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Research Article

Analysing Social Epidemics by Delayed Stochastic Models

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We investigate the dynamics of a delayed stochastic mathematical model to understand the evolution of the alcohol consumption in Spain. Sufficient condition for stability in probability of the equilibrium point of the dynamic model with aftereffect and stochastic perturbations is obtained via Kolmanovskii and Shaikhet general method of Lyapunov functionals construction. We conclude that alcohol consumption in Spain will be constant (with stability) in time with around 36.47% of nonconsumers, 62.94% of nonrisk consumers, and 0.59% of risk consumers. This approach allows us to emphasize the possibilities of the dynamical models in order to study human behaviour.

1. Introduction

In this paper, we propose a mathematical framework to model social epidemics. To be precise, we propose delayed and stochastic consideration on mathematical models to analyze human behaviors related to addictions.

Hereditary systems or systems with delays are very popular in researches (see, e.g., [1–4] and the references therein). In this paper, a nonlinear dynamic alcohol consumption model [5] is generalized via adding distributed delays. Sufficient conditions for existence of the positive equilibrium points of this system are obtained. It is also supposed that this nonlinear system is exposed to additive stochastic perturbations of white noise that are directly proportional to the deviation of the system current state from the equilibrium point. Such type of stochastic perturbations first was proposed in [6, 7] and successfully used later in [8–11]. One of the important point of this assumption is that the equilibrium point is the solution of the stochastic system too. In this case, the influence of the stochastic perturbations on the considered system is small enough in the neighbourhood of the equilibrium point and big enough if the system state is far enough from the equilibrium point.

The considered nonlinear system is linearized in the neighborhood of the positive point of equilibrium, and sufficient condition for asymptotic mean square stability of the zero solution of the constructed linear system is obtained via Kolmanovskii and Shaikhet general method of Lyapunov functionals construction (GMLFC) that is used for stability investigation of stochastic functional-differential and difference equations [12–22]. Since the order of nonlinearity more than 1, this condition is also sufficient one [23, 24] for stability in probability of the initial nonlinear system by stochastic perturbations.

This way of stability investigation was successfully used for investigation of different mathematical models of systems with delays: SIR epidemic model [6], predator-prey model [7, 11], Nicholson blowflies model [9], inverted pendulum [25–27].

The present paper is organized as follows. Section 2 presents the delayed model including the study of its equilibrium points. Stochastics characteristics of the delayed model are shown in Section 3. Section 4 is related to the study of stability of the equilibrium point of the delayed stochastics model proposed. The main conclusions derived from the study are presented in the last section.

2. Delayed Mathematical Model

Taking into account the proposal presented by Rosenquist et al. in [28], we consider alcohol consumption habit as susceptible to be transmitted by peer pressure or social contact. This fact lead us to propose an epidemiological type mathematical model to study this social epidemic.

Let $A(t)$ be nonconsumers, individuals that have never consumed alcohol or they infrequently have alcohol consumption, and $M(t)$ nonrisk consumers, individuals with regular low consumption; to be precise, men who consume less than 50 cc of alcohol every day and women who consume less than 30 cc of alcohol every day. Let $R(t)$ be risk consumers, individuals with regular high consumption, that is, men who consume more than 50 cc of alcohol every day and women who consume more than 30 cc of alcohol every day.

Considering homogeneous mixing [29], that is, each individual can contact with any other individual (peer pressure), dynamic alcohol consumption model is given by the following nonlinear system of ordinary differential equations with distributed delay:

$$\dot{A}(t) = \mu P(t) + \gamma R(t) - d_A A(t) - \beta A(t) \int_0^\infty f(s) \left(\frac{M(t-s) + R(t-s)}{P(t-s)} \right) ds, \tag{2.1}$$

$$\dot{M}(t) = \beta A(t) \int_0^\infty f(s) \left(\frac{M(t-s) + R(t-s)}{P(t-s)} \right) ds - dM(t) - \alpha M(t), \tag{2.2}$$

$$\dot{R}(t) = \alpha M(t) - \gamma R(t) - dR(t), \tag{2.3}$$

$$P(t) = A(t) + M(t) + R(t), \tag{2.4}$$

where

(i) μ: birth rate in Spain;

(ii) γ: rate at which a risk consumer becomes a nonconsumer;

(iii) d_A: death rate in Spain;

(iv) β: transmission rate due to social pressure to increase alcohol consumption (family, friends, marketing, TV, etc.);

(v) d: augmented death rate due to alcohol consumption. Accidents at work, traffic accidents and diseases derived by alcohol consumption are considered. The information available to calculate the augmented death rate due to alcohol consumption is aggregate. This fact does not allow us to consider the difference between the augmented death rate in nonrisk and risk consumers;

(vi) α: rate at which a nonrisk consumer moves to the risk consumption subpopulation.

It is supposed that the parameters $\alpha, \beta, \gamma, \mu, d_A, d$ and the function $f(s)$ are nonnegative and the following condition holds:

$$\int_0^{\infty} f(s)ds = 1. \tag{2.5}$$

In the particular case $f(s) = \delta(s - h)$, where $h > 0$, $\delta(s)$ is Dirac's function, system (2.1) is a system with discrete delay h. The case of a system without delay ($h = 0$) is considered in [5].

