) *is asymptotically mean square stable.*

Theorem 2.5. *Let there exist a functional $V(t,\varphi)\in C_{1,2}$ such that*

$$c_1|y(t)|^2\le V(t,y_t)\le c_2\sup_{s\le 0}|y(t+s)|^2,\quad LV(t,y_t)\le 0 \tag{2.8}$$

for $c_i>0$, $i=1,2$ and for any $\varphi\in H$ such that $P\{\|\varphi\|\le\delta\}=1$, where $\delta>0$ is sufficiently small. Then, the trivial solution of (2.3) *is stochastically stable.*

Consider the linear part of (2.2)

$$\begin{aligned}\dot{y}_1&=-(\beta I^*+\mu_1)y_1-\beta S^*\int_0^h f(s)y_2(t-s)ds+\alpha y_3+\sigma_1y_1\dot{w}_1(t),\\ \dot{y}_2&=\beta I^*y_1-\beta S^*y_2+\beta S^*\int_0^h f(s)y_2(t-s)ds+\sigma_2y_2\dot{w}_2(t),\\ \dot{y}_3&=\lambda y_2-(\mu_3+\alpha)y_3+\sigma_3y_3\dot{w}_3(t).\end{aligned} \tag{2.9}$$

Theorem 2.6. *Assume that $R_0>1$ and the parameters of system* (2.2) *satisfy conditions*

$$\begin{gathered}0\le\sigma_1^2<2\mu_1-\frac{\alpha(1+q)}{q},\\ 0\le\sigma_2^2<\frac{q(2\beta S^*-\alpha)}{1+q}=\frac{q[2(\mu_2+\lambda)-\alpha]}{1+q},\\ 0\le\sigma_3^2<2\mu_3+\alpha-\lambda,\\ \sqrt{\frac{2\alpha q}{2\mu_3+\alpha-\lambda-\sigma_3^2}}<\min\left\{\frac{(2\mu_1-\sigma_1^2)q-\alpha(1+q)}{\beta S^*},p^*\right\},\end{gathered} \tag{2.10}$$

where $p^*=(-\beta S^*+\sqrt{((\beta S^*)^2-4\lambda[(1+q)\sigma_2^2-2q\beta S^*+q\alpha])})/2\lambda$. Then, the trivial solution of system (2.9) *is asymptotically mean square stable.*

Proof. Set

$$V_1 = py_1^2 + y_2^2 + p^2y_3^2 + q(y_1 + y_2)^2 \tag{2.11}$$

for some $p > 0$ and $q > 0$. Let L be the generating operator of the system (2.9), then

$$\begin{aligned}
LV_1 &= \left[-(\beta I^* + \mu_1)y_1 - \beta S^* \int_0^h f(s)y_2(t-s)ds + \alpha y_3\right][2py_1 + 2q(y_1 + y_2)] \\
&\quad + \left[\beta I^* y_1 - \beta S^* y_2 + \beta S^* \int_0^h f(s)y_2(t-s)ds\right][2y_2 + 2q(y_1 + y_2)] \\
&\quad + 2p^2y_3[\lambda y_2 - (\mu_3 + \alpha)y_3] + (p+q)\sigma_1^2y_1^2 + 2q\sigma_1\sigma_2y_1y_2 + (1+q)\sigma_2^2y_2^2 + p^2\sigma_3^2y_3^2 \\
&= \left[\left(\sigma_1^2 - 2\mu_1\right)(p+q) - 2p\beta I^*\right]y_1^2 + (1+q)\left(\sigma_2^2 - 2\beta S^*\right)y_2^2 \\
&\quad + p^2\left[\sigma_3^2 - 2(\mu_3 + \alpha)\right]y_3^2 + 2\alpha(p+q)y_1y_3 + 2\left(q\alpha + p^2\lambda\right)y_2y_3 \\
&\quad + 2[(\sigma_1\sigma_2 - \beta S^* - \mu_1)q + \beta I^*]y_1y_2 + 2\beta S^*(y_2 - py_1)\int_0^h f(s)y_2(t-s)ds.
\end{aligned} \tag{2.12}$$

