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Research Article

Existence of Solution to a Second-Order Boundary Value Problem via Noncompactness Measures

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The existence and uniqueness of the solutions to the Dirichlet boundary value problem in the Banach spaces is discussed by using the fixed point theory of condensing mapping, doing precise computation of measure of noncompactness, and calculating the spectral radius of linear operator.

1. Introduction

This paper is mainly concerned with the following second-order Dirichlet boundary value problem:

$$\begin{gathered} -u''(t) = f(t, u(t)), \quad t \in I = [0,1], \\ u(0) = u(1) = \theta, \end{gathered} \tag{1.1}$$

in a Banach space E, where $f(t, x) \in C(I \times E, E)$, θ is the zero element of E.

In the last several decades, there has been much attention focused on the boundary value problems for various nonlinear ordinary differential equations, difference equations, and functional differential equations, see [1–20] and the references therein. The existence of solutions for Neumann boundary value problems has been considerably investigated in many publications such as [2–5, 8–10]. Dirichlet boundary value problems have deserved the attention of many researchers, see [11–20] and the references therein.

In particular, the authors in [11] have studied the following two-point boundary value problem:

$$x'' = H(t, x, x'), \quad 0 < t < 1, \qquad ax(0) - bx'(0) = x_0, \qquad cx(1) + dx'(1) = x_1, \tag{1.2}$$

where $a, b, c, d \geq 0$ and $ad + bc > 0$. They obtained the existence of solutions by means of the Darbo fixed point theorem and properties of the measure of noncompactness.

We would like to mention the results due to [11]. First, we point out that many authors applied the famous Sadovskii's fixed point theorem to investigate similar problems and used the following hypothesis with respect to the Kuratowski measure of noncompactness $\alpha(\cdot)$: there exists a constant $k > 0$ such that for any bounded and equicontinuous set $A, B \subset C(I, E)$ and $t \in I$, $\alpha(H(I \times A \times B)) \leq k \max\{\alpha(A), \alpha(B)\}$. What is more, they required a stronger condition, that is, $\|H(t, x, y)\| \leq L$ for $(t, x, y) \in I \times E \times E$ and the constant k satisfies $0 < k < 1/2$ (see Remarks 3.2–3.6).

The authors in [15, 18] have studied the following boundary value problem:

$$x'' = f(t, x, x'), \qquad c_1 x(a) - d_1 x'(a) = x_1, \qquad c_2 x(b) + d_2 x'(b) = x_2, \tag{1.3}$$

where X is a real Banach space, $J = [a, b] \subset \mathbb{R}$, $f : J \times X^2 \rightarrow X$ is continuous, $c_i, d_i \in \mathbb{R}$, and $x_i \in X$ for $i = 1, 2$. They obtained the existence of solutions by means of Sadovskii's fixed point theorem and properties of the measure of noncompactness.

Motivated by the above-mentioned work [11, 15, 18], the main aim of this paper is to study the existence and uniqueness of solutions for the problem (1.1) under the new conditions. The main new features presented in this paper are as follows First, the existence and uniqueness of solutions to Banach space's Dirichlet boundary value problem is proved precisely calculating the spectral radius of linear operation. Second, the conditions imposed on the BVP (1.1) are weak. Third, the main tools used in the analysis are Sadovskii's fixed point theorem and precise computation of measure of noncompactess. Our results can be seen as a supplement of the results in [11] (see Remarks 3.2–3.6).

This paper is organized as follows. In Section 2, we provide some basic definitions, preliminaries facts, and various lemmas which will be used throughout this paper. In Section 3, we give main results in this paper.

2. Preliminaries and Lemmas

Let E be a real Banach space and P be a cone in E which defines a partial ordering in E by $x \leq y$ if and only if $y - x \in P \cdot P$ is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of E, and the smallest N is called the normal constant of P (it is clear, $N \geq 1$). If $x \leq y$ and $x \neq y$, we write $x < y$. For details on cone theory, see the monograph [7].

Let $I = [0, 1]$. By $C(I, E)$ we denote the Banach space of all continuous functions from I into E with the norm

$$\|y\|_c := \max\{|y(t)| : t \in I\}. \tag{2.1}$$

Definition 2.1 (see [7]). Assume that S is a bounded set in E. Let $\alpha(S) = \inf\{\delta > 0 : S$ be expressed as the union $S = \cup_{i=1}^{m} S_i$ of a finite number of sets S_i with diameter $\operatorname{diam}(S_i) \leq \delta\}$.