It is assumed that when a nonconsumer individual is *infected* by alcohol consumers, there is a time s during which the alcohol consumption habit develops in nonconsumer and it is only after that time that nonconsumer individual becomes nonrisk consumer. It is also assumed that at any time t alcohol consumers population is simply proportional to alcohol consumers population at time $t - s$. Let $f(s)$ be the fraction of nonconsumer population that takes time s to become alcohol consumers (nonrisk consumer).

2.1. Normalization of the Delayed Model

Put

$$a(t) = \frac{A(t)}{P(t)}, \qquad m(t) = \frac{M(t)}{P(t)}, \qquad r(t) = \frac{R(t)}{P(t)}. \tag{2.6}$$

From (2.1) and (2.6), it follows that

$$a(t) + m(t) + r(t) = 1. \tag{2.7}$$

Adding the first three equations (2.1) by virtue of (2.7), we obtain

$$\frac{\dot{P}(t)}{P(t)} = \mu - d + (d - d_A)a(t). \tag{2.8}$$

It is easy to see that

$$\begin{aligned} \dot{a}(t) &= \frac{\dot{A}(t)P(t) - A(t)\dot{P}(t)}{P^2(t)} = \frac{\dot{A}(t)}{P(t)} - \frac{A(t)}{P(t)} \times \frac{\dot{P}(t)}{P(t)} \\ &= \frac{\dot{A}(t)}{P(t)} - a(t)\left[\mu - d + (d - d_A)a(t)\right], \end{aligned} \tag{2.9}$$

and similarly

$$\begin{aligned} \dot{m}(t) &= \frac{\dot{M}(t)}{P(t)} - m(t)\left[\mu - d + (d - d_A)a(t)\right], \\ \dot{r}(t) &= \frac{\dot{R}(t)}{P(t)} - r(t)\left[\mu - d + (d - d_A)a(t)\right]. \end{aligned} \tag{2.10}$$

Thus, putting

$$I(a_t) = \int_0^\infty f(s)a(t-s)ds, \tag{2.11}$$

from (2.1), (2.5), (2.7), (2.9), (2.10), we have

$$\begin{aligned} \dot{a}(t) &= \mu + \gamma r(t) + \beta a(t)I(a_t) - a(t)\left[\beta + \mu - (d - d_A)(1 - a(t))\right], \\ \dot{m}(t) &= \beta a(t) - \beta a(t)I(a_t) - m(t)\left[\alpha + \mu + (d - d_A)a(t)\right], \\ \dot{r}(t) &= \alpha m(t) - r(t)\left[\gamma + \mu + (d - d_A)a(t)\right]. \end{aligned} \tag{2.12}$$

Via (2.7), the last equation can be rejected, and, as a result, we obtain the system of two differential equations

$$\begin{aligned} \dot{a}(t) &= \mu + \gamma - \gamma m(t) + \beta a(t)I(a_t) - a(t)\left[\beta + \mu + \gamma - (d - d_A)(1 - a(t))\right], \\ \dot{m}(t) &= \beta a(t) - \beta a(t)I(a_t) - m(t)\left[\alpha + \mu + (d - d_A)a(t)\right]. \end{aligned} \tag{2.13}$$

2.2. Existence of the Equilibrium Point

Via (2.5), (2.7), (2.13), the point of equilibrium (a^*, m^*, r^*) is defined by the system of the algebraic equations:

$$(\mu + \gamma)(1 - a^*) = a^*(\beta - d + d_A)(1 - a^*) + \gamma m^*, \tag{2.14}$$

$$\beta a^*(1 - a^*) = m^*\left[\alpha + \mu + (d - d_A)a^*\right], \tag{2.15}$$

$$1 = a^* + m^* + r^*. \tag{2.16}$$

Lemma 2.1. *If $d \in [d_A, \beta + d_A)$, then system* (2.14) *has the unique positive solution* (a^*, m^*, r^*) *if and only if*

$$\beta > d - d_A + \mu + \frac{\alpha\gamma}{\alpha + \gamma + \mu + d - d_A}. \tag{2.17}$$

If $d \geq \beta + d_A$, then system (2.14) *does not have positive solutions.*

Proof. From two first equations of system (2.14), we have

$$\mu + \gamma - a^*(\beta - d + d_A) = \frac{\beta\gamma a^*}{\alpha + \mu + (d - d_A)a^*}, \tag{2.18}$$

or

$$\begin{gathered} Q(a^*)^2 + Ba^* - C = 0, \\ B = (\beta - d + d_A)(\alpha + \gamma + \mu) - \mu(d - d_A), \\ Q = (\beta - d + d_A)(d - d_A), \qquad C = (\mu + \alpha)(\mu + \gamma). \end{gathered} \tag{2.19}$$

Thus, via (2.14), the equilibrium point (a^*, m^*, r^*) is defined by the system of the algebraic equations (2.19) and

$$m^* = \frac{\beta a^*(1 - a^*)}{\alpha + \mu + (d - d_A)a^*}, \qquad r^* = 1 - a^* - m^*. \tag{2.20}$$

