Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 745697, 17 pages
doi:10.1155/2012/745697

Research Article

Complete Asymptotic Analysis of a Two-Nation Arms Race Model with Piecewise Constant Nonlinearities

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Received 30 August 2011; Accepted 25 October 2011

Academic Editor: Guang Zhang

A discrete time two-nation arms race model involving a piecewise constant nonlinear control function is formulated and studied. By elementary but novel arguments, we are able to give a complete analysis of its asymptotic behavior when the threshold parameter in the control function varies from 0^+ to ∞. We show that all solutions originated from positive initial values tend to limit one or two cycles. An implication is that when devastating weapons are involved, "terror equilibrium" can be achieved and escalated race avoided. It is hoped that our analysis will provide motivation for further studying of discrete-time equations with piecewise smooth nonlinearities.

1. Introduction

In [1, pages 87–90], a simple dynamical model of a two-nation arms race based on Richardson's ideas in [2] is explained, and several interesting conclusions are drawn which can be used to explain stable and escalated arms races. Roughly, let $\mathbf{N} = \{0, 1, 2, \dots\}$, and let A_n and B_n be the amount spent on armaments by two respective countries A and B in year $n \in \mathbf{N}$. Assuming, A has a fixed amount of distrust of the other country, causing it to retain arms, then

$$A_n = (1 - r_A)A_{n-1} + s_A B_{n-1} + u, \tag{1.1}$$

where the constant s_A measures country A's distrust of country B in that it reacts to the way B arms itself, $r_A \in (0, 1)$ is a measure of A's own economy, and u is the basic annual expenditure (e.g., maintenance expense). If we now assume a similar situation for country B, then we also have

$$B_n = (1 - r_B)B_{n-1} + s_B A_{n-1} + v. \tag{1.2}$$

Under the assumption that $r_A = r_B = r$ and $s_A = s_B = s$, it is shown that if the initial total expenditure $A_{-1} + B_{-1}$ is large and that the distrust factor is also so large that $s > r$, then no two countries can sustain exponentially increasing expenditures on arms, and the alternative is war or negotiation. While this model is an oversimplification one, it could help to understand plausible reasons behind World War I (see [2, 3] in which various aspects of arms race modeling are discussed).

The above model cannot explain some of the observations we can make nowadays. Therefore, we need to build various models and analyze their asymptotic behaviors. In this paper, we will build one such model based on the idea that although the distrust factor is the same as in the previous model, the expenditure by the other country in year $n-1$ in (1.1) is replaced by $s_A f_\lambda(B_{n-1})$ where

$$f_\lambda(B_{n-1}) = \begin{cases} 1 & \text{if } B_{n-1} \in (0, \lambda], \\ 0 & \text{if } B_{n-1} \in (-\infty, 0] \cup (\lambda, \infty), \end{cases} \tag{1.3}$$

and the term A_{n-1} in (1.2) is replaced by a similar one. The "discontinuous" function f_λ has a clear physical meaning. Indeed, the positive parameter λ may be treated as a cutoff threshold indicator so that when the competitor is already spending an unreasonable amount of money (such as stocking of hundreds of nuclear missiles that can annihilate our mother earth) or is not spending any, there is no need to add the budget anymore. With this function at hand, we may rewrite our new model as follows:

$$\begin{aligned} x_n &= a x_{n-\alpha} + b f_\lambda(y_{n-1}) + c, \\ y_n &= r y_{n-\beta} + s f_\tau(x_{n-1}) + t, \end{aligned} \tag{1.4}$$

where we have introduced two "delays" α and β in order to reflect the fact that the expenditure in a previous accounting period may not be recorded precisely, and hence historical expenditure records may be more reliable for use in making future decisions.

Although (1.4) may seem to be a simple model, there are still too many parameters involved. We therefore make further (reasonable) assumptions as follows:

$$c, t = 0, \qquad \alpha = \beta = 2, \qquad \tau = \lambda > 0, \qquad r = a \in (0,1), \qquad b = s = (1 - a). \tag{1.5}$$

The assumption that $c, t = 0$ means that the fixed expenditures are relatively low in both countries, while we assume that $\alpha = \beta = 2$ so as to use the best up-to-date and "reliable" accounting records x_{n-2} and y_{n-2}. By choosing $a \in (0,1)$ and $b = 1 - a$, country A is making a decision based on a convex combination of the expenditures x_{n-2} and the blanket-ceiling sum $f_\lambda(y_{n-1})$. If country B takes on a similar decision policy, then we end up with

$$y_n = a^* y_{n-2} + (1 - a^*) f_\lambda(x_{n-1}), \tag{1.6}$$

where a^* may or may not differ from a. The case $a^* \neq a$ is only more technically difficult, and therefore, in this paper, we will assume the case $a = a^*$ (which is already nontrivial as we will see) so that both countries play "symmetric" roles in the interactions.

By adopting these assumptions, we then settle on the following dynamical system:

$$x_n = ax_{n-2} + (1-a)f_\lambda(y_{n-1}),$$
$$y_n = ay_{n-2} + (1-a)f_\lambda(x_{n-1}), \tag{1.7}$$

for $n \in \mathbf{N}$, where in this model, $a \in (0,1)$, $\lambda > 0$. Note that if we let $z = (x, y)$ and

$$F_\lambda(z) = (f_\lambda(y), f_\lambda(x)), \tag{1.8}$$

then the above system (1.7) can be written as

$$z_n = az_{n-2} + a' F_\lambda(z_{n-1}), \quad n \in \mathbf{N}, \tag{1.9}$$

where we write $z_n = (x_n, y_n)$ and $a' = 1 - a$ for the sake of convenience.

The above vector equation is a three-term recurrence relation. Hence, for given z_{-2} and z_{-1} in the plane, a unique sequence $\{z_k\}_{k=-2}^{\infty}$ can be calculated from it. Such a sequence is called a solution of (1.9) determined by z_{-2} and z_{-1}. Among different z_{-2} and z_{-1}, those lying in the positive quadrant are of special interests since expenditures are always positive. Therefore, our subsequent interests are basically the asymptotic behaviors of all such solutions determined by z_{-2} and z_{-1} with positive components.

We remark that system (1.9) can be regarded as a discrete dynamical system with piecewise smooth nonlinearities. Such systems have not been explored extensively (see, e.g., the discussions on "polymodal" discrete systems in [4], and there are only several recent studies on scalar equations with piecewise smooth nonlinearities [5–9])! Therefore, a complete asymptotic analysis of our equation is essential in the further development of discontinuous (in particular, polymodal) discrete time dynamical systems.

We need to be more precise about the statements to be made later. To this end, we first note that given any z_{-2}, z_{-1} in the quadrant $(0,\infty)^2$, the solution $\{z_n\}_{n-2}^{\infty}$ determined by it also lies in the same quadrant (in the sense that $z_n \in (0,\infty)^2$ for $n \in \mathbf{N}$). Depending on the locations of z_{-2} and z_{-1}, it is clear that the behavior of the corresponding solution may differ. For this reason, it is convenient to distinguish various parts of the first quadrant in the following manner:

$$A = (0,\lambda]^2, \qquad B = (\lambda,\infty) \times (0,\lambda], \qquad C = (\lambda,\infty) \times (\lambda,\infty), \qquad D = (0,\lambda] \times (\lambda,\infty), \tag{1.10}$$

where λ is a fixed positive number, then

$$\mathfrak{P} = \{A, B, C, D\} \tag{1.11}$$

is a partition of the quadrant $(0,\infty)^2$.

Note that these subsets depend on λ, but this dependence is not emphasized in the sequel for the sake of convenience.

For any solution $\{z_n\}_{n=-2}^{\infty}$ originated from z_{-2} and z_{-1} in the above subsets, our main conclusion in this paper is that $\{z_{2n}\}$ will tend to some vector u and $\{z_{2n+1}\}$ will tend to another vector v (which may or may not be equal to u). This implication is important since it says that escalated arms race cannot happen and World War III should not happen if our model is correct!

For the sake of convenience, we record this fact by means of

$$z_n \longrightarrow \langle u, v\rangle. \tag{1.12}$$

In case $u = v$, $\{z_n\}$ is convergent to u, and hence we may also write

$$z_n \longrightarrow u \quad \text{or} \quad z_n \longrightarrow \langle u\rangle. \tag{1.13}$$

To arrive at our main conclusion, we note, however, that since f_λ is a discontinuous function, the standard theories that employ continuous arguments cannot be applied to yield asymptotic criteria. Fortunately, we may resort to elementary arguments as to be seen below.