$\alpha(S)$ is said to be the *Kuratowski* measure of noncompactness and is called the noncompactness measure for short. For details and properties of the noncompactness measure see [7].

Definition 2.2 (see [7]). The mapping A is said to be a condensing operator if A is continuous, bounded, and for any nonrelatively compact and bounded set $S \subset D$,

$$\alpha(A(S)) < \alpha(S). \tag{2.2}$$

The following lemmas are of great importance in the proof of our main results.

Lemma 2.3. *Suppose that $M \notin \{-n^2\pi^2 \mid n = 1,2,3,\ldots\}$. Then for any $h \in C(I,E)$, the linear boundary value problem*

$$\begin{gathered} -u''(t) + Mu(t) = h(t), \quad t \in I = [0,1], \\ u(0) = u(1) = \theta, \end{gathered} \tag{2.3}$$

has a unique solution $u := T_M h \in C^2(I,E)$, and $T_M : C(I,E) \mapsto C(I,E)$ is bounded linear operator.

Proof. Note that $M \notin \{-n^2\pi^2 \mid n = 1,2,3,\ldots\}$, which assures second-order boundary value problem

$$\begin{gathered} -\gamma''(t) + M\gamma(t) = 0, \\ \gamma(0) = \gamma(1) = 0, \end{gathered} \tag{2.4}$$

has only a zero solution. To obtain a solution of the problem (2.3), we require a mapping whose kernel $G_M(t,s) : I \times I \to \mathbb{R}$ is the Green's function of the boundary value problem (2.4). Let $\beta = \sqrt{|M|}$, we consider three cases.

Case 1. if $M > 0$, we have

$$G_M(t,s) = \begin{cases} \dfrac{\sinh(\beta t)\sinh(\beta(1-s))}{\beta \sinh \beta}, & \text{if } 0 \leq t \leq s \leq 1, \\ \dfrac{\sinh(\beta s)\sinh(\beta(1-t))}{\beta \sinh \beta}, & \text{if } 0 \leq s \leq t \leq 1. \end{cases} \tag{2.5}$$

Case 2. if $M = 0$, we have

$$G_0(t,s) = \begin{cases} t(1-s), & \text{if } 0 \leq t \leq s \leq 1, \\ s(1-t), & \text{if } 0 \leq s \leq t \leq 1. \end{cases} \tag{2.6}$$

Case 3. if $M < 0$, $M \neq -n^2\pi^2$, $n = 1, 2, 3, \ldots$, we have

$$G_M(t,s) = \begin{cases} \dfrac{\sin(\beta t)\sin(\beta(1-s))}{\beta \sin\beta}, & \text{if } 0 \leq t \leq s \leq 1, \\ \dfrac{\sin(\beta s)\sin(\beta(1-t))}{\beta \sin\beta}, & \text{if } 0 \leq s \leq t \leq 1. \end{cases} \tag{2.7}$$

After direct computations, it is easy to see that

$$u(t) = \int_0^1 G_M(t,s)h(s)ds := (T_M h)(t) \tag{2.8}$$

is continuously differentiable, and $u(t)$ is a solution of (2.3).

We now claim that solution of the boundary value problem (2.3) is unique. The proof is as follows. If possible, suppose that $v(t) \in C^2(I,E)$ is another solution of the problem (2.3). For any $\varphi \in E^*$ (E^* denotes the dual space of E), let $p(t) = \varphi(u(t) - v(t))$, thus we obtain $p(t) \in C^2(I)$, $p''(t) = \varphi(u''(t) - v''(t))$. By (2.3), we have

$$\begin{gathered} -p''(t) + Mp(t) = 0, \\ p(0) = p(1) = 0, \end{gathered} \tag{2.9}$$

that is, $p(t)$ is a solution of the boundary value problem (2.4). However, on the other hand, problem (2.4) has only a zero solution, therefore we have $\varphi(u(t)-v(t)) = 0$. Thus, we get $u(t)-v(t) = \theta$, hence $u(t) \equiv v(t)$ in I, which implies that solution of the problem (2.3) is unique, say, $u := T_M h$, and $T_M : C(I,E) \to C(I,E)$.

It is easy to see that T_M is bounded linear operator. This completes the proof. □

Remark 2.4. If $M > -\pi^2$, it is easy to see that $G_M(t,s) \geq 0$.