It is easy to check that, by condition $d \in [d_A, \beta + d_A)$ (or $Q \geq 0$), the existence of the solution a^* of (2.19) in the interval $(0, 1)$ is equivalent to the condition $C < Q + B$ that is equivalent to (2.17). If $d \geq \beta + d_A$, then $Q \leq 0$ and $B < 0$. So, (2.19) cannot have positive roots. The proof is completed. □

Example 2.2. Consider the values of the parameters $\alpha, \beta, \gamma, \mu, d, d_A$ from [5]:

$$\begin{gathered} \alpha = 0.000110247, \qquad \beta = 0.0284534, \qquad \gamma = 0.00144, \\ \mu = 0.01, \qquad d = 0.009, \qquad d_A = 0.008. \end{gathered} \tag{2.21}$$

Then, condition (2.17) for these values of the parameters holds, and the solution of systems (2.19) and (2.20) is

$$a^* = 0.3647389407, \qquad m^* = 0.6293831151, \qquad r^* = 0.005877944497 \tag{2.22}$$

or in the percents $a^* = 36.47\%$, $m^* = 62.94\%$, $r^* = 0.59\%$.

3. Stochastic Perturbations, Centralization, and Linearization

Let us suppose that system (2.13) is exposed to stochastic perturbations type of white noise $(\dot{w}_1(t), \dot{w}_2(t))$, which are directly proportional to the deviation of system (2.13) state $(a(t), m(t))$ from the point (a^*, m^*), that is,

$$\begin{aligned}\dot{a}(t) &= \mu + \gamma - \gamma m(t) + \beta a(t) I(a_t) - a(t)\left[\beta + \mu + \gamma - (d - d_A)(1 - a(t))\right] \\ &\quad + \sigma_1 (a(t) - a^*)\dot{w}_1(t), \\ \dot{m}(t) &= \beta a(t) - \beta a(t) I(a_t) - m(t)\left[\alpha + \mu + (d - d_A)a(t)\right] + \sigma_2(m(t) - m^*)\dot{w}_2(t).\end{aligned} \tag{3.1}$$

Here, $w_1(t), w_2(t)$ are the mutually independent standard Wiener processes, the stochastic differential equations of system (3.1) are understanding in Ito sense [30].

To centralize system (3.1) in the equilibrium point, put now

$$x_1(t) = a(t) - a^*, \qquad x_2(t) = m(t) - m^*. \tag{3.2}$$

Then, from (3.1), it follows that

$$\begin{aligned}\dot{x}_1(t) &= \mu + \gamma - \gamma(m^* + x_2(t)) + \beta(a^* + x_1(t))(a^* + I(x_{1t})) \\ &\quad - (a^* + x_1(t))\left[\beta + \mu + \gamma - (d - d_A)(1 - a^* - x_1(t))\right] + \sigma_1 x_1(t)\dot{w}_1(t), \\ \dot{x}_2(t) &= \beta(a^* + x_1(t)) - \beta(a^* + x_1(t))(a^* + I(x_{1t})) \\ &\quad - (m^* + x_2(t))\left[\alpha + \mu + (d - d_A)(a^* + x_1(t))\right] + \sigma_2 x_2(t)\dot{w}_2(t),\end{aligned} \tag{3.3}$$

or

$$\begin{aligned}\dot{x}_1(t) &= \mu(1 - a^*) + \gamma(1 - a^* - m^*) - a^*(1 - a^*)\left(\beta - d + d_A\right) - \mu x_1(t) \\ &\quad + \gamma(-x_1(t) - x_2(t)) + x_1(t)(1 - 2a^*)(d - d_A) - \beta x_1(t)(1 - a^*) + \beta a^* I(x_{1t}) \\ &\quad - x_1^2(t)(d - d_A) + \beta x_1(t) I(x_{1t}) + \sigma_1 x_1(t)\ \dot{w}_1(t), \\ \dot{x}_2(t) &= \beta a^*(1 - a^*) - m^*\left[\alpha + \mu + (d - d_A)a^*\right] + \beta x_1(t)(1 - a^*) \\ &\quad - m^* x_1(t)(d - d_A) - \beta a^* I(x_{1t}) - x_2(t)\left[\alpha + \mu + (d - d_A)a^*\right] \\ &\quad - \beta x_1(t) I_1(x_{1t}) - x_2(t)x_1(t)(d - d_A) + \sigma_2 x_2(t)\dot{w}_2(t).\end{aligned} \tag{3.4}$$

Via (2.14) from (2.16), it follows that

$$\begin{aligned}\dot{x}_1(t) &= -a_{11}x_1(t) - \gamma x_2(t) + \beta a^* I(x_{1t}) \\ &\quad + \beta x_1(t) I(x_{1t}) - (d - d_A)x_1^2(t) + \sigma_1 x_1(t)\dot{w}_1(t), \\ \dot{x}_2(t) &= a_{21}x_1(t) - a_{22}x_2(t) - \beta a^* I(x_{1t}) \\ &\quad - \beta x_1(t) I(x_{1t}) - x_1(t)x_2(t)(d - d_A) + \sigma_2 x_2(t)\dot{w}_2(t),\end{aligned} \tag{3.5}$$

where

$$a_{11} = \mu + \gamma + (\beta - d + d_A)(1 - a^*) + (d - d_A)a^*,$$
$$a_{21} = \frac{\beta(\alpha + \mu)(1 - a^*)}{\alpha + \mu + (d - d_A)a^*}, \qquad a_{22} = \alpha + \mu + (d - d_A)a^*. \tag{3.6}$$

Note that, for $d \in [d_A, \beta + d_A)$, the numbers a_{11}, a_{21}, a_{22} are positive.