Before doing so, let us make a few remarks. First, note that our system (1.9) is autonomous (time invariant) and also symmetric in the sense that under two sets of "symmetric initial conditions," the behaviors of the corresponding solutions are also "symmetric." This statement can be made more precise in mathematical terms. However, a simple example is sufficient to illustrate this: suppose that $\lambda = 1$. If $\{z_n\}_{n=-2}^{\infty}$ is a solution of (1.9) with $(z_{-2}, z_{-1}) \in A \times B$, then as will be seen below, $z_{2n} \rightarrow (1,0)$ and $z_{2n+1} \rightarrow (1,1)$. If we now replace the condition $(z_{-2}, z_{-1}) \in A \times B$ with the symmetric initial condition $(z_{-2}, z_{-1}) \in A \times D$, then we will end up with the conclusion that $z_{2n} \rightarrow (0,1)$ and $z_{2n+1} \rightarrow (1,1)$. Such two conclusions will be referred to as dual results. We will see some tables which contain some obvious *dual results* later.

Next, by (1.8), we may easily see that

$$F_\lambda(A) = \mathbf{k}, \qquad F_\lambda(B) = \mathbf{i}, \qquad F_\lambda(C) = \mathbf{0}, \qquad F_\lambda(D) = \mathbf{j}, \tag{1.14}$$

where

$$\mathbf{0} = (0,0), \quad \mathbf{i} = (1,0), \quad \mathbf{j} = (0,1), \quad \mathbf{k} = (1,1). \tag{1.15}$$

Therefore, in case $\{z_k\}$ is a solution of (1.9) such that, say, $z_k \in A$ for all large k, then (1.9) is reduced to

$$z_n = a z_{n-2} + a' \mathbf{k}, \tag{1.16}$$

for all large n. Hence, linear systems and their related properties will also appear in later discussions. More precisely, two groups of bounding quantities α_j and β_j are needed:

$$\alpha_j = 1 + \frac{\lambda - 1}{a^j}, \quad \beta_j = \frac{\lambda}{a^j}, \quad j \in \mathbf{N}. \tag{1.17}$$

Note that they satisfy $\alpha_0 = \beta_0 = \lambda$ and the recurrence relations

$$\alpha_{j+1} = \frac{1}{a}\alpha_j + \frac{a-1}{a}, \quad \beta_{j+1} = \frac{1}{a}\beta_j, \quad j \in \mathbf{N}. \tag{1.18}$$

We also need the following two properties of linear systems. Let $\{x_k\}_{k=-2}^{\infty}$ be real scalar (or vector) sequences that satisfy

$$x_{2k} = ax_{2k-2} + d, \quad k \in \mathbf{N}, \tag{1.19}$$

$$x_{2k+1} = ax_{2k-1} + d, \quad k \in \mathbf{N}, \tag{1.20}$$

where $a \in (0,1)$, and d is a real number (resp., a real vector).

(i) If $\{x_n\}_{n=-2}^{\infty}$ is a sequence which satisfies (1.19), then

$$x_{2k} = a^{k+1}x_{-2} + \frac{(1-a^{k+1})}{1-a}d, \quad k \in \mathbf{N}. \tag{1.21}$$

(ii) If $\{x_n\}_{n=-2}^{\infty}$ is a sequence which satisfies (1.20), then

$$x_{2k+1} = a^{k+1}x_{-1} + \frac{(1-a^{k+1})}{1-a}d, \quad k \in \mathbf{N}. \tag{1.22}$$

Finally, we need to consider various ordering arrangements for three or four nonnegative integers k, p, l, and m. First, the ordering arrangements of three integers $k, p,$ and l can be classified into 6 cases: (1) $k = p \le l$, (2) $l = k < p$, (3) $p = l < k$, (4) $k < \min\{p, l\}$, (5) $p < \min\{k, l\}$, and (6) $l < \min\{k, p\}$. In fact, let $a, b \in R$. Then either $a < b, a = b$, or $a > b$. Let $a, b, c \in R$, then

$$\begin{aligned} a < b &\Longrightarrow c \in (-\infty, a), \quad c = a, \quad c \in (a,b), \quad c = b \quad \text{or} \quad c \in (b,\infty), \\ a = b &\Longrightarrow c \in (-\infty, a), \quad c = a \quad \text{or} \quad c \in (a,\infty), \\ a > b &\Longrightarrow c \in (-\infty, b), \quad c = b, \quad c \in (b,a), \quad c = a \quad \text{or} \quad c \in (a,\infty). \end{aligned} \tag{1.23}$$

These are equivalent to

$$\begin{gathered} a = b \le c, \qquad c = a < b, \qquad b = c < a, \\ a < \min\{b,c\}, \qquad b < \min\{a,c\}, \qquad c < \min\{a,b\}, \end{gathered} \tag{1.24}$$

by comparing the two sets of statements.

By similar reasoning, there are 12 ordering arrangements for four integers $k, p, l,$ and m: (1) $k = p \leq \min\{l, m\}$, (2) $k < \min\{p, l, m\}$, (3) $p < \min\{k, l, m\}$, (4) $p = l < \min\{k, m\}$, (5) $p = l = m < k$, (6) $l < \min\{p, k, m\}$, (7) $l = m < \min\{k, p\}$, (8) $l = m = k < p$, (9) $m < \min\{l, k, p\}$, (10) $m = k < \min\{l, p\}$, (11) $p = m < \min\{l, k\}$, and (12) $k = l < \min\{p, m\}$.

Our following plan is quite simple. We will treat our λ as a bifurcation parameter and distinguish four different cases (i) $\lambda > 1$, (ii) $\lambda = 1$, (iii) $0 < \lambda < 1 - a$, and (iv) $1 - a \leq \lambda < 1$, and consider different z_{-2}, z_{-1} in A, B, C, or D and discuss the precise asymptotic behaviors of the corresponding solutions determined by them.

2. The Case $\lambda > 1$

This case is relatively simple.

Theorem 2.1. *Suppose, $\lambda > 1$. Let $\{z_k\}_{k=-2}^{\infty}$ be any solution of* (1.9) *with $(z_{-2}, z_{-1}) \in (0, \infty)^2$. Then $z_n \to \mathbf{k}$.*

Proof. By (1.9), we may see that $x_n \leq a x_{n-2} + a'$ and $y_n \leq a y_{n-2} + a'$ for $n \in \mathbf{N}$, then $\lim_n x_n \leq 1 < \lambda$ and $\lim_n y_n \leq 1 < \lambda$. Thus, there exists an integer m such that $(z_k, z_{k+1}) \in (0, \lambda]^2$ for all $k \geq m$. Therefore, $z_{k+2} = a z_k + a'\mathbf{k}$ for all $k \geq m$. In view of (1.21) and (1.22), $z_n \to \mathbf{k}$. The proof is complete. □

3. The Case $\lambda = 1$

In this section, we assume that $\lambda = 1$. If $\{z_k\}_{k=-2}^{\infty}$ is a solution of (1.9) and if $z_k \in A$ and $z_{k+1} \in (0, \infty)^2$, then in view of (1.14), $F_\lambda(z_{k+1}) \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{0}\}$, and hence

$$z_{k+2} = a z_k + a' F_\lambda(z_{k+1}) \in A. \tag{3.1}$$

By similar reasoning, we may consider all other possible cases and collect our findings in tables. To simplify the description of these tables, we first note that $\{\beta_j\}_{j=0}^{\infty} = \{1/a^j\}_{j=0}^{\infty}$ is a strictly increasing and divergent sequence. Therefore, if we let

$$I^{(i)} = (\beta_i, \beta_{i+1}], \qquad B^{(i)} = I^{(i)} \times (0, 1], \qquad D^{(i)} = (0, 1] \times I^{(i)}, \qquad C^{(i,j)} = I^{(i)} \times I^{(j)}, \tag{3.2}$$

then

$$(1, \infty) = \bigcup_{i=0}^{\infty} I^{(i)}, \qquad B = \bigcup_{i=0}^{\infty} B^{(i)}, \qquad D = \bigcup_{i=0}^{\infty} D^{(i)}, \qquad C = \bigcup_{i,j \in N} C^{(i,j)}. \tag{3.3}$$

First of all, the fact that $z_k \in A$ and $z_{k+1} \in A$ implies, $z_{k+2} \in A$ is recorded as the (A, A) entry in Table 1. In this table, we may also find other entries which are self-explanatory.