Let

$$q = \frac{\beta I^*}{\beta S^* + \mu_1 - \sigma_1\sigma_2}. \tag{2.13}$$

Since $\sigma_1\sigma_2 \le (\sigma_1^2 + \sigma_2^2)/2 < \mu_1 + \beta S^*$, it means that $q > 0$. By using the inequality $2|uv| \le u_1^2 + u_2^2$ and $2\alpha py_1y_3 \le \alpha p(y_1^2/p + py_3^2) = \alpha y_1^2 + \alpha p^2y_3^2$, we find that

$$\begin{aligned}
LV_1 &\le \left[\left(\sigma_1^2 - 2\mu_1\right)q + \alpha(1+q) + p\beta S^*\right]y_1^2 \\
&\quad + \left[(1+q)\left(\sigma_2^2 - 2\beta S^*\right) + q\alpha + p^2\lambda + \beta S^*\right]y_2^2 \\
&\quad + \left[p^2\left(\sigma_3^2 - 2\mu_3 - \alpha\right) + 2\alpha q + p^2\lambda\right]y_3^2 \\
&\quad + (1+p)\beta S^* \int_0^h f(s)y_2^2(t-s)ds.
\end{aligned} \tag{2.14}$$

We now choose the functional V_2 to eliminate the term with delay

$$V_2 = (1+p)\beta S^* \int_0^h f(s) \int_{t-s}^t y_2^2(\tau)d\tau ds. \tag{2.15}$$

Then for functional $V = V_1 + V_2$, we obtain

$$\begin{aligned} LV \leq & \left[\left(\sigma_1^2 - 2\mu_1\right)q + \alpha(1+q) + p\beta S^*\right]y_1^2 \\ & + \left[p^2\lambda + p\beta S^* + (1+q)\sigma_2^2 - 2q\beta S^* + q\alpha\right]y_2^2 \\ & + \left[p^2\left(\sigma_3^2 - 2\mu_3 - \alpha + \lambda\right) + 2\alpha q\right]y_3^2. \end{aligned} \tag{2.16}$$

If the first condition of (2.10) holds, then $(\sigma_1^2 - 2\mu_1)q + \alpha(1+q) < 0$. Set $F(p) = p^2\lambda + p\beta S^* + (1+q)\sigma_2^2 - 2q\beta S^* + q\alpha$, and if the second condition of (2.10) is true, then $F(0) < 0$, thus $F(p) = 0$ has one positive root $p^* = (-\beta S^* + \sqrt{((\beta S^*)^2 - 4\lambda[(1+q)\sigma_2^2 - 2q\beta S^* + q\alpha])})/2\lambda$, for any $0 < p < p^*$, $F(p) < 0$. From (2.10), there exists a $p > 0$, such that

$$\sqrt{\frac{2\alpha q}{2\mu_3 + \alpha - \lambda - \sigma_3^2}} < p < \min\left\{\frac{(2\mu_1 - \sigma_1^2)q - \alpha(1+q)}{\beta S^*}, p^*\right\}. \tag{2.17}$$

Therefore, there exists a $c > 0$ such that $LV \leq -c|y|^2$, where $y = (y_1, y_2, y_3)$. From Theorem 2.4, we can conclude that the zero solution of system (2.9) is asymptotically mean square stable. The theorem is proved. □

Remark 2.7. If $\alpha = 0$, then the system (2.1) becomes an SIR model, which has been discussed in [8]. The conditions (2.10) of Theorem 2.6 reduce to

$$0 \leq \sigma_1^2 < 2\mu_1, \qquad 0 \leq \sigma_2^2 < \frac{2q(\mu_2 + \lambda)}{1+q}, \qquad 0 \leq \sigma_3^2 < 2\mu_3 - \lambda. \tag{2.18}$$

The constant p in the proof of Theorem 2.6 is $0 < p < \min\{((2\mu_1 - \sigma_1^2)q)/(\beta S^*), p_1^*\}$ with $p_1^* = (-\beta S^* + \sqrt{((\beta S^*)^2 - 4\lambda[(1+q)\sigma_2^2 - 2q\beta S^*])})/2\lambda$. The first two conditions in (2.18) are the same as those in Theorem 7 of [8]. Since for $\alpha > 0$, we use different inequality to zoom up the term $2(q\alpha + p^2\lambda)y_2y_3$, then the third condition in (2.18) is different from that in Theorem 7 of [8].

Theorem 2.8. *Assume that $R_0 > 1$ and that conditions* (2.10) *are satisfied. Then the trivial solution of system* (2.2) *is stochastically stable.*

The proof is omitted because of the fact that the initial system (2.2) has a nonlinearity order more than one, then the conditions sufficient for asymptotic mean square stability of the trivial solution of the linear part of this system are sufficient for stochastic stability of the trivial solution of the initial system [9, 10]. Thus, if the conditions (2.10) hold, then the trivial solution of system (2.2) is stochastically stable.

3. Conclusions

In this paper, we have extended the well-known SIRS epidemic model with time delays by introducing a white noise term in it. We want to examine how environmental fluctuations affect the stability of system (1.2). By constructing Liapunov functional, we obtain sufficient conditions for the stochastic stability of the endemic equilibrium E^*. Our main results extend the corresponding results in paper [8], which discussed an SIR epidemic model.

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