Rejecting the nonlinear terms in (3.5), we obtain the linear part of system (3.5)

$$\dot{x}_1(t) = -a_{11}x_1(t) - \gamma x_2(t) + \beta a^* I(x_{1t}) + \sigma_1 x_1(t)\dot{w}_1(t),$$
$$\dot{x}_2(t) = a_{21}x_1(t) - a_{22}x_2(t) - \beta a^* I(x_{1t}) + \sigma_2 x_2(t)\dot{w}_2(t). \tag{3.7}$$

4. Stability of the Equilibrium Point

Note that nonlinear system (3.5) has the order of nonlinearity more than 1. Thus, as it follows from [23, 24], sufficient conditions for asymptotic mean square stability of the zero solution of linear part (3.7) of nonlinear system (3.5) at the same time are sufficient conditions for stability in probability of the zero solution of nonlinear system (3.5) and therefore are sufficient conditions for stability in probability of the solution (a^*, m^*) of system (3.1).

To get sufficient conditions for asymptotic mean square stability of the zero solution of system (3.7), rewrite this system in the form

$$\dot{x}(t) = Ax(t) + B(x_t) + \sigma(x(t))\dot{w}(t), \tag{4.1}$$

where

$$x(t) = (x_1(t), x_2(t))', \qquad w(t) = (w_1(t), w_2(t))',$$
$$B(x_t) = (\beta a^* I(x_t), -\beta a^* I(x_t))',$$
$$A = \begin{pmatrix} -a_{11} & -\gamma \\ a_{21} & -a_{22} \end{pmatrix}, \qquad \sigma(x(t)) = \begin{pmatrix} \sigma_1 x_1(t) & 0 \\ 0 & \sigma_2 x_2(t) \end{pmatrix}. \tag{4.2}$$

Definition 4.1. The trivial solution of (2.20) is called as follows.

(i) *Mean square stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x(t,\phi)|^2 < \varepsilon$ for any $t \geq 0$ provided that the initial function $x(s) = \phi(s)$, $s \leq 0$, satisfies the condition $\sup_{s\leq 0} \mathbf{E}|\phi(s)|^2 < \delta$.

(ii) *Asymptotically mean square stable* if it is mean square stable and satisfies the condition $\lim_{t\to\infty} \mathbf{E}|x(t,\phi)|^2 = 0$ for each initial function ϕ.

(iii) *Stable in probability* if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that the solution $x(t,\phi)$ of (2.20) satisfies the condition $\mathbf{P}\{\sup_{t\geq 0}|x(t,\phi)| > \varepsilon_1/\mathfrak{F}_0\} < \varepsilon_2$ for any initial function ϕ such that $\mathbf{P}\{\sup_{s\leq 0}|\phi(s)| \leq \delta\} = 1$.

Following the GMLFC [12–14] for stability investigation of (4.1), consider the auxiliary equation without memory

$$\dot{y}(t) = Ay(t) + \sigma(y(t))\dot{w}(t), \tag{4.3}$$

and the matrix equation

$$A'P + PA + P_\sigma = -C, \tag{4.4}$$

where

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}, \qquad P_\sigma = \begin{pmatrix} p_{11}\sigma_1^2 & 0 \\ 0 & p_{22}\sigma_2^2 \end{pmatrix}, \qquad C = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.5}$$

$c > 0$, the matrix A is defined in (3.6), (4.2).

If matrix equation (4.4) has a positive definite solution P, then the function $v(y) = y'Py$ is a Lyapunov function for (4.3).

Note that matrix equation (4.4) can be represented as the system of the equations

$$\begin{aligned} 2(-p_{11}a_{11} + p_{12}a_{21}) + p_{11}\sigma_1^2 &= -c, \\ 2(-p_{12}\gamma - p_{22}a_{22}) + p_{22}\sigma_2^2 &= -1, \\ -p_{11}\gamma - p_{12}(a_{11} + a_{22}) + p_{22}a_{21} &= 0, \end{aligned} \tag{4.6}$$

with the solution

$$p_{11} = \frac{c + 2a_{21}p_{12}}{2A_{11}}, \qquad p_{12} = \frac{a_{21}A_{11} - c\gamma A_{22}}{2A}, \qquad p_{22} = \frac{1 - 2\gamma p_{12}}{2A_{22}}, \tag{4.7}$$

where

$$\begin{gathered} A_{ii} = a_{ii} - \delta_i, \quad \delta_i = \frac{1}{2}\sigma_i^2, \quad i = 1,2, \\ A = A_{11}A_{12}A_{22} + \gamma a_{21}(A_{11} + A_{22}), \quad A_{12} = a_{11} + a_{22}. \end{gathered} \tag{4.8}$$

Lemma 4.2. *If $\delta_1 < a_{11}$ and $\delta_2 < a_{22}$, that is,*

$$A_{11} > 0, \qquad A_{22} > 0, \tag{4.9}$$

then the matrix $P = \|p_{ij}\|$ with entries (4.7) is a positive definite one.