Table 1

	A	$B^{(s)}$	$C^{(s,t)}$	$D^{(s)}$
A	A	A	A	A
$B^{(i)}$	B	B	$aB^{(i)}$	$aB^{(i)}+a'\mathbf{j}$
$C^{(i,j)}$	C	$aC^{(i,j)}+a'\mathbf{i}$	$aC^{(i,j)}$	$aC^{(i,j)}+a'\mathbf{j}$
$D^{(i)}$	D	$aD^{(i)}+a'\mathbf{i}$	$aD^{(i)}$	D

Table 2

Initial condition	Condition	Conclusion
$(z_{-2},z_{-1})\in B^{(k)}\times C^{(s,p)}$	$0\le k\le p$	$(z_{2k},z_{2k+1})\in A\times C$
$(z_{-2},z_{-1})\in B^{(k)}\times C^{(s,p)}$	$0\le p<k$	$(z_{2p},z_{2p+1})\in B\times B$
$(z_{-2},z_{-1})\in B^{(k)}\times D^{(p)}$	$0\le k\le p$	$(z_{2k},z_{2k+1})\in A\times D$
$(z_{-2},z_{-1})\in B^{(k)}\times D^{(p)}$	$0\le p<k$	$(z_{2p},z_{2p+1})\in B\times A$
$(z_{-2},z_{-1})\in C^{(s,k)}\times B^{(p)}$	$0\le k\le p$	$(z_{2k},z_{2k+1})\in B\times B$
$(z_{-2},z_{-1})\in C^{(s,k)}\times B^{(p)}$	$0\le p<k$	$(z_{2p},z_{2p+1})\in C\times A$
$(z_{-2},z_{-1})\in D^{(k)}\times C^{(p,s)}$	$0\le k\le p$	$(z_{2k},z_{2k+1})\in A\times C$
$(z_{-2},z_{-1})\in D^{(k)}\times C^{(p,s)}$	$0\le p<k$	$(z_{2p},z_{2p+1})\in D\times D$
$(z_{-2},z_{-1})\in D^{(k)}\times B^{(p)}$	$0\le k\le p$	$(z_{2k},z_{2k+1})\in A\times B$
$(z_{-2},z_{-1})\in D^{(k)}\times B^{(p)}$	$0\le p<k$	$(z_{2p},z_{2p+1})\in D\times A$
$(z_{-2},z_{-1})\in C^{(k,s)}\times D^{(p)}$	$0\le k\le p$	$(z_{2k},z_{2k+1})\in D\times D$
$(z_{-2},z_{-1})\in C^{(k,s)}\times D^{(p)}$	$0\le p<k$	$(z_{2p},z_{2p+1})\in C\times A$

Table 3: Initial condition: $(z_{-2},z_{-1})\in C^{(k,p)}\times C^{(l,m)}$.

Condition	Conclusion
$0\le k=p\le\min\{l,m\}$	$(z_{2k},z_{2k+1})\in A\times C$
$0\le k<\min\{l,m,p\}$	$(z_{2k},z_{2k+1})\in D^{(p-k-1)}\times C^{(l-k-1,t)}$ for some $t\in\mathbf{N}$
$0\le p<\min\{k,l,m\}$	$(z_{2p},z_{2p+1})\in B^{(k-p-1)}\times C^{(t,m-p-1)}$ for some $t\in\mathbf{N}$
$0\le p=l<\min\{m,k\}$	$(z_{2p},z_{2p+1})\in B^{(k-p-1)}\times C^{(t,m-p-1)}$ for some $t\in\mathbf{N}$
$0\le p=l=m<k$	$(z_{2p},z_{2p+1})\in B\times B$
$0\le l<\min\{m,k,p\}$	$(z_{2l},z_{2l+1})\in C^{(k-l-1,p-l-1)}\times D^{(m-l-1)}$
$0\le l=m<\min\{p,k\}$	$(z_{2l},z_{2l+1})\in C\times A$
$0\le l=k=m<p$	$(z_{2l},z_{2l+1})\in D\times D$
$0\le m<\min\{l,k,p\}$	$(z_{2m},z_{2m+1})\in C^{(k-m-1,p-m-1)}\times B^{(l-m-1)}$
$0\le k=m<\min\{l,p\}$	$(z_{2k},z_{2k+1})\in D^{(p-k-1)}\times C^{(l-k-1,t)}$ for some $t\in\mathbf{N}$
$0\le p=m<\min\{l,k\}$	$(z_{2p},z_{2p+1})\in B\times B$
$0\le k=l<\min\{p,m\}$	$(z_{2k},z_{2l+1})\in D\times D$

By similar considerations, we may also obtain Tables 1 and 2.

For instance, let us show the third data row in Table 2. Suppose that $z_{-2}\in B^{(k)}$ and $z_{-1}\in D^{(p)}$, where $0\le k\le p$. By Table 1, if $0=k\le p$, then

$$\begin{aligned} z_0 &= az_{-2}+a'F_\lambda(z_{-1}) &&= az_{-2}+a'\mathbf{j}\in aB^{(0)}+a'\mathbf{j}\subseteq A,\\ z_1 &= az_{-1}+a'F_\lambda(z_0) &&= az_{-1}+a'\mathbf{k}\in aD^{(p)}+a'\mathbf{k}\subseteq D. \end{aligned} \tag{3.4}$$

If $0 < k \leq p$, then

$$
\begin{aligned}
z_0 &= az_{-2} + a' F_\lambda(z_{-1}) = az_{-2} + a'\mathbf{j} \in aB^{(k)} + a'\mathbf{j} \subseteq B^{(k-1)},\\
z_1 &= az_{-1} + a' F_\lambda(z_0) \ = az_{-1} + a'\mathbf{i} \in aD^{(p)} + a'\mathbf{i} \subseteq D^{(p-1)},\\
z_2 &= az_0 + a' F_\lambda(z_1) = az_0 + a'\mathbf{j} \in aB^{(k-1)} + a'\mathbf{j} \subseteq B^{(k-2)},\\
z_3 &= az_1 + a' F_\lambda(z_2) = az_1 + a'\mathbf{i} \in aD^{(p-1)} + a'\mathbf{i} \subseteq D^{(p-2)},
\end{aligned}
\tag{3.5}
$$

and by induction,

$$
\begin{aligned}
z_{2k} &= az_{2k-2} + a' F_\lambda(z_{2k-1}) = az_{2k-2} + a'\mathbf{j} \in aB^{(0)} + a'\mathbf{j} \subseteq A,\\
z_{2k+1} &= az_{2k-1} + a' F_\lambda(z_{2k}) \ = az_{2k-1} + a'\mathbf{k} \in aD^{(p-k)} + a'\mathbf{k} \subseteq D^{(p-k-1)},
\end{aligned}
\tag{3.6}
$$

that is, $z_{2k} \in A$ and $z_{2k+1} \in D$.

As another example, let us show the second data row in Table 3. Suppose that $(z_{-2}, z_{-1}) \in C^{(k,p)} \times C^{(l,m)}$, where $0 \leq k < \min\{l, m, p\}$, then by Table 1, if $0 = k < \min\{l, m, p\}$,

$$
\begin{aligned}
z_0 &= az_{-2} + a' F_\lambda(z_{-1}) = az_{-2} \in aC^{(0,p)} \subseteq D^{(p-1)},\\
z_1 &= az_{-1} + a' F_\lambda(z_0) = az_{-1} + a'\mathbf{j} \in aC^{(l,m)} + a'\mathbf{j} \subseteq I^{(l-1)} \times (1, \infty).
\end{aligned}
\tag{3.7}
$$

If $0 < k < \min\{l, m, p\}$, then

$$
\begin{aligned}
z_0 &= az_{-2} + a' F_\lambda(z_{-1}) = az_{-2} \in aC^{(k,p)} \subseteq C^{(k-1,p-1)},\\
z_1 &= az_{-1} + a' F_\lambda(z_0) = az_{-1} \in aC^{(l,m)} \subseteq C^{(l-1,m-1)},\\
z_2 &= az_0 + a' F_\lambda(z_1) = az_0 \in aC^{(k-1,p-1)} \subseteq C^{(k-2,p-2)},\\
z_3 &= az_1 + a' F_\lambda(z_2) = az_1 \in aC^{(l-1,m-1)} \subseteq C^{(l-2,m-2)},
\end{aligned}
\tag{3.8}
$$

and by induction,

$$
\begin{aligned}
z_{2k} &= az_{2k-2} + a' F_\lambda(z_{2k-1}) = az_{2k-2} \in aC^{(0,p-k)} \subseteq D^{(p-k-1)},\\
z_{2k+1} &= az_{2k-1} + a' F_\lambda(z_{2k}) = az_{2k-1} + a'\mathbf{j} \in aC^{(l-k,m-k)} + a'\mathbf{j} \subseteq I^{(l-k-1)} \times (1, \infty),
\end{aligned}
\tag{3.9}
$$

that is, $z_{2k} \in D^{(p-k-1)}$ and $z_{2k+1} \in C^{(l-k-1,t)}$ for some $t \in N$.