Lemma 2.5. *Assume that $M > -\pi^2$, and $T_M : C(I,E) \mapsto C(I,E)$ is given by* (2.8). *Then*

(1) *the spectral radius $r(T_M) = 1/(M + \pi^2)$;*

(2) *If E is an ordered Banach space, then T_M is a positive operator, that is, if $h \geq 0$, then $T_M h \geq 0$.*

Proof. (1) Define operator $\mathcal{L} : \mathfrak{D}(\mathcal{L}) \mapsto C(I,E)$ by

$$\mathcal{L}u := -u'' + Mu, \tag{2.10}$$

where $\mathfrak{D}(\mathcal{L}) = \{u \in C^2(I,E) \mid u(0) = u(1) = \theta\}$. By Lemma 2.3, we have that T_M is bounded invertible operator of $\mathcal{L}$, and if $\lambda \neq M + n^2\pi^2, n = 1, 2, 3, \ldots$, then $\mathcal{L} - \lambda I$ has a bounded invertible operator, thus $\lambda \in \rho(\mathcal{L})$.

Let $\lambda = \lambda_n = M + n^2\pi^2$, λ_n is a eigenvalue of $\mathcal{L}$. For any $x \in E$, $x \neq \theta$, since $\sin\sqrt{M + n^2\pi^2}t$ is eigenvector of λ_n, then the spectrum of operator $\mathcal{L}$ is $\sigma(\mathcal{L}) = \{M + n^2\pi^2 \mid n = 1, 2, 3, \ldots\}$. By the spectral mapping theorem [21], we get

$$\sigma(T_M) = \{\theta\} \cup \left\{ \frac{1}{M + n^2\pi^2} \mid n \in \mathcal{N} \right\}, \tag{2.11}$$

so $r(T_M) = 1/(M + \pi^2)$.

(2) If $h \geq 0$, by definition of $G_M(t, s)$ and $M > -\pi^2$, then we get $G_M(t, s) \geq 0$, so by (2.8), we have $T_M h \geq 0$, that is, T_M is a positive operator. This achieves the proof. □

Remark 2.6. In particular. If $E = \mathbb{R}^1$, $M = 0$, then by (1) of Lemma 2.5, we get $r(T_0) = 1/\pi^2$, where T_0 is an operator in $C(I)$:

$$(T_0\varphi)(t) = \int_0^1 G_0(t, s)\varphi(s)ds, \tag{2.12}$$

and $\|T_0\|_c = 1/8$. In fact:

$$\left|(T_0\varphi)(t)\right| = \left|\int_0^1 G_0(t, s)\varphi(s)ds\right| \leq \int_0^1 G_0(t, s)ds \cdot \|\varphi\|_c = \frac{1}{2}t(t-1) \cdot \|\varphi\|_c \leq \frac{1}{8} \cdot \|\varphi\|_c. \tag{2.13}$$

This means that $\|T_0\varphi\|_c \leq (1/8)\|\varphi\|_c$, therefore $\|T_0\|_c \leq 1/8$. However, on the other hand, $\|T_0(1)\|_c = 1/8$. As a result, we obtain $\|T_0\|_c = 1/8$.

Lemma 2.7. *Let $J = [a, b]$, $u \in C(J, E)$, $\varphi \in C(J, \mathbb{R}^+)$. Then*

$$\int_a^b \varphi(s)u(s)ds \in \left(\int_a^b \varphi(s)ds\right) \cdot \overline{\mathrm{co}}u(J), \tag{2.14}$$

where $u(J) = \{u(t) \mid t \in J\}$, $\overline{\mathrm{co}}u(J)$ is closed convex hull of $u(J)$.

Proof. If $\int_a^b \varphi(s)ds = 0$, then (2.14) is true. We suppose that $\int_a^b \varphi(s)ds > 0$, and take a partition of $[a, b]$:

$$\Delta_n : a = t_0^{(n)} < t_1^{(n)} < t_2^{(n)} < \cdots < t_m^{(n)} = b. \tag{2.15}$$

Let $\Delta t_i^{(n)} = t_i^{(n)} - t_{i-1}^{(n)}$, $\|\Delta_n\| = \max\{\Delta t_i^{(n)} : 1 \leq i \leq m_n\}$, by definition of *Riemann* integral, we get

$$\begin{gathered}\int_a^b \varphi(s)u(s)ds = \lim_{n\to\infty}\sum_{n=1}^{m_n}\varphi\left(t_i^{(n)}\right)u\left(t_i^{(n)}\right)\Delta t_i^{(n)}, \\ \int_a^b \varphi(t)dt = \lim_{n\to\infty}\sum_{n=1}^{m_n}\varphi\left(t_i^{(n)}\right)\Delta t_i^{(n)}.\end{gathered} \tag{2.16}$$