Proof. Note that via (4.9) $\gamma a_{21}A_{ii} < A$, $i = 1,2$. Thus, from (4.7), it follows that for arbitrary $c > 0$

$$p_{12} < \frac{a_{21}A_{11}}{2A} < \frac{1}{2\gamma}, \qquad c + 2a_{21}p_{12} = c\left(1 - \frac{\gamma a_{21}A_{22}}{A}\right) + \frac{a_{21}^2 A_{11}}{A} > 0. \tag{4.10}$$

Therefore, via (4.9), $p_{11} > 0$, $p_{22} > 0$. Let us show that $p_{11}p_{22} > p_{12}^2$. Really, from

$$\frac{(c + 2a_{21}p_{12})(1 - 2\gamma p_{12})}{4A_{11}A_{22}} > p_{12}^2, \tag{4.11}$$

it follows that $4Bp_{12}^2 - 2(a_{21} - c\gamma)p_{12} < c$ by $B = A_{11}A_{22} + \gamma a_{21}$. Substituting into this inequality p_{12} from (4.7), we have $B(a_{21}A_{11} - c\gamma A_{22})^2 - A(a_{21} - c\gamma)(a_{21}A_{11} - c\gamma A_{22}) < cA^2$ or

$$c^2\gamma^2 A_{22}(A - BA_{22}) + cA_{11}A_{22}(AA_{12} + 2\gamma a_{21}B) + a_{21}^2 A_{11}(A - BA_{11}) > 0. \tag{4.12}$$

Since $BA_{ii} < A$, $i = 1, 2$, the obtained inequality holds for arbitrary $c > 0$. Thus, for arbitrary $c > 0$, the matrix P with entries (4.7) is a positive definite one. The proof is completed. □

Theorem 4.3. *If conditions* (4.9) *hold and for some $c > 0$ the matrix P entries* (4.7) *satisfy the condition*

$$(\beta a^* |p_{12} - p_{22}|)^2 + 2\beta a^* |p_{11} - p_{12}| < c \tag{4.13}$$

then the solution (a^*, m^*) of system (3.1) *is stable in probability.*

Proof. Note that the order of nonlinearity of system (3.1) is more than one. Therefore, via [23, 24] to get for this system conditions of stability in probability, it is enough to get conditions for asymptotic mean square stability of the zero solution of linear part (3.7) of this system. Following the GMLFC [13–15], we will construct a Lyapunov functional for system (3.7) in the form $V = V_1 + V_2$, where $V_1 = x'Px$, $x = (x_1, x_2)'$, P is a positive definite solution of system (4.6) with entries (4.7) and V_2 will be chosen below.

Let L be the infinitesimal operator [30] of system (3.7). Then, via (3.7), (4.6),

$$\begin{aligned} LV_1 &= 2(p_{11}x_1(t) + p_{12}x_2(t))(-a_{11}x_1(t) - \gamma x_2(t) + \beta a^* I(x_{1t})) + p_{11}\sigma_1^2 x_1^2(t) \\ &\quad + 2(p_{12}x_1(t) + p_{22}x_2(t))(a_{21}x_1(t) - a_{22}x_2(t) - \beta a^* I(x_{1t})) + p_{22}\sigma_2^2 x_2^2(t) \\ &= -cx_1^2(t) - x_2^2(t) + 2\beta a^* [(p_{11} - p_{12})x_1(t) + (p_{12} - p_{22})x_2(t)] I(x_{1t}). \end{aligned} \tag{4.14}$$

Via (2.11), (2.5), we have $2x_1(t)I(x_{1t}) \le x_1^2(t) + I(x_{1t}^2)$ and $2x_2(t)I(x_{1t}) \le \alpha x_2^2(t) + \alpha^{-1}I(x_{1t}^2)$ for some $\alpha > 0$. Using these inequalities, we obtain

$$\begin{aligned} LV_1 &\le -cx_1^2(t) - x_2^2(t) + \beta a^* |p_{11} - p_{12}| \left(x_1^2(t) + I\left(x_{1t}^2\right)\right) \\ &\quad + \beta a^* |p_{12} - p_{22}| \left(\alpha x_2^2(t) + \alpha^{-1} I\left(x_{1t}^2\right)\right) \\ &= (\beta a^* |p_{11} - p_{12}| - c)x_1^2(t) + (\beta a^* |p_{12} - p_{22}|\alpha - 1)x_2^2(t) + qI\left(x_{1t}^2\right), \end{aligned} \tag{4.15}$$

where

$$q = \beta a^* \left(|p_{11} - p_{12}| + |p_{12} - p_{22}|\alpha^{-1}\right). \tag{4.16}$$

