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Research Article

On Reciprocal Series of Generalized Fibonacci Numbers with Subscripts in Arithmetic Progression

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We investigate formulas for closely related series of the forms: $\sum_{n=0}^{\infty} 1/(U_{an+b}+c)$, $\sum_{n=0}^{\infty}(-1)^n U_{an+b}/(U_{an+b}+c)^2$, $\sum_{n=0}^{\infty} U_{2(an+b)}/(U_{an+b}^2+c)^2$ for certain values of a, b, and c.

1. Introduction

Let p be a nonzero integer such that $\Delta = p^2+4 \neq 0$. The generalized Fibonacci and Lucas sequences are defined by the following recurrences:

$$U_{n+1} = pU_n + U_{n-1},$$
$$V_{n+1} = pV_n + V_{n-1}, \tag{1.1}$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, respectively. When $p = 1$, $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

If α and β are the roots of equation $x^2 - px - 1 = 0$, the Binet formulas of the sequences $\{U_n\}$ and $\{V_n\}$ have the forms:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n, \tag{1.2}$$

respectively.

In [1], Backstrom developed formulas for closely related series of the form:

$$\sum_{n=0}^{\infty} \frac{1}{F_{an+b}+c}, \tag{1.3}$$

for certain values of a, b, and c. For example, he obtained the following series:

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + F_K} = \frac{K\sqrt{5}}{2L_K},$$
$$\sum_{n=0}^{\infty} \frac{1}{F_{(2n+1)K+2t} + F_K} = \begin{cases} \dfrac{\left(\sqrt{5} - 5F_t/L_t\right)}{2L_K}, & t \text{ even}, \\ \dfrac{\left(\sqrt{5} - L_t/F_t\right)}{2L_K}, & t \text{ odd}, \end{cases} \tag{1.4}$$

where K represents an odd integer and t is an integer in the range $-(K-1)/2$ to $(K-1)/2$ inclusive. Also, he gave the similar results for Lucas numbers.

In [2], Popov found in explicit form series of the form:

$$\sum_{n=0}^{\infty} \frac{1}{F_{an+b} \pm c}, \qquad \sum_{n=0}^{\infty} \frac{1}{F_{an+b}F_{cn+d}}, \qquad \sum_{n=0}^{\infty} \frac{1}{F_{an+b}^2 \pm F_{cn+d}^2}, \tag{1.5}$$

for certain values of a, b, c, and d.

In [3], Popov generalized some formulas of Backstrom [1] related to sums of reciprocal series of Fibonacci and Lucas numbers. For example,

$$\Delta \sum_{n=0}^{\infty} \frac{(-q)^{s+nr}}{V_{(2n+1)r+2s} - (-q)^{s+nr}V_r} = \begin{cases} \dfrac{\beta^s}{U_rU_s}, & \left|\dfrac{\beta}{\alpha}\right| < 1, \\ \dfrac{\alpha^s}{U_rU_s}, & \left|\dfrac{\alpha}{\beta}\right| < 1, \end{cases} \tag{1.6}$$

where s and r are integers.

In [4], Gauthier found the closed form expressions for the following sums:

$$\sum_{k=1}^{m} \frac{(-1)^{kn} f_{(2k+1)n}}{f_{(k+1)n}^2 f_{kn}^2}, \quad m, n \geq 1,$$
$$\sum_{k=0}^{m} \frac{(-1)^{kn} f_{(2k+1)n}}{l_{(k+1)n}^2 l_{kn}^2}, \quad m, n \geq 0, \tag{1.7}$$

where for $x \neq 0$ an indeterminate, the generalized Fibonacci and Lucas polynomials $\{f_n\}_n$ and $\{l_n\}_n$ are given by the following recurrences:

$$f_{n+2} = xf_{n+1} + f_n, \quad f_0 = 0,\ f_1 = 1,\ n \geq 0,$$
$$l_{n+2} = xl_{n+1} + l_n, \quad l_0 = 2,\ l_1 = x,\ n \geq 0, \tag{1.8}$$

respectively.