By means of the information obtained so far, let $\{z_n\}_{n=-2}^{\infty}$ be a solution of (1.9), we will be able to show the following result.

Theorem 3.1. *Suppose that $\lambda = 1$. Let $\{z_n\}_{n=-2}^{\infty}$ be a solution of* (1.9) *originated from* $(0, \infty)^2$*. Then*

$$
z_n \longrightarrow \langle \mathbf{i} \rangle, \langle \mathbf{j} \rangle, \langle \mathbf{k} \rangle, \langle \mathbf{0}, \mathbf{k} \rangle, \langle \mathbf{k}, \mathbf{0} \rangle, \langle \mathbf{i}, \mathbf{k} \rangle, \langle \mathbf{k}, \mathbf{i} \rangle, \langle \mathbf{j}, \mathbf{k} \rangle \quad \text{or } \langle \mathbf{k}, \mathbf{j} \rangle. \tag{3.10}
$$

Table 4

	A	$B^{(s)}$	$C^{(s,t)}$	$D^{(s)}$
A	$\langle \mathbf{k}\rangle$	$\langle \mathbf{i},\mathbf{k}\rangle$	$\langle \mathbf{0},\mathbf{k}\rangle$	$\langle \mathbf{j},\mathbf{k}\rangle$
$B^{(i)}$	$\langle \mathbf{k},\mathbf{i}\rangle$	$\langle \mathbf{i}\rangle$	$\langle \mathbf{0},\mathbf{k}\rangle$ or $\langle \mathbf{i}\rangle$	$\langle \mathbf{j},\mathbf{k}\rangle$ or $\langle \mathbf{k},\mathbf{i}\rangle$
$C^{(i,j)}$	$\langle \mathbf{k},\mathbf{0}\rangle$	$\langle \mathbf{i}\rangle$ or $\langle \mathbf{k},\mathbf{0}\rangle$	$\langle \mathbf{0},\mathbf{k}\rangle$, $\langle \mathbf{k},\mathbf{0}\rangle$, $\langle \mathbf{i}\rangle$ or $\langle \mathbf{j}\rangle$	$\langle \mathbf{j}\rangle$ or $\langle \mathbf{k},\mathbf{0}\rangle$
$D^{(i)}$	$\langle \mathbf{k},\mathbf{j}\rangle$	$\langle \mathbf{i},\mathbf{k}\rangle$ or $\langle \mathbf{k},\mathbf{i}\rangle$	$\langle \mathbf{0},\mathbf{k}\rangle$ or $\langle \mathbf{i}\rangle$	$\langle \mathbf{j}\rangle$

To this end, let us consider first the case where $(z_{-2}, z_{-1}) \in A \times A$, then by the (A, A) entry in Table 1, $z_0 \in A$. By induction, we may then see that $z_k \in A$ for all $k \geq -2$. Hence, by (1.9), we see that

$$z_n = az_{n-2} + (1-a)\mathbf{k}, \quad n \in N, \tag{3.11}$$

from which we easily obtain

$$\begin{aligned} z_{2k} &= a^{k+1} z_{-2} + \left(1 - a^{k+1}\right)\mathbf{k}, \quad n \in N, \\ z_{2k+1} &= a^{k+1} z_{-1} + \left(1 - a^{k+1}\right)\mathbf{k}, \quad n \in N. \end{aligned} \tag{3.12}$$

By (1.21) and (1.22), we see that $z_{2k}, z_{2k+1} \to \mathbf{k}$ so that $z_n \to \mathbf{k}$. We record this conclusion in the (A, A) entry of Table 4.

Consider another case where $z_{-2} \in A$ and $z_{-1} \in B^{(j)}$. Then by the $(A, B^{(j)})$ entry of Table 1, we see that $z_0 \in A$. Since $z_{-1} \in B^{(j)}$ and $z_0 \in A$, then by Table 1 again,

$$z_1 \in aB^{(j)} + a'\mathbf{k} \in B^{(i)}, \tag{3.13}$$

for some $i \in N$. By induction, we may then see that $z_{2k} \in A$ and $z_{2k+1} \in B$ for $k \geq -1$. Hence, by (1.9), we see that

$$\begin{aligned} z_{2n} &= az_{2n-2} + a'\mathbf{i}, \\ z_{2n+1} &= az_{2n-1} + a'\mathbf{k}, \end{aligned} \tag{3.14}$$

for $n \in N$. We may then easily see that $z_{2n} \to \mathbf{i}$ and $z_{2n+1} \to \mathbf{k}$.

By similar arguments, we may then derive the $(A, C^{(s,t)})$, $(A, D^{(s)})$, $(B^{(i)}, A)$, $(B^{(i)}, B^{(s)})$, $(C^{(i,j)}, A)$, $(D^{(i)}, A)$ and $(D^{(i)}, D^{(s)})$ entries in Table 4.

To see why the other entries are correct, we consider a typical case where $z_{-2} \in B^{(i)}$ and $z_{-1} \in D^{(s)}$ for some $i, s \in N$. Suppose that $0 \leq i \leq s$. By Table 2, $z_{2i} \in A$ and $z_{2i+1} \in D$. Hence by the (A, D) entry in Table 4, we see that $z_{2k} \to \mathbf{j}$ and $z_{2k+1} \to \mathbf{k}$. While if $0 \leq s < i$, then $z_{2s} \in B$ and $z_{2s+1} \in A$. Hence, by the (B, A) entry in Table 4, we see that $z_{2k} \to \mathbf{k}$ and $z_{2k+1} \to \mathbf{i}$.

Table 5

	A	B	C	D
A	C	B	A	D
B	C	B	aB	$aB + a'\mathbf{j}$
C	C	$aC + a'\mathbf{i}$	aC	$aC + a'\mathbf{j}$
D	C	$aD + a'\mathbf{i}$	aD	D

4. The Case $\lambda \in (0, 1-a)$

Suppose that $\lambda \in (0, 1-a)$. Then $a\lambda + 1 - a > \lambda$, and $\{\alpha_j\}_{j=0}^{\infty}$ is a strictly decreasing sequence which diverges to $-\infty$. Hence, there exists $M \in \mathbf{N}$ such that $\alpha_0, \dots, \alpha_M > 0$ and $\alpha_{M+1} \le 0$. Let $D_j = \alpha_j$ for $j = 0, \dots, M$, and let $D_{M+1} = 0$, then

$$(0, \lambda] = \bigcup_{j=0}^{M} (D_{j+1}, D_j]. \tag{4.1}$$

We denote

$$\begin{gathered}
A^{(p,\lambda)} = (D_{p+1}, D_p] \times (0, \lambda], \quad A^{(\lambda,p)} = (0, \lambda] \times (D_{p+1}, D_p], \quad A^{(p,q)} = (D_{p+1}, D_p] \times (D_{q+1}, D_q], \\
B^{(\lambda,p)} = (\lambda, \infty) \times (D_{p+1}, D_p], \quad B^{(p,\lambda)} = I^{(p)} \times (0, \lambda], \quad B^{(p,q)} = I^{(p)} \times (D_{q+1}, D_q], \\
C^{(\lambda,p)} = (\lambda, \infty) \times I^{(p)}, \quad C^{(p,\lambda)} = I^{(p)} \times (\lambda, \infty), \quad C^{(p,q)} = I^{(p)} \times I^{(q)}, \\
D^{(\lambda,p)} = (0, \lambda] \times I^{(p)}, \quad D^{(p,\lambda)} = (D_{p+1}, D_p] \times (\lambda, \infty), \quad D^{(p,q)} = (D_{p+1}, D_p] \times I^{(q)},
\end{gathered} \tag{4.2}$$

then

$$\begin{gathered}
A = \bigcup_{p=0}^{M} A^{(p,\lambda)} = \bigcup_{p=0}^{M} A^{(\lambda,p)} = \bigcup_{p=0}^{M} \bigcup_{q=0}^{M} A^{(p,q)}, \qquad B = \bigcup_{p=0}^{\infty} B^{(p,\lambda)} = \bigcup_{p=0}^{M} B^{(\lambda,p)} = \bigcup_{p=0}^{\infty} \bigcup_{q=0}^{M} B^{(p,q)}, \\
C = \bigcup_{p=0}^{\infty} C^{(p,\lambda)} = \bigcup_{p=0}^{\infty} C^{(\lambda,p)} = \bigcup_{p=0}^{\infty} \bigcup_{q=0}^{\infty} C^{(p,q)}, \qquad D = \bigcup_{p=0}^{M} D^{(p,\lambda)} = \bigcup_{p=0}^{\infty} D^{(\lambda,p)} = \bigcup_{p=0}^{M} \bigcup_{q=0}^{\infty} D^{(p,q)}.
\end{gathered} \tag{4.3}$$

In Table 5, we record the fact that $z_\alpha \in B$ and $z_{\alpha+1} \in B$ which implies $z_{\alpha+2} \in B$ as the (B, B) entry, and so forth.