Putting

$$V_2 = q\int_0^\infty f(s)\int_{t-s}^t x_1^2(\theta)d\theta\, ds, \tag{4.17}$$

via (2.5), (2.11), we have $LV_2 = q(x_1^2(t) - I(x_{1t}^2))$. Therefore, via (4.15), (4.16) for the functional $V = V_1 + V_2$, we have

$$LV \leq \left(2\beta a^*\left|p_{11} - p_{12}\right| + \beta a^*\left|p_{12} - p_{22}\right|\alpha^{-1} - c\right)x_1^2(t) + \left(\beta a^*\left|p_{12} - p_{22}\right|\alpha - 1\right)x_2^2(t). \tag{4.18}$$

Thus, if

$$2\beta a^*\left|p_{11} - p_{12}\right| + \beta a^*\left|p_{12} - p_{22}\right|\alpha^{-1} < c, \quad \beta a^*\left|p_{12} - p_{22}\right|\alpha < 1, \tag{4.19}$$

then the zero solution of system (3.7) is [3] asymptotically mean square stable. From (4.19), it follows that

$$\frac{\beta a^*\left|p_{12} - p_{22}\right|}{c - 2\beta a^*\left|p_{11} - p_{12}\right|} < \alpha < \frac{1}{\beta a^*\left|p_{12} - p_{22}\right|}. \tag{4.20}$$

Thus, if for some $c > 0$ condition (4.13) holds, then there exists $\alpha > 0$ such that conditions (4.20) (or (4.19)) hold too, and therefore the zero solution of system (3.7) is asymptotically mean square stable. From here and [23, 24], it follows that the zero solution of system (3.5) and therefore the equilibrium point of system (3.1) is stable in probability. The proof is completed. □

Example 4.4. Consider system (3.1) with the value of the parameters α, β, γ, μ, d, d_A and the equilibrium point (a^*, m^*, r^*) given in (2.21) and (2.22) and the levels of noises $\sigma_1 = 0.028969$, $\sigma_2 = 0.142252$. We consider this value for σ_1 and σ_2 as an example.

From (3.6), it follows that the values of system (3.7) parameters are $a_{11} = 0.029245$, $a_{21} = 0.017446$, $a_{22} = 0.010475$, and conditions (4.9), hold: $\delta_1 = 0.00042 < a_{11}$, $\delta_2 = 0.010118 < a_{22}$. Put $c = 20$. Then, via (4.7), $p_{11} = 477.4438$, $p_{12} = 215.6615$, $p_{22} = 530.4124$ and condition (4.13) holds: $(\beta a^*|p_{12} - p_{22}|)^2 + 2\beta a^*|p_{11} - p_{12}| = 16.1036 < 20$. Thus, system (3.1) solution is stable in probability.

Example 4.5. Consider system (3.1) with the previous values of all parameters except for the levels of noises that are $\sigma_1 = 0.0075$, $\sigma_2 = 0.0077$. These values of σ_1 and σ_2 are selected taking into account sample errors of the monitoring of the epidemic (alcohol consumption) in Spain in the period 199720100332007. More details about these sample errors are shown in [31].

Then, conditions (4.9) hold: $\delta_1 = 0.000028 < a_{11}$, $\delta_2 = 0.000030 < a_{22}$. Put $c = 4$. Then, via (4.7), $p_{11} = 78.6856$, $p_{12} = 17.1347$, $p_{22} = 45.5060$, and condition (4.13) holds: $(\beta a^*|p_{12} - p_{22}|)^2 + 2\beta a^*|p_{11} - p_{12}| = 1.36 < 4$. Thus, system (3.1) solution is stable in probability.

Let us get now two corollaries from Theorem 4.3 which simplify verification of stability condition (4.13). Put

$$B_0 = \frac{A_{22}(A_{12}+\gamma)+\gamma a_{21}}{2A}, \qquad B_1 = \frac{a_{21}(a_{21}-A_{11})}{2A},$$
$$D_0 = \frac{\gamma(A_{22}+\gamma)}{2A}, \qquad D_1 = \frac{A_{11}(A_{12}-a_{21})+\gamma a_{21}}{2A},$$
$$f(c) = (\beta a^*)^2(D_0c+D_1)^2 + 2\beta a^*|B_0c+B_1| - c, \tag{4.21}$$
$$S = (\beta a^* D_0)^2\left(\frac{D_1}{D_0}-\frac{B_1}{B_0}\right)^2 + \frac{B_1}{B_0},$$
$$R = 2\beta a^* B_0\left(\frac{1-2\beta a^* B_0}{2(\beta a^* D_0)^2} - \frac{D_1}{D_0} + \frac{B_1}{B_0}\right), \qquad Q = \frac{1}{4(\beta a^* D_0)^2} - \frac{D_1}{D_0} - \frac{B_0^2}{D_0^2}.$$

Remark 4.6. From (4.7) and (4.21), it follows that $p_{22}-p_{12} = D_0c+D_1$ and $p_{11}-p_{12} = B_0c+B_1$. Thus, condition (4.13) is equivalent to the condition $f(c) < 0$.