In this paper, we investigate formulas for closely related series of the forms:

$$\sum_{n=0}^{\infty} \frac{1}{U_{an+b} + c}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n U_{an+b}}{(U_{an+b} + c)^2}, \qquad \sum_{n=0}^{\infty} \frac{U_{2(an+b)}}{(U_{an+b}^2 + c)^2}, \tag{1.9}$$

for certain values of a, b and c.

2. On Some Series of Reciprocals of Generalized Fibonacci Numbers

In this section, firstly, we will give the following lemmas for further use.

Lemma 2.1. *Let n be an arbitrary nonzero integer. For integer $m \geq 1$,*

$$\sum_{k=1}^{m} \frac{(-1)^{kn} U_{(2k+1)n}}{U_{(k+1)n}^2 U_{kn}^2} = \frac{1}{4U_n} \left(\frac{V_n^2}{U_n^2} - \frac{V_{(m+1)n}^2}{U_{(m+1)n}^2} \right), \tag{2.1}$$

and for integer $m \geq 0$,

$$\sum_{k=0}^{m} \frac{(-1)^{kn} U_{(2k+1)n}}{V_{(k+1)n}^2 V_{kn}^2} = \frac{U_{(m+1)n}^2}{4U_n V_{(m+1)n}^2}. \tag{2.2}$$

Proof. We give the proof of Lemma 2.1 as the proofs of the sums in [4], using the following equalities:

$$\begin{aligned} \frac{U_{(2k+1)n}}{U_{(k+1)n} U_{kn}} &= \frac{1}{2} \left(\frac{V_{kn}}{U_{kn}} + \frac{V_{(k+1)n}}{U_{(k+1)n}} \right), \\ \frac{U_{(2k+1)n}}{V_{(k+1)n} V_{kn}} &= \frac{1}{2} \left(\frac{U_{(k+1)n}}{V_{(k+1)n}} + \frac{U_{kn}}{V_{kn}} \right), \\ \frac{(-1)^{kn} U_n}{U_{(k+1)n} U_{kn}} &= \frac{1}{2} \left(\frac{V_{kn}}{U_{kn}} - \frac{V_{(k+1)n}}{U_{(k+1)n}} \right), \\ \frac{(-1)^{kn} U_n}{V_{(k+1)n} V_{kn}} &= \frac{1}{2} \left(\frac{U_{(k+1)n}}{V_{(k+1)n}} - \frac{U_{kn}}{V_{kn}} \right). \end{aligned} \tag{2.3}$$

□

Lemma 2.2. *For arbitrary integers n and t,*

$$\begin{aligned} V_{2n} - (-1)^{n-t} V_{2t} &= \Delta U_{n-t} U_{n+t}, \\ V_{2n} + (-1)^{n-t} V_{2t} &= V_{n-t} V_{n+t}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} U_n^2 - (-1)^{n-t} U_t^2 &= U_{n-t} U_{n+t}, \\ V_n^2 - (-1)^{n-t} V_t^2 &= \Delta U_{n-t} U_{n+t}. \end{aligned} \tag{2.5}$$

Proof. From Binet formulas of sequences $\{U_n\}$ and $\{V_n\}$, the desired results are obtained. □

Theorem 2.3. *For an odd integer t,*

$$\begin{aligned} \sum_{n=1}^{m} \frac{1}{U_{(2n+1)t} + U_t} &= \frac{1}{2V_t} \left(\frac{2 - V_{2t}}{U_{2t}} - \frac{2 - V_{2(m+1)t}}{U_{2(m+1)t}} \right), \\ \sum_{n=1}^{m} \frac{1}{U_{(2n+1)t} - U_t} &= \frac{1}{2V_t} \left(\frac{2 + V_{2t}}{U_{2t}} - \frac{2 + V_{2(m+1)t}}{U_{2(m+1)t}} \right). \end{aligned} \tag{2.6}$$