We take n sufficiently large, such that $\|\Delta_n\| \to 0$, then we get

$$\frac{\int_a^b \varphi(s)u(s)ds}{\int_a^b \varphi(s)ds} = \lim_{n\to\infty}\sum_{n=1}^{m_n}\frac{\varphi\left(t_i^{(n)}\right)u\left(t_i^{(n)}\right)\Delta t_i^{(n)}}{\sum_{n=1}^{m_n}\varphi\left(t_i^{(n)}\right)\Delta t_i^{(n)}} \in \overline{\mathrm{co}}u(J). \tag{2.17}$$

This finishes the proof. □

Lemma 2.8. *Suppose that D is a bounded set in E, then there exists a countable subset D_1 of D, such that*

$$\alpha(D) \le 2\alpha(D_1). \tag{2.18}$$

Proof. Let $\alpha(D) > 0$, $D \ne \emptyset$. For $r_n = \alpha(D)(1-1/n)$, take $x_1^{(n)} \in D$, then $D \setminus B(x_1^{(n)}, r_n/2)$ is not a cover of D. Take $x_2^{(n)} \in D \setminus B(x_2^{(n)}, r_n/2)$, then $B(x_2^{(n)}, r_n/2)$ is not a cover of D. Continuing this process, take $x_{k+1}^{(n)} \in D \setminus \cup_{i=1}^{k} B(x_i^{(n)}, r_n/2)$, then $B(x_i^{(n)}, r_n/2)$ $(i = 1,2,3,\dots,k)$ is not a cover of D.

Set $D_n = \{x_k^{(n)} \mid k = 1,2,3,\dots\}$, then we get $d(s,t) \ge r_n/2$, where $d(s,t)$ denote the distance between two points s and t of D_n. Thus it follows that $\alpha(D_n) \ge r_n/2$.

Setting $D_1 = \cup_{i=1}^{\infty} D_n$, choose n sufficiently large such that $\alpha(D_1) \ge \alpha(D_n) \ge r_n/2 \to \alpha(D)/2$, that is, $\alpha(D) \le 2\alpha(D_1)$. The proof is completed. □

Lemma 2.9. *If B is a bounded set in $C(I,E)$, $B(I) = \{u(t) \mid u \in B,\ t \in I\} \subset E$. Then*

$$\alpha(B(I)) \le 2\alpha(B). \tag{2.19}$$

Proof. For any $\varepsilon > 0$, there exists a partition $B = \cup_{i=1}^{n} B_i$ such that

$$\mathrm{diam}(B_i) < \alpha(B) + \varepsilon, \tag{2.20}$$

for $i = 1,2,3,\dots,n$. Choose $u_i \in B_i$ $(i = 1,2,3,\dots,n)$. Since u_i is uniformly continuous on I, there exists $\delta > 0$, such that $t', t'' \in I$, and $|t' - t''| < \delta$, we have

$$\left\|u_i(t') - u_i(t'')\right\| < \varepsilon. \tag{2.21}$$

Let $\Delta : 0 = t_0 < t_1 < t_2 < t_3 < \cdots < t_m = 1$ be a partition of I, and $\|\Delta\| < \delta$. Set $I_j = [t_{j-1}, t_j]$, $D_{ij} = B_i(I_j) = \{u(t) \mid u \in B_i, t \in I_j\}$. Clearly, we have

$$D = B(I) = \bigcup_{i=1}^{n} B_i(I) = \bigcup_{j=1}^{m}\bigcup_{i=1}^{n} B_i(I_j). \tag{2.22}$$

For any $u, v \in B_i$ and $t, s \in I_j$, it follows from (2.20) and (2.21) that

$$\begin{aligned} \|u(t) - v(s)\| &\leq \|u(t) - u_i(t)\| + \|u_i(t) - u_i(s)\| + \|v(s) - u_i(s)\| \\ &\leq \|u - u_i\|_c + \varepsilon + \|v - u_i\|_c \\ &\leq 2\,\mathrm{diam}(B_i) + \varepsilon \\ &< 2\alpha(B) + 3\varepsilon. \end{aligned} \tag{2.23}$$