Corollary 4.7. *If conditions* (4.9) *hold and* $S < 0$, *then the solution of system* (3.1) *is stable in probability.*

Proof. Note that from $S < 0$ follows that $B_1 < 0$. Putting $c_0 = -B_1/B_0 > 0$, we obtain $f(c_0) = S < 0$, that is, condition (4.13) holds. The proof is completed. □

Corollary 4.8. *If conditions* (4.9) *hold and* $0 \le R < Q$, *then the solution of system* (3.1) *is stable in probability.*

Proof. Let us suppose that $B_0c + B_1 \ge 0$. Then, the minimum of the function $f(c)$ is reached by

$$c_0 = 1 - \frac{2\beta a^* B_0}{2(\beta a^* D_0)^2} - \frac{D_1}{D_0} \ge -\frac{B_1}{B_0}. \tag{4.22}$$

Substituting c_0 into the function $f(c)$, we obtain that the condition $f(c_0) < 0$ is equivalent to the condition $0 \le R < Q$. The proof is completed. □

Example 4.9. Consider system (3.1) with the values of the parameters from Example 4.4. Calculating S, R, Q, we obtain, $S = 4.56 > 0$, $R = 740.0 < Q = 1313.59$. From Corollary 4.8, it follows that the solution of system (3.1) is stable in probability.

Example 4.10. Consider system (3.1) with the values of the parameters from Example 4.5. Calculating S, R, Q, we obtain $S = -0.38 < 0$, $R = 2499.07 < Q = 4708.10$. From both Corollary 4.7 and Corollary 4.8 it follows that the solution of system (3.1), is stable in probability.

5. Conclusions

In this work, a modelling approach based on delayed and stochastic differential equations is proposed to understand social behaviours and their evolutionary trends. Taking into account this approach, we can know how social habits can evolve in the future. Considering the study proposed related to the alcohol consumption habit, we note that around 63.5% of Spanish population is (and will be) alcohol consumers (risk or nonrisk consumers). The existence of a equilibrium point (stable in probability) in $a^* = 36.47\%$ (a: nonconsumers), $m^* = 62.94\%$ (m: nonrisk consumers), $r^* = 0.59\%$ (r: riskconsumers), see (2.22), allows us to confirm it.

This mathematical approach is an example as to how delayed and stochastic models can also be an useful tool to model human behaviour. We consider that this approach can be an interesting framework for public health authorities and policy makers.

Note also that the system (2.1) with delay is considered here first. The conditions for the existence of the equilibrium point of the considered system by stochastic perturbations are new result too. The main result of the paper is the conditions for stability in probability of the equilibrium point by stochastic perturbations. The here proposed research method can be used for stability investigation of other important models.

References

[1] V. B. Kolmanovskiĭ and A. D. Myshkis, *Applied Theory of Functional-Differential Equations*, Kluwer Academic, Boston, Mass, USA, 1992.

[2] V. B. Kolmanovskiĭ and A. D. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Academic, Dodrecht, The Netherlands, 1999.

[3] V. B. Kolmanovskiĭ and V. R. Nosov, *Stability of Functional-Differential Equations*, Academic Press, New York, NY, USA, 1986.

[4] V. B. Kolmanovskiĭ and L. E. Shaĭkhet, *Control of Systems with Aftereffect*, vol. 157 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1996.

[5] F.-J. Santonja, E. Sánchez, M. Rubio, and J.-L. Morera, "Alcohol consumption in Spain and its economic cost: a mathematical modeling approach," *Mathematical and Computer Modelling*, vol. 52, no. 7-8, pp. 999–1003, 2010.

[6] E. Beretta, V. B. Kolmanovskiĭ, and L. E. Shaĭkhet, "Stability of epidemic model with time delays influenced by stochastic perturbations," *Mathematics and Computers in Simulation*, vol. 45, no. 3-4, pp. 269–277, 1998.

[7] L. E. Shaĭkhet, "Stability of predator-prey model with aftereffect by stochastic perturbation," *Stability and Control*, vol. 1, no. 1, pp. 3–13, 1998.

[8] M. Bandyopadhyay and J. Chattopadhyay, "Ratio-dependent predator-prey model: effect of environmental fluctuation and stability," *Nonlinearity*, vol. 18, no. 2, pp. 913–936, 2005.

[9] N. Bradul and L. E. Shaĭkhet, "Stability of the positive point of equilibrium of Nicholson's blowflies equation with stochastic perturbations: numerical analysis," *Discrete Dynamics in Nature and Society*, Article ID 92959, 25 pages, 2007.

[10] M. Carletti, "On the stability properties of a stochastic model for phage-bacteria interaction in open marine environment," *Mathematical Biosciences*, vol. 175, no. 2, pp. 117–131, 2002.