Proof. By replacing n with $(2n+1)t$ in (2.5), we have

$$U^2_{(2n+1)t} - U_t^2 = U_{2nt}U_{2(n+1)t}, \tag{2.7}$$

or

$$\frac{1}{U_{(2n+1)t} + U_t} = \frac{U_{(2n+1)t} - U_t}{U_{2nt}U_{2(n+1)t}}. \tag{2.8}$$

Taking $r = (2n+1)t$ and $s = t$ in the equality $V_sU_r = U_{r+s} + (-1)^s U_{r-s}$ [5], the equality (2.8) is rewritten as follows:

$$\frac{1}{U_{(2n+1)t} + U_t} = \frac{1}{V_t}\left(\frac{1}{U_{2nt}} + \frac{(-1)^t}{U_{2(n+1)t}}\right) - \frac{U_t}{U_{2nt}U_{2(n+1)t}}. \tag{2.9}$$

We have the sum

$$\sum_{n=1}^{m} \frac{1}{U_{(2n+1)t} + U_t} = \frac{1}{V_t}\sum_{n=1}^{m}\left(\frac{1}{U_{2nt}} + \frac{(-1)^t}{U_{2(n+1)t}}\right) - U_t\sum_{n=1}^{m}\frac{1}{U_{2nt}U_{2(n+1)t}}. \tag{2.10}$$

For an odd integer t, we have

$$\sum_{n=1}^{m}\left(\frac{1}{U_{2nt}} - \frac{1}{U_{2(n+1)t}}\right) = \frac{1}{U_{2t}} - \frac{1}{U_{2(m+1)t}}, \tag{2.11}$$

and taking $s = 2nt$ and $r = 2t$ in identity [5]:

$$U_{s+r}V_s - U_sV_{s+r} = 2(-1)^s U_r, \tag{2.12}$$

we get

$$\sum_{n=1}^{m}\frac{1}{U_{2nt}U_{2(n+1)t}} = \frac{1}{2U_{2t}}\sum_{n=1}^{m}\left(\frac{V_{2nt}}{U_{2nt}} - \frac{V_{2(n+1)t}}{U_{2(n+1)t}}\right) = \frac{1}{2U_{2t}}\left(\frac{V_{2t}}{U_{2t}} - \frac{V_{2(m+1)t}}{U_{2(m+1)t}}\right). \tag{2.13}$$

Substituting (2.11) and (2.13) in (2.10), we have the desired result. □

For example, if we take $t = 1$ and $p = 1$ in (2.6), we have

$$\sum_{n=1}^{m}\frac{1}{F_{2n+1} + 1} = \frac{F_{2m+1} - 1}{F_{2(m+1)}}. \tag{2.14}$$

Note that

$$F_{2(m+1)}\sum_{n=1}^{m}\frac{1}{F_{2n+1} + 1} = \sum_{n=1}^{m} F_{2n}. \tag{2.15}$$

Corollary 2.4. *For an odd integer t,*

$$\sum_{n=1}^{m}\frac{1}{U_{(2n+1)t}+U_t}=\begin{cases}\dfrac{1}{2V_t}\left(\dfrac{V_{(m+1)t}}{U_{(m+1)t}}-\dfrac{V_t}{U_t}\right), & m \text{ is even},\\[2ex] \dfrac{1}{2V_t}\left(\dfrac{\Delta U_{(m+1)t}}{V_{(m+1)t}}-\dfrac{V_t}{U_t}\right), & m \text{ is odd},\end{cases}$$
$$\sum_{n=1}^{m}\frac{1}{U_{(2n+1)t}-U_t}=\begin{cases}\dfrac{\Delta}{2V_t}\left(\dfrac{U_t}{V_t}-\dfrac{U_{(m+1)t}}{V_{(m+1)t}}\right), & m \text{ is even},\\[2ex] \dfrac{1}{2V_t}\left(\dfrac{\Delta U_t}{V_t}-\dfrac{V_{(m+1)t}}{U_{(m+1)t}}\right), & m \text{ is odd}.\end{cases} \tag{2.16}$$