Tables 6 and 7 are similar to Tables 2 and 3.

For example, let us show the first data row in Table 6. Suppose that $z_{-2} \in C^{(\lambda,p)}$ and $z_{-1} \in B^{(m,\lambda)}$ where $0 \le p \le m$. Then by Table 5, if $0 = p \le m$,

$$\begin{aligned}
z_0 &= az_{-2} + a'F_\lambda(z_{-1}) = az_{-2} + a'\mathbf{i} \in aC^{(\lambda,0)} + a'\mathbf{i} \subseteq B, \\
z_1 &= az_{-1} + a'F_\lambda(z_0) = az_{-1} + a'\mathbf{i} \in aB^{(m,\lambda)} + a'\mathbf{i} \subseteq B.
\end{aligned} \tag{4.4}$$

Table 6

Initial condition	Condition	Conclusion
$(z_{-2},z_{-1}) \in C^{(\lambda,p)} \times B^{(m,\lambda)}$	$0 \le p \le m$	$(z_{2p}, z_{2p+1}) \in B \times B$
$(z_{-2},z_{-1}) \in C^{(\lambda,p)} \times B^{(m,\lambda)}$	$0 \le m < p$	$(z_{2m}, z_{2m+1}) \in C \times A$
$(z_{-2},z_{-1}) \in B^{(p,\lambda)} \times C^{(\lambda,m)}$	$0 \le p \le m$	$(z_{2p}, z_{2p+1}) \in A \times C$
$(z_{-2},z_{-1}) \in B^{(p,\lambda)} \times C^{(\lambda,m)}$	$0 \le m < p$	$(z_{2m}, z_{2m+1}) \in B \times B$
$(z_{-2},z_{-1}) \in D^{(\lambda,k)} \times B^{(p,\lambda)}$	$0 \le k \le p$	$(z_{2k}, z_{2k+1}) \in B \times B$
$(z_{-2},z_{-1}) \in D^{(\lambda,k)} \times B^{(p,\lambda)}$	$0 \le p < k$	$(z_{2p}, z_{2p+1}) \in C \times A$
$(z_{-2},z_{-1}) \in C^{(p,\lambda)} \times D^{(\lambda,m)}$	$0 \le p \le m$	$(z_{2p}, z_{2p+1}) \in D \times D$
$(z_{-2},z_{-1}) \in C^{(p,\lambda)} \times D^{(\lambda,m)}$	$0 \le m < p$	$(z_{2m}, z_{2m+1}) \in C \times A$
$(z_{-2},z_{-1}) \in D^{(\lambda,p)} \times C^{(m,\lambda)}$	$0 \le p \le m$	$(z_{2p}, z_{2p+1}) \in A \times C$
$(z_{-2},z_{-1}) \in D^{(\lambda,p)} \times C^{(m,\lambda)}$	$0 \le m < p$	$(z_{2m}, z_{2m+1}) \in D \times D$
$(z_{-2},z_{-1}) \in B^{(k,\lambda)} \times D^{(\lambda,p)}$	$0 \le k \le p$	$(z_{2k}, z_{2k+1}) \in D \times D$
$(z_{-2},z_{-1}) \in B^{(k,\lambda)} \times D^{(\lambda,p)}$	$0 \le p < k$	$(z_{2p}, z_{2p+1}) \in C \times A$

Table 7: Initial condition: $(z_{-2}, z_{-1}) \in C^{(k,p)} \times C^{(l,m)}$.

Condition	Conclusion
$0 \le k = p \le \min\{l, m\}$	$(z_{2k}, z_{2k+1}) \in A \times C$
$0 \le k < \min\{l, m, p\}$	$(z_{2k}, z_{2k+1}) \in D^{(\lambda,p-k-1)} \times C^{(l-k-1,\lambda)}$
$0 \le p < \min\{k, l, m\}$	$(z_{2p}, z_{2p+1}) \in B^{(k-p-1,\lambda)} \times C^{(\lambda,m-p-1)}$
$0 \le p = l < \min\{m, k\}$	$(z_{2p}, z_{2p+1}) \in B^{(k-p-1,\lambda)} \times C^{(\lambda,m-p-1)}$
$0 \le p = l = m < k$	$(z_{2p}, z_{2p+1}) \in B \times B$
$0 \le l < \min\{m, k, p\}$	$(z_{2l}, z_{2l+1}) \in C^{(k-l-1,\lambda)} \times D^{(\lambda,m-l-1)}$
$0 \le l = m < \min\{p, k\}$	$(z_{2l}, z_{2l+1}) \in C \times A$
$0 \le l = k = m < p$	$(z_{2l}, z_{2l+1}) \in D \times D$
$0 \le m < \min\{l, k, p\}$	$(z_{2m}, z_{2m+1}) \in C^{(\lambda,p-m-1)} \times B^{(l-m-1,\lambda)}$
$0 \le k = m < \min\{l, p\}$	$(z_{2m}, z_{2m+1}) \in D^{(\lambda,p-k-1)} \times C^{(l-k-1,\lambda)}$
$0 \le p = m < \min\{l, k\}$	$(z_{2p}, z_{2p+1}) \in B \times B$
$0 \le k = l < \min\{p, m\}$	$(z_{2k}, z_{2l+1}) \in D \times D$

Table 8

	A	B	C	D
A	$\langle \mathbf{k}, \mathbf{0} \rangle$	$\langle \mathbf{i} \rangle$	$\langle \mathbf{0}, \mathbf{k} \rangle$	$\langle \mathbf{j} \rangle$
B	$\langle \mathbf{k}, \mathbf{0} \rangle$	$\langle \mathbf{i} \rangle$	$\langle \mathbf{0}, \mathbf{k} \rangle$ or $\langle \mathbf{i} \rangle$	$\langle \mathbf{j} \rangle$ or $\langle \mathbf{k}, \mathbf{0} \rangle$
C	$\langle \mathbf{k}, \mathbf{0} \rangle$	$\langle \mathbf{i} \rangle$ or $\langle \mathbf{k}, \mathbf{0} \rangle$	$\langle \mathbf{0}, \mathbf{k} \rangle$, $\langle \mathbf{k}, \mathbf{0} \rangle$, $\langle \mathbf{i} \rangle$ or $\langle \mathbf{j} \rangle$	$\langle \mathbf{j} \rangle$ or $\langle \mathbf{k}, \mathbf{0} \rangle$
D	$\langle \mathbf{k}, \mathbf{0} \rangle$	$\langle \mathbf{i} \rangle$ or $\langle \mathbf{k}, \mathbf{0} \rangle$	$\langle \mathbf{0}, \mathbf{k} \rangle$ or $\langle \mathbf{j} \rangle$	$\langle \mathbf{j} \rangle$

Table 9

Assumption	Conclusion
$(z_{-2}, z_{-1}) \in B \times B$	$(z_{2k}, z_{2k+1}) \in B \times B$
$(z_{-2}, z_{-1}) \in A \times C$	$(z_{2k}, z_{2k+1}) \in A \times C$
$(z_{-2}, z_{-1}) \in D \times D$	$(z_{2k}, z_{2k+1}) \in D \times D$
$(z_{-2}, z_{-1}) \in C \times A$	$(z_{2k}, z_{2k+1}) \in C \times A$