So,

$$\mathrm{diam}(D_{ij}) = \sup_{u,v\in B_i;\, t,s\in I_j} \|u(t) - v(s)\| \leq 2\alpha(B) + 3\varepsilon. \tag{2.24}$$

Thus it follows that $\alpha(D) \leq 2\alpha(B) + 3\varepsilon$. Therefore, by using the arbitrariness of ε, we have $\alpha(B(I)) \leq 2\alpha(B)$. The lemma is proved. □

Lemma 2.10 (see [7]). *Assume that $H \subset C[J, E]$ is bounded and equicontinuous. Then $\alpha(H(t))$ is continuous on J and*

$$\alpha\left(\left\{\int_J x(t)dt : x \in H\right\}\right) \leq \int_J \alpha(H(t))dt. \tag{2.25}$$

Lemma 2.11 (see [7]). *Suppose that H is a countable family of strongly measurable functions $x : J \mapsto E$. If there exists a function $M \in L[J, \mathbb{R}^+]$ such that $\|x(t)\| \leq M(t)$ for a.e. $t \in J$, then $\alpha(H(t)) \in L[J, \mathbb{R}^+]$ and*

$$\alpha\left(\left\{\int_J x(t)dt : x \in H\right\}\right) \leq 2\int_J \alpha(H(t))dt. \tag{2.26}$$

Lemma 2.12. *Assume that Ω_1 is equicontinuous in $C(I, E)$. Then $\overline{\mathrm{co}}(\Omega_1)$ is equicontinuous.*

Proof. For any $\varepsilon > 0$, it follows from the equicontinuity of Ω_1 that there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies $\|u(t_1) - u(t_2)\| < \varepsilon/3$ for all $t_1, t_2 \in I$ and $u \in \Omega_1$.

For any $h \in \overline{\mathrm{co}}(\Omega_1)$, by virtue of definition of $\overline{\mathrm{co}}(\Omega_1)$, we have

$$\forall u_1, u_2, \ldots, u_n \in \Omega_1, \lambda_1, \lambda_2, \ldots, \lambda_n > 0, \quad \sum_{i=1}^n \lambda_i = 1, \quad \left\|\sum_{i=1}^n \lambda_i u_i - h\right\|_c < \frac{\varepsilon}{3}. \tag{2.27}$$

Thus, we get

$$\begin{aligned}
\|h(t_1) - h(t_2)\| &\leq \left\| h(t_1) - \sum_{i=1}^{n} \lambda_i u_i(t_1) \right\| + \left\| \sum_{i=1}^{n} \lambda_i u_i(t_2) - h(t_2) \right\| \\
&\quad + \left\| \sum_{i=1}^{n} \lambda_i [u_i(t_1) - u_i(t_2)] \right\| \\
&\leq 2 \left\| \sum_{i=1}^{n} \lambda_i u_i - h \right\|_c + \sum_{i=1}^{n} \lambda_i \|u_i(t_1) - u_i(t_2)\| \\
&< \frac{2}{3}\varepsilon + \sum_{i=1}^{n} \lambda_i \cdot \frac{1}{3}\varepsilon \\
&= \varepsilon.
\end{aligned} \tag{2.28}$$

Hence, $\overline{\text{co}}(\Omega_1)$ is equicontinuous. This finishes the proof. □

Lemma 2.13 (see [7]). *If $H \subset C[I, E]$ is bounded and equicontinuous, then $\alpha(B(t))$ is continuous in I and $\alpha(B) = \alpha(B(I)) = \max_{x \in I} \alpha(B(t))$.*

Lemma 2.14 (see [7] (Sadovskii's Theorem)). *Assume that D is a nonempty bounded, closed, and convex set. If a mapping $A : D \mapsto D$ is condensing, then A has a fixed point in D.*

3. Main Results

In this section, we present and prove our main results.

Theorem 3.1. *Let E be a Banach space. Suppose that $f(t, x) \in C(I \times E, E)$ and the following conditions hold:*

(H_1) *there exist two positive numbers c_0 and c_1, such that*

$$\|f(t, x)\| \leq c_0 + c_1 \|x\|, \quad \forall t \in I, \ x \in E, \tag{3.1}$$

(H_2) *for any a bounded set D in E, there exists a constant $L > 0$ such that*

$$\alpha\big(f(I \times D)\big) \leq L\alpha(D), \tag{3.2}$$

(H_3) *there exist two positive numbers c_1 and L with $c_1 < \pi^2$, $L < 4$.*

Then problem (1.1) *has at least one solution.*

Proof. Define the integral operator $A : C(I, E) \mapsto C(I, E)$

$$(Au)(t) = \int_0^1 G_0(t, s) f(s, u(s)) ds = T_0\big(f(\cdot, u)\big). \tag{3.3}$$

Then $A : C(I, E) \mapsto C(I, E)$ is continuous, and it is clear that u is a solution of the problem (1.1) if and only if u is a fixed point of A.