Proof. Using the equalities $V_{2n}=V_n^2-2(-1)^n=\Delta U_n^2+2(-1)^n$ and $U_{2n}=U_nV_n$ in Theorem 2.3, the results are obtained. □

Corollary 2.5. *Let t be an odd integer. For $|\beta/\alpha|<1, t>0$ and $|\alpha/\beta|<1, t<0$,*

$$\sum_{n=1}^{\infty}\frac{1}{U_{(2n+1)t}+U_t}=\frac{1}{2V_t}\left(\sqrt{\Delta}-\frac{V_t}{U_t}\right),$$
$$\sum_{n=1}^{\infty}\frac{1}{U_{(2n+1)t}-U_t}=\frac{\Delta}{2V_t}\left(\frac{U_t}{V_t}-\frac{1}{\sqrt{\Delta}}\right), \tag{2.17}$$

and for $|\beta/\alpha|<1, t<0$ and $|\alpha/\beta|<1, t>0$,

$$\sum_{n=1}^{\infty}\frac{1}{U_{(2n+1)t}+U_t}=\frac{-1}{2V_t}\left(\sqrt{\Delta}+\frac{V_t}{U_t}\right),$$
$$\sum_{n=1}^{\infty}\frac{1}{U_{(2n+1)t}-U_t}=\frac{\Delta}{2V_t}\left(\frac{U_t}{V_t}+\frac{1}{\sqrt{\Delta}}\right). \tag{2.18}$$

Proof. Since

$$\lim_{n\to\infty}\left(\frac{V_{an+b}}{U_{an+b}}\right)=\begin{cases}\sqrt{\Delta} & \left|\dfrac{\beta}{\alpha}\right|<1, a>0, \quad \left|\dfrac{\alpha}{\beta}\right|<1, a<0,\\[2ex] -\sqrt{\Delta} & \left|\dfrac{\beta}{\alpha}\right|<1, a<0, \quad \left|\dfrac{\alpha}{\beta}\right|<1, a>0,\end{cases} \tag{2.19}$$

the results are easily seen by equalities (2.16). □

Theorem 2.6. *For an integer $m\geq 1$ and an arbitrary nonzero integer t,*

$$\sum_{n=1}^{m}\frac{(-1)^{nt}U_{(2n+1)t}}{\left(V_{(2n+1)t}-(-1)^{nt}V_t\right)^2}=\frac{1}{4\Delta^2U_t}\left(\frac{V_t^2}{U_t^2}-\frac{V_{(m+1)t}^2}{U_{(m+1)t}^2}\right). \tag{2.20}$$

Proof. By replacing n with $(2n+1)t/2$ and t with $t/2$ in (2.4), we have

$$V_{(2n+1)t} - (-1)^{nt} V_t = \Delta U_{nt} U_{(n+1)t}, \tag{2.21}$$

or

$$\frac{1}{V_{(2n+1)t} - (-1)^{nt} V_t} = \frac{1}{\Delta U_{nt} U_{(n+1)t}}. \tag{2.22}$$

Multiplying equality (2.22) by $(-1)^{nt} U_{(2n+1)t} / U_{nt} U_{(n+1)t}$, we get

$$\frac{(-1)^{nt} U_{(2n+1)t}}{U_{nt} U_{(n+1)t} \left(V_{(2n+1)t} - (-1)^{nt} V_t\right)} = \frac{(-1)^{nt} U_{(2n+1)t}}{\Delta U_{nt}^2 U_{(n+1)t}^2}. \tag{2.23}$$

We have the sum:

$$\sum_{n=1}^{m} \frac{(-1)^{nt} U_{(2n+1)t}}{U_{nt} U_{(n+1)t} \left(V_{(2n+1)t} - (-1)^{nt} V_t\right)} = \frac{1}{\Delta} \sum_{n=1}^{m} \frac{(-1)^{nt} U_{(2n+1)t}}{U_{tn}^2 U_{t(n+1)}^2}. \tag{2.24}$$