If $0 < p \le m$, then

$$\begin{aligned}
z_0 &= az_{-2} + a'F_\lambda(z_{-1}) = az_{-2} + a'\mathbf{i} \in aC^{(\lambda,p)} + a'\mathbf{i} \subseteq C^{(\lambda,p-1)},\\
z_1 &= az_{-1} + a'F_\lambda(z_0) = az_{-1} \in aB^{(m,\lambda)} \subseteq B^{(m-1,\lambda)},\\
z_2 &= az_0 + a'F_\lambda(z_1) = az_0 + a'\mathbf{i} \in aC^{(\lambda,p-1)} + a'\mathbf{i} \subseteq C^{(\lambda,p-2)},\\
z_3 &= az_1 + a'F_\lambda(z_2) = az_1 \in aB^{(m-1,\lambda)} \subseteq B^{(m-2,\lambda)},
\end{aligned} \tag{4.5}$$

and by induction,

$$\begin{aligned}
z_{2p} &= az_{2p-2} + a'F_\lambda\left(z_{2p-1}\right) = az_{2p-2} + a'\mathbf{i} \in aC^{(\lambda,0)} + a'\mathbf{i} \subseteq B,\\
z_{2p+1} &= az_{2p-1} + a'F_\lambda\left(z_{2p}\right) = az_{2p-1} + a'\mathbf{i} \in aB^{(m-p,\lambda)} + a'\mathbf{i} \subseteq B,
\end{aligned} \tag{4.6}$$

that is, $z_{2p} \in B$ and $z_{2p+1} \in B$.

As another example, let us show the second data row in Table 7. Suppose that $(z_{-2}, z_{-1}) \in C^{(k,p)} \times C^{(l,m)}$, where $0 \le k < \min\{l, m, p\}$. Then by Table 5, if $0 = k < \min\{l, m, p\}$,

$$\begin{aligned}
z_0 &= az_{-2} + a'F_\lambda(z_{-1}) = az_{-2} \in aC^{(0,p)} \subseteq D^{(\lambda,p-1)},\\
z_1 &= az_{-1} + a'F_\lambda(z_0) = az_{-1} + a'\mathbf{j} \in aC^{(l,m)} + a'\mathbf{j} \subseteq I^{(l-1)} \times (\lambda, \infty).
\end{aligned} \tag{4.7}$$

If $0 < k < \min\{l, m, p\}$, then

$$\begin{aligned}
z_0 &= az_{-2} + a'F_\lambda(z_{-1}) = az_{-2} \in aC^{(k,p)} \subseteq C^{(k-1,p-1)},\\
z_1 &= az_{-1} + a'F_\lambda(z_0) = az_{-1} \in aC^{(l,m)} \subseteq C^{(l-1,m-1)},\\
z_2 &= az_0 + a'F_\lambda(z_1) = az_0 \in aC^{(k-1,p-1)} \subseteq C^{(k-2,p-2)},\\
z_3 &= az_1 + a'F_\lambda(z_2) = az_1 \in aC^{(l-1,m-1)} \subseteq C^{(l-2,m-2)},
\end{aligned} \tag{4.8}$$

and by induction,

$$\begin{aligned}
z_{2k} &= az_{2k-2} + a'F_\lambda(z_{2k-1}) = az_{2k-2} \in aC^{(0,p-k)} \subseteq D^{(\lambda,p-k-1)},\\
z_{2k+1} &= az_{2k-1} + a'F_\lambda(z_{2k}) = az_{2k-1} + a'\mathbf{j} \in aC^{(l-k,p-k)} + a'\mathbf{j} \subseteq I^{(l-k-1)} \times (\lambda, \infty),
\end{aligned} \tag{4.9}$$

that is, $z_{2k} \in D^{(\lambda,p-k-1)}$ and $z_{2k+1} \in C^{(l-k-1,\lambda)}$.

Theorem 4.1. *Suppose that* $\lambda \in (0, 1-a)$. *Let* $\{z_n\}_{n=-2}^{\infty}$ *be a solution of* (1.9) *originated from* $(0,\infty)^2$. *Then*

$$z_n \longrightarrow \langle \mathbf{i} \rangle, \langle \mathbf{j} \rangle, \langle \mathbf{0}, \mathbf{k} \rangle \text{ or } \langle \mathbf{k}, \mathbf{0} \rangle. \tag{4.10}$$

Table 10

Initial condition	Condition	Conclusion
$A^{(p,\lambda)} \times B^{(\lambda,m)}$	$0 \le p \le m$	$(z_{2p}, z_{2p+1}) \in B \times B$
$A^{(p,\lambda)} \times B^{(\lambda,m)}$	$0 \le m < p$	$(z_{2m}, z_{2m+1}) \in A \times C$
$C^{(\lambda,p)} \times B^{(m,\lambda)}$	$0 \le p \le m$	$(z_{2p}, z_{2p+1}) \in B \times B$
$C^{(\lambda,p)} \times B^{(m,\lambda)}$	$0 \le m < p$	$(z_{2m}, z_{2m+1}) \in C \times A$
$D^{(p,\lambda)} \times A^{(\lambda,m)}$	$0 \le p \le m$	$(z_{2p}, z_{2p+1}) \in C \times A$
$D^{(p,\lambda)} \times A^{(\lambda,m)}$	$0 \le m < p$	$(z_{2m}, z_{2m+1}) \in D \times D$
$B^{(p,\lambda)} \times C^{(\lambda,m)}$	$0 \le p \le m$	$(z_{2p}, z_{2p+1}) \in A \times C$
$B^{(p,\lambda)} \times C^{(\lambda,m)}$	$0 \le m < p$	$(z_{2m}, z_{2m+1}) \in B \times B$
$A^{(\lambda,p)} \times D^{(m,\lambda)}$	$0 \le p \le m$	$(z_{2p}, z_{2p+1}) \in D \times D$
$A^{(\lambda,p)} \times D^{(m,\lambda)}$	$0 \le m < p$	$(z_{2m}, z_{2m+1}) \in A \times C$
$C^{(p,\lambda)} \times D^{(\lambda,m)}$	$0 \le p \le m$	$(z_{2p}, z_{2p+1}) \in D \times D$
$C^{(p,\lambda)} \times D^{(\lambda,m)}$	$0 \le m < p$	$(z_{2m}, z_{2m+1}) \in C \times A$
$B^{(\lambda,p)} \times A^{(m,\lambda)}$	$0 \le p \le m$	$(z_{2p}, z_{2p+1}) \in C \times A$
$B^{(\lambda,p)} \times A^{(m,\lambda)}$	$0 \le m < p$	$(z_{2m}, z_{2m+1}) \in B \times B$
$D^{(\lambda,p)} \times C^{(m,\lambda)}$	$0 \le p \le m$	$(z_{2p}, z_{2p+1}) \in A \times C$
$D^{(\lambda,p)} \times C^{(m,\lambda)}$	$0 \le m < p$	$(z_{2m}, z_{2m+1}) \in D \times D$

Table 11: Assumption: $(z_{-2}, z_{-1}) \in A^{(k,p)} \times A^{(l,m)}$.

Condition	Conclusion
$0 \le k = p \le \min\{l, m\}$	$(z_{2k}, z_{2k+1}) \in C \times A$
$0 \le p = l = m < k$	$(z_{2p}, z_{2p+1}) \in D \times D$
$0 \le l = m < \min\{k, p\}$	$(z_{2l}, z_{2l+1}) \in A \times C$
$0 \le l = m = k < p$	$(z_{2l}, z_{2l+1}) \in B \times B$
$0 \le p = m < \min\{k, l\}$	$(z_{2p}, z_{2p+1}) \in D \times D$
$0 \le k = l < \min\{p, m\}$	$(z_{2k}, z_{2k+1}) \in B \times B$
$0 \le k < \min\{p, l, m\}$	$(z_{2k}, z_{2k+1}) \in B^{(\lambda,p-k-1)} \times A^{(l-k-1,\lambda)}$
$0 \le p < \min\{k, l, m\}$	$(z_{2p}, z_{2p+1}) \in D^{(k-p-1,\lambda)} \times A^{(\lambda,m-p-1)}$
$0 \le p = l < \min\{k, m\}$	$(z_{2p}, z_{2p+1}) \in D^{(k-p-1,\lambda)} \times A^{(\lambda,m-p-1)}$
$0 \le l < \min\{p, k, m\}$	$(z_{2l}, z_{2l+1}) \in A^{(k-l-1,\lambda)} \times B^{(\lambda,m-l-1)}$
$0 \le m < \min\{p, k, l\}$	$(z_{2m}, z_{2m+1}) \in A^{(\lambda,p-m-1)} \times D^{(l-m-1,\lambda)}$
$0 \le k = m < \min\{p, l\}$	$(z_{2m}, z_{2m+1}) \in B^{(\lambda,p-k-1)} \times A^{(l-k-1,\lambda)}$

As in the proof of Theorem 3.1, we may construct Table 8.