[11] L. E. Shaĭkhet, "Stability of a positive point of equilibrium of one nonlinear system with aftereffect and stochastic perturbations," *Dynamic Systems and Applications*, vol. 17, no. 1, pp. 235–253, 2008.

[12] V. B. Kolmanovskiĭ and L. E. Shaĭkhet, "About one application of the general method of Lyapunov functionals construction," *International Journal of Robust and Nonlinear Control*, vol. 13, no. 9, pp. 805–818, 2003.

[13] V. B. Kolmanovskiĭ and L. E. Shaĭkhet, "About some features of general method of Lyapunov functionals construction," *Stability and Control*, vol. 6, no. 1, pp. 49–76, 2004.

[14] V. B. Kolmanovskiĭ and L. E. Shaĭkhet, "A method for constructing Lyapunov functionals for stochastic differential equations of neutral type," *Differentialniye Uravneniya*, vol. 31, no. 11, pp. 1851–1857, 1995 (Russian), Translation in Differential Equations vol. 31, no. 11, pp. 1819–1825, 1995.

[15] V. B. Kolmanovskiĭ and L. E. Shaĭkhet, "A method for constructing Lyapunov functionals for stochastic systems with aftereffect," *Differentialniye Uravneniya*, vol. 29, no. 11, pp. 1909–1920, 1993 (Russian), Translation in Differential Equations vol. 29, no. 11, pp. 1657–1666, 1993.

[16] V. B. Kolmanovskiĭ and L. E. Shaĭkhet, "Construction of Lyapunov functionals for stochastic hereditary systems: a survey of some recent results," *Mathematical and Computer Modelling*, vol. 36, no. 6, pp. 691–716, 2002.

[17] V. B. Kolmanovskiĭ and L. E. Shaĭkhet, "General method of Lyapunov functionals construction for stability investigation of stochastic difference equations," in *Dynamical Systems and Applications*, vol. 4, pp. 397–439, 1995.

[18] V. B. Kolmanovskiĭ and L. E. Shaĭkhet, "New results in stability theory for stochastic functional-differential equations (SFDEs) and their applications," in *Proceedings of the Dynamic Systems and Applications*, vol. 1, pp. 167–171, Atlanta, Ga, USA, 1993.

[19] V. B. Kolmanovskiĭ and L. E. Shaĭkhet, "Some peculiarities of the general method of Lyapunov functionals construction," *Applied Mathematics Letters*, vol. 15, no. 3, pp. 355–360, 2002.

[20] V. B. Kolmanovskiĭ and L. E. Shaĭkhet, "Stability of stochastic systems with aftereffect," *Avtomatika i Telemekhanika*, no. 7, pp. 66–85, 1993 (Russian), Translation in Automation and Remote Control vol. 54, no. 7, part 1, pp.1087–1107, 1993.

[21] L. E. Shaĭkhet, "Modern state and development perspectives of Lyapunov functionals method in the stability theory of stochastic hereditary systems," *Theory of Stochastic Processes*, vol. 2, no. 18 (1-2), pp. 248–259, 1996.

[22] L. E. Shaĭkhet, "Necessary and sufficient conditions of asymptotic mean square stability for stochastic linear difference equations," *Applied Mathematics Letters*, vol. 10, no. 3, pp. 111–115, 1997.

[23] L. E. Shaĭkhet, "Stability in probability of nonlinear stochastic hereditary systems," *Dynamic Systems and Applications*, vol. 4, no. 2, pp. 199–204, 1995.

[24] L. E. Shaĭkhet, "Stability in probability of nonlinear stochastic systems with delay," *Matematicheskie Zametki*, vol. 57, no. 1, pp. 142–146, 1995 (Russian), Translation in Math Notes vol. 57, no. 1-2, pp. 103–106, 1995.

[25] P. Borne, V. B. Kolmanovskiĭ, and L. E. Shaĭkhet, "Steady-state solutions of nonlinear model of inverted pendulum," in *Proceedings of the 3rd Ukrainian-Scandinavian Conference in Probability Theory and Mathematical Statistics*, Kyiv, Ukraine, June 1999.

[26] P. Borne, V. B. Kolmanovskiĭ, and L. E. Shaĭkhet, "Steady-state solutions of nonlinear model of inverted pendulum," *Theory of Stochastic Processes*, vol. 5, no. 21 (1-2), pp. 203–209, 1999.

[27] P. Borne, V. Kolmanovskiĭ, and L. E. Shaĭkhet, "Stabilization of inverted pendulum by control with delay," *Dynamic Systems and Applications*, vol. 9, no. 4, pp. 501–514, 2000.

[28] J. N. Rosenquist, J. Murabito, J. H. Fowler, and N. A. Christakis, "The spread of alcohol consumption behavior in a large social network," *Annals of Internal Medicine*, vol. 152, no. 7, pp. 426–433, 2010.

[29] J. D. Murray, *Mathematical Biology*, Springer, New York, NY, USA, 3rd edition, 2002.

[30] I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations, I, II, III*, Springer, 1972.

[31] Spanish Ministry of Health, "National Drug Observatory Reports," 2008, http://www.pnsd.msc.es/Categoria2/observa/estudios/home.htm.