Using the equalities (2.1) and (2.21), the proof is obtained. □

Corollary 2.7. *For an arbitrary nonzero integer t,*

$$\sum_{n=1}^{\infty} \frac{(-1)^{nt} U_{(2n+1)t}}{\left(V_{(2n+1)t} - (-1)^{nt} V_t\right)^2} = \frac{1}{4\Delta^2 U_t} \left(\frac{V_t^2}{U_t^2} - \Delta \right). \tag{2.25}$$

Proof. Taking $m \to \infty$ in Theorem 2.6 and using (2.19), the result is easily obtained. □

Theorem 2.8. *For an integer $m \geq 0$ and an arbitrary nonzero integer t,*

$$\sum_{n=0}^{m} \frac{(-1)^{nt} U_{(2n+1)t}}{\left(V_{(2n+1)t} + (-1)^{nt} V_t\right)^2} = \frac{U_{(m+1)t}^2}{4 U_t V_{(m+1)t}^2}. \tag{2.26}$$

Proof. The proof of the theorem is similar to the proof of Theorem 2.6. □

Corollary 2.9. *For an arbitrary nonzero integer t,*

$$\sum_{n=0}^{\infty} \frac{(-1)^{nt} U_{(2n+1)t}}{\left(V_{(2n+1)t} + (-1)^{nt} V_t\right)^2} = \frac{1}{4\Delta U_t}. \tag{2.27}$$

Proof. Taking $m \to \infty$ in Theorem 2.8 and using (2.19), the result is easily obtained. □

For example, if we take $t = 3$ and $p = 1$ in (2.27), we have

$$\sum_{n=0}^{\infty} \frac{(-1)^{n} F_{3(2n+1)}}{\left(L_{3(2n+1)} + (-1)^{n} 4\right)^2} = \frac{1}{40}. \tag{2.28}$$

Theorem 2.10. *For an integer $m \geq 1$ and an arbitrary nonzero integer t,*

$$\sum_{n=1}^{m} \frac{U_{2(2n+1)t}}{\left(U_{(2n+1)t}^2 - U_t^2\right)^2} = \frac{1}{4U_{2t}} \left(\frac{V_{2t}^2}{U_{2t}^2} - \frac{V_{2(m+1)t}^2}{U_{2(m+1)t}^2} \right),$$
$$\sum_{n=1}^{m} \frac{U_{2(2n+1)t}}{\left(V_{(2n+1)t}^2 - V_t^2\right)^2} = \frac{1}{4\Delta^2 U_{2t}} \left(\frac{V_{2t}^2}{U_{2t}^2} - \frac{V_{2(m+1)t}^2}{U_{2(m+1)t}^2} \right). \tag{2.29}$$

Proof. The proof of theorem is similar to the proof of Theorem 2.6. □

Corollary 2.11. *For an arbitrary nonzero integer t,*

$$\sum_{n=1}^{\infty} \frac{U_{2(2n+1)t}}{\left(U_{(2n+1)t}^2 - U_t^2\right)^2} = \frac{1}{4U_{2t}} \left(\frac{V_{2t}^2}{U_{2t}^2} - \Delta \right),$$
$$\sum_{n=1}^{\infty} \frac{U_{2(2n+1)t}}{\left(V_{(2n+1)t}^2 - V_t^2\right)^2} = \frac{1}{4\Delta^2 U_{2t}} \left(\frac{V_{2t}^2}{U_{2t}^2} - \Delta \right). \tag{2.30}$$

Proof. Taking $m \to \infty$ in Theorem 2.10 and using (2.19), the result is easily obtained. □

For example, if we take $t = 2$ in the equality (2.30), we have

$$\sum_{n=1}^{\infty} \frac{U_{4(2n+1)}}{\left(V_{2(2n+1)}^2 - V_2^2\right)^2} = \frac{1}{\Delta^2 U_4^3}. \tag{2.31}$$

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