We now show that A is a condensing operator. Let B be bounded in $C(I, E)$, by (H_1), we claim that $\{-(Au)'' \mid u \in B\}$ is bounded. Since $(Au)(0) = 0$, we know that $\{(Au)' \mid u \in B\}$ is bounded, this means that $A(B)$ is equicontinuous. Therefore, it follows from Lemma 2.13 that $\alpha(A(B)) = \max_{x\in I}\alpha(A(B)(t))$.

For any $u \in B$, $t \in I$, from Lemma 2.7, we have

$$\begin{aligned}(Au)(t) &= \int_0^1 G_0(t,s) f(s,u(s))ds \\ &\in \left(\int_0^1 G_0(t,s)ds\right) \cdot \overline{\mathrm{co}}\{f(s,u(s)) \mid s \in I\} \\ &\subset \left(\int_0^1 G_0(t,s)ds\right) \cdot \overline{\mathrm{co}}\{f(I \times B(I))\}.\end{aligned} \tag{3.4}$$

Hence,

$$(A(B))(t) \subset \left(\int_0^1 G_0(t,s)ds\right) \cdot \overline{\mathrm{co}}\{f(I \times B(I))\}. \tag{3.5}$$

Using the properties of the noncompactness measure together with (H_2), we obtain

$$\begin{aligned}\alpha(A(B)(t)) &\le \int_0^1 G_0(t,s)ds \cdot \alpha\big(f(I \times B(I))\big) \\ &\le L\alpha(B(I)) \cdot \left(\int_0^1 G_0(t,s)ds\right) \\ &= \frac{1}{2}t(1-t) \cdot L\alpha(B(I)) \\ &\le \frac{1}{8}L\alpha(B(I)).\end{aligned} \tag{3.6}$$

By Lemma 2.9, we have

$$\alpha(A(B)(t)) \le \frac{1}{8}L \cdot \alpha(B(I)) \le \frac{1}{8}L \cdot 2\alpha(B) = \frac{1}{4}L \cdot \alpha(B). \tag{3.7}$$

Hence,

$$\alpha(A(B)) \le \frac{1}{4}L \cdot \alpha(B). \tag{3.8}$$

By (H_3), we get $0 < L/4 < 1$, therefore A is condensing.

Let $\Omega := B_c(\theta, R) = \{u \in C(I, E) : \|u\| < R\}$, we will prove that $u - \lambda Au \ne \theta$, $0 < \lambda \le 1$, $u \in \partial\Omega$ for R sufficiently large. By means of the homotopy invariance theorem, we have

$\deg(I - A, \Omega, \theta) = 1$. By virtue of the solvability of Kronecker [6], we know that A has a fixed point in Ω, and the fixed point of A is a solution of the problem (1.1).

Indeed. If there exists a constant $\lambda_0 \in (0,1]$, $\lambda_0 \in \partial\Omega$ such that $u_0 - \lambda_0 A u_0 = \theta$, then u_0 satisfies

$$u_0 = \lambda_0 \int_0^1 G_0(t,s) f(s, u_0(s)) ds. \tag{3.9}$$

Let $\varphi_0(t) = \|u_0(t)\|$, and T_0 is an operator in $C(I)$ defined by

$$(T_0\varphi)(t) = \int_0^1 G_0(t,s)\varphi(s) ds. \tag{3.10}$$

Then by Lemma 2.5 we have $r(T_0) = 1/\pi^2$.

By (3.9) and (H_1), we have

$$\begin{aligned} \varphi_0(t) &\le \int_0^1 G_0(t,s)\big(c_0 + c_1\varphi_0(s)\big) ds \\ &= c_0 T_0(1) + c_1 \big(T_0\varphi_0\big)(t) \\ &\le c_0\|T_0\| + c_1\big(T_0\varphi_0\big)(t) \\ &\le c_0 + c_1\big(T_0\varphi_0\big)(t). \end{aligned} \tag{