For example, the (B, B) entry states that if $(z_{-2}, z_{-1}) \in B \times B$, then the solution $\{z_n\}$ of (1.9) originated from it will tend to $\langle \mathbf{i} \rangle$. Indeed, by Table 5, $z_0 \in B$, and then by induction, $z_k \in B$ for all $k \ge -2$. Hence, by (1.9), we see that

$$z_n = a z_{n-2} + (1 - a)\mathbf{i}, \quad n \in N, \tag{4.11}$$

from which we easily obtain

$$\begin{aligned} z_{2k} &= a^{k+1} z_{-2} + \left(1 - a^{k+1}\right)\mathbf{i}, \quad n \in N, \\ z_{2k+1} &= a^{k+1} z_{-1} + \left(1 - a^{k+1}\right)\mathbf{i}, \quad n \in N. \end{aligned} \tag{4.12}$$

Table 12: Assumption: $(z_{-2}, z_{-1}) \in C^{(k,p)} \times C^{(l,m)}$.

Condition	Conclusion
$0 \le k = p \le \min\{l, m\}$	$(z_{2k}, z_{2k+1}) \in A \times C$
$0 \le p = l = m < k$	$(z_{2p}, z_{2p+1}) \in B \times B$
$0 \le l = m < \min\{k, p\}$	$(z_{2l}, z_{2l+1}) \in C \times A$
$0 \le l = m = k < p$	$(z_{2l}, z_{2l+1}) \in D \times D$
$0 \le p = m < \min\{k, l\}$	$(z_{2p}, z_{2p+1}) \in B \times B$
$0 \le k = l < \min\{p, m\}$	$(z_{2k}, z_{2k+1}) \in D \times D$
$0 \le k < \min\{p, l, m\}$	$(z_{2k}, z_{2k+1}) \in D^{(\lambda, p-k-1)} \times C^{(l-k-1,\lambda)}$
$0 \le p < \min\{k, l, m\}$	$(z_{2p}, z_{2p+1}) \in B^{(p-k-1,\lambda)} \times C^{(\lambda, m-p-1)}$
$0 \le p = l < \min\{k, m\}$	$(z_{2p}, z_{2p+1}) \in B^{(k-p-1,\lambda)} \times C^{(\lambda, m-p-1)}$
$0 \le l < \min\{p, k, m\}$	$(z_{2l}, z_{2l+1}) \in C^{(k-l-1,\lambda)} \times D^{(\lambda, m-l-1)}$
$0 \le m < \min\{p, k, l\}$	$(z_{2m}, z_{2m+1}) \in C^{(\lambda, p-m-1)} \times B^{(l-m-1,\lambda)}$
$0 \le k = m < \min\{p, l\}$	$(z_{2m}, z_{2m+1}) \in D^{(\lambda, p-k-1)} \times C^{(l-k-1,\lambda)}$

Hence $z_{2k}, z_{2k+1} \to \mathbf{i}$ so that $z_n \to \mathbf{i}$. By similar reasoning, we may show the validity of the $(A, A), (A, B), (A, C), (A, D), (B, A), (C, A), (D, A)$, and (D, D) entries.

Next, suppose that $(z_{-2}, z_{-1}) \in B^{(p,\lambda)} \times C^{(\lambda,m)}$. Then the solution $\{z_n\}$ of (1.9) originated from it will tend to $\langle \mathbf{0}, \mathbf{k} \rangle$ or $\langle \mathbf{i} \rangle$. Indeed, by Table 6, if $0 \le p \le m$, then $(z_{2p}, z_{2p+1}) \in A \times C$ and by induction, $(z_{2n+2p}, z_{2n+2p+1}) \in A \times C$ for all $n \in N$. Hence, by (1.9), we have

$$z_{2n+2p} = a^n z_{2p}, \quad n \in N,$$
$$z_{2n+2p+1} = a^n z_{2p+1} + (1 - a^n)\mathbf{k}, \quad n \in N. \tag{4.13}$$

Hence, $z_{2n} \to \mathbf{0}, z_{2n+1} \to \mathbf{k}$. If $0 \le m < p$, then $(z_{2m}, z_{2m+1}) \in B \times B$. By previous argument, we see $z_n \to \mathbf{i}$. By similar reasoning, we may show the correctness of the other entries. The proof is complete.

5. The Case $\lambda \in [1 - a, 1)$

Suppose that $\lambda \in [1-a, 1)$, then $a\lambda + 1 - a > \lambda$. Therefore, we may continue to use the notations described in the previous case $\lambda \in (0, 1 - a)$ and proceed as in the previous two sections and derive Tables 9, 10, 11, 12, 13 and 14.

By means of these tables, we may then derive the following result.

Theorem 5.1. *Suppose that* $\lambda \in [1 - a, 1)$. *Let* $\{z_n\}$ *be a solution of* (1.9) *originated from* $(0, \infty)^2$. *Then*

$$z_n \longrightarrow \langle \mathbf{i} \rangle, \langle \mathbf{j} \rangle, \langle \mathbf{k}, \mathbf{0} \rangle \text{ or } \langle \mathbf{0}, \mathbf{k} \rangle. \tag{5.1}$$

The proof amounts to showing the validity of Table 15.

Table 13: Assumption: $(z_{-2}, z_{-1}) \in D^{(k,p)} \times B^{(l,m)}$.

Condition	Conclusion
$0 \le k = p \le \min\{l, m\}$	$(z_{2k}, z_{2k+1}) \in B \times B$
$0 \le p = l = m < k$	$(z_{2p}, z_{2p+1}) \in A \times C$
$0 \le l = m < \min\{k, p\}$	$(z_{2l}, z_{2l+1}) \in D \times D$
$0 \le l = m = k < p$	$(z_{2l} z_{2l+1}) \in C \times A$
$0 \le p = m < \min\{k, l\}$	$(z_{2p}, z_{2p+1}) \in A \times C$
$0 \le k = l < \min\{p, m\}$	$(z_{2k}, z_{2k+1}) \in C \times A$
$0 \le k < \min\{p, l, m\}$	$(z_{2k}, z_{2k+1}) \in C^{(\lambda,p-k-1)} \times B^{(l-k-1,\lambda)}$
$0 \le p < \min\{k, l, m\}$	$(z_{2p}, z_{2p+1}) \in A^{(k-p-1,\lambda)} \times B^{(\lambda,m-p-1)}$
$0 \le p = l < \min\{k, m\}$	$(z_{2p}, z_{2p+1}) \in A^{(k-p-1,\lambda)} \times B^{(\lambda,m-p-1)}$
$0 \le l < \min\{p, k, m\}$	$(z_{2l}, z_{2l+1}) \in D^{(k-l-1,\lambda)} \times A^{(\lambda,m-l-1)}$
$0 \le m < \min\{p, k, l\}$	$(z_{2m}, z_{2m+1}) \in D^{(\lambda,p-m-1)} \times C^{(l-m-1,\lambda)}$
$0 \le k = m < \min\{p, l\}$	$(z_{2m}, z_{2m+1}) \in C^{(\lambda,p-k-1)} \times B^{(l-k-1,\lambda)}$

Table 14: Assumption: $(z_{-2}, z_{-1}) \in B^{(p,k)} \times D^{(m,l)}$.

Condition	Conclusion
$0 \le k = p \le \min\{l, m\}$	$(z_{2k}, z_{2k+1}) \in D \times D$
$0 \le p = l = m < k$	$(z_{2p}, z_{2p+1}) \in A \times C$
$0 \le l = m < \min\{k, p\}$	$(z_{2l}, z_{2l+1}) \in B \times B$
$0 \le l = m = k < p$	$(z_{2l}, z_{2l+1}) \in C \times A$
$0 \le p = m < \min\{k, l\}$	$(z_{2p}, z_{2p+1}) \in A \times C$
$0 \le k = l < \min\{p, m\}$	$(z_{2k}, z_{2k+1}) \in C \times A$
$0 \le k < \min\{p, l, m\}$	$(z_{2k}, z_{2k+1}) \in C^{(p-k-1,\lambda)} \times D^{(\lambda,l-k-1)}$
$0 \le p < \min\{k, l, m\}$	$(z_{2p}, z_{2p+1}) \in A^{(\lambda,k-p-1)} \times D^{(m-p-1,\lambda)}$
$0 \le p = l < \min\{k, m\}$	$(z_{2p}, z_{2p+1}) \in A^{(\lambda,k-p-1)} \times D^{(m-p-1,\lambda)}$
$0 \le l < \min\{p, k, m\}$	$(z_{2l}, z_{2l+1}) \in B^{(\lambda,k-l-1)} \times A^{(m-l-1,\lambda)}$
$0 \le m < \min\{p, k, l\}$	$(z_{2m}, z_{2m+1}) \in B^{(p-m-1,\lambda)} \times C^{(\lambda,l-m-1)}$
$0 \le k = m < \min\{p, l\}$	$(z_{2m}, z_{2m+1}) \in C^{(p-k-1,\lambda)} \times D^{(\lambda,l-k-1)}$

For example, the $(B^{(p,k)}, D^{(m,l)})$ entry states that if $(z_{-2}, z_{-1}) \in B^{(p,k)} \times D^{(m,l)}$ where $0 \le p < \min\{k, l, m\}$, then the solution $\{z_n\}$ of (1.9) originated from it will tend to $\langle \mathbf{j} \rangle$ or $\langle \mathbf{0}, \mathbf{k} \rangle$. Indeed, by Table 14, $(z_{2p}, z_{2p+1}) \in A^{(\lambda,k-p-1)} \times D^{(m-p-1,\lambda)}$. Furthermore, by Table 10, if $0 \le k - p - 1 \le m - p - 1$, then $(z_{2k}, z_{2k+1}) \in D \times D$. Hence, by (1.9), we see that

$$z_n = a z_{n-2} + (1 - a)\mathbf{j}, \quad n \ge 2k + 2, \tag{5.2}$$

from which we easily obtain

$$\begin{aligned} z_{2n+2k} &= a^n z_{2k} + (1 - a^n)\mathbf{j}, \quad n \in \mathbf{N}, \\ z_{2n+2k+1} &= a^n z_{2k+1} + (1 - a^n)\mathbf{j}, \quad n \in \mathbf{N}. \end{aligned} \tag{5.3}$$

Hence, $z_n \to \mathbf{j}$.

Table 15

	A	B	C	D
A	$\langle \mathbf{k}, \mathbf{0}\rangle, \langle \mathbf{0}, \mathbf{k}\rangle, \langle \mathbf{i}\rangle$ or $\langle \mathbf{j}\rangle$	$\langle \mathbf{i}\rangle$ or $\langle \mathbf{0}, \mathbf{k}\rangle$	$\langle \mathbf{0}, \mathbf{k}\rangle$	$\langle \mathbf{j}\rangle$ or $\langle \mathbf{0}, \mathbf{k}\rangle$
B	$\langle \mathbf{k}, \mathbf{0}\rangle$ or $\langle \mathbf{i}\rangle$	$\langle \mathbf{i}\rangle$	$\langle \mathbf{0}, \mathbf{k}\rangle$ or $\langle \mathbf{i}\rangle$	$\langle \mathbf{k}, \mathbf{0}\rangle, \langle \mathbf{0}, \mathbf{k}\rangle, \langle \mathbf{i}\rangle$ or $\langle \mathbf{j}\rangle$
C	$\langle \mathbf{k}, \mathbf{0}\rangle$	$\langle \mathbf{i}\rangle$ or $\langle \mathbf{k}, \mathbf{0}\rangle$	$\langle \mathbf{0}, \mathbf{k}\rangle, \langle \mathbf{k}, \mathbf{0}\rangle, \langle \mathbf{i}\rangle$ or $\langle \mathbf{j}\rangle$	$\langle \mathbf{j}\rangle$ or $\langle \mathbf{k}, \mathbf{0}\rangle$
D	$\langle \mathbf{k}, \mathbf{0}\rangle$ or $\langle \mathbf{j}\rangle$	$\langle \mathbf{k}, \mathbf{0}\rangle, \langle \mathbf{0}, \mathbf{k}\rangle, \langle \mathbf{i}\rangle$ or $\langle \mathbf{j}\rangle$	$\langle \mathbf{0}, \mathbf{k}\rangle$ or $\langle \mathbf{j}\rangle$	$\langle \mathbf{j}\rangle$

If $0 \le m-p-1 < k-p-1$, then $(z_{2m}, z_{2m+1}) \in A\times C$. By Table 9 again, $(z_{2n}, z_{2n+1}) \in A\times C$ for all $n \ge m$. Hence, by (1.9), we see that

$$z_{2n} = az_{2n-2}, \quad n > m,$$
$$z_{2n+1} = az_{2n-1} + (1-a)\mathbf{k}, \quad n > m, \tag{5.4}$$

from which we easily obtain

$$z_{2n+2m} = a^n z_{2m}, \quad n \in \mathbf{N},$$
$$z_{2n+2m+1} = a^n z_{2m+1} + (1-a^n)\mathbf{k}, \quad n \in \mathbf{N}. \tag{5.5}$$

Hence, $z_{2n} \to \mathbf{0}, z_{2n} \to \mathbf{k}$. By similar reasoning, we may derive the other entries of Table 15. The proof is complete.

6. Conclusions and Remarks

We have discussed a simple two-nation arms race model with a positive threshold λ hidden in a nonlinear piecewise constant control function. Treating λ as a bifurcation parameter which varies from 0^+ to $+\infty$, we have discussed the limiting behaviors of all possible solutions of (1.9) originated from the positive orthant $(0,\infty)^2$.

Let $\mathbf{0} = (0,0)$, $\mathbf{i} = (1,0)$, $\mathbf{j} = (0,1)$, and $\mathbf{k} = (1,1)$.

(i) For $\lambda > 1$, all solutions of (1.9) originated from the positive orthant tend to $\langle \mathbf{k}\rangle$.

(ii) For $\lambda = 1$, all such solutions must either be tending to $\langle \mathbf{i}\rangle$, $\langle \mathbf{j}\rangle$, or $\langle \mathbf{k}\rangle$ or to $\langle \mathbf{0}, \mathbf{k}\rangle, \langle \mathbf{k}, \mathbf{0}\rangle, \langle \mathbf{i}, \mathbf{k}\rangle, \langle \mathbf{k}, \mathbf{i}\rangle, \langle \mathbf{j}, \mathbf{k}\rangle$, or $\langle \mathbf{k}, \mathbf{j}\rangle$.

(iii) For $0 < \lambda < 1$, all such solutions must either be tending to the steady states $\langle \mathbf{i}\rangle$ or $\langle \mathbf{j}\rangle$, or to $\langle \mathbf{k}, \mathbf{0}\rangle$ or $\langle \mathbf{0}, \mathbf{k}\rangle$.

Recall that a sequence $\{z_n\}_{n=-2}^{\infty}$ is asymptotically ω-periodic if it can be expressed as the sum of two sequences $\{p_n\}_{n=-2}^{\infty}$ and $\{q_n\}_{n=-2}^{\infty}$, where $p_n \to 0$ and $\{q_n\}_{n=-2}^{\infty}$ is periodic with prime period ω. Therefore, as a direct consequence of our investigations, all solutions of (1.9) originated from the positive orthants must either tend to limit 1-cycles $\langle \mathbf{i}\rangle$, $\langle \mathbf{j}\rangle$, or $\langle \mathbf{k}\rangle$ or to limit 2-cycles $\langle \mathbf{0}, \mathbf{k}\rangle$, $\langle \mathbf{i}, \mathbf{k}\rangle$, or $\langle \mathbf{k}, \mathbf{j}\rangle$. Such a conclusion meets our expectation of a "terror equilibrium" in nuclear arms races.

We conclude our investigations with the following remarks. One may object that the above east-west view of conflicts has less bite nowadays and that this way of thinking of the problem is far too simple. Indeed, there is now a second-generation literature that incorporates real strategic thinking, with sound foundation in decision theory and game theory (see, for instance, [10–12]). Yet to the best of our knowledge, there is no complete mathematical analysis similar to those described above. We hope that our results will be useful in furthering the mathematical investigation of arms race models based on more recent and realistic social models.

Acknowledgment

The first author is supported by the National Science Foundation (no. 11161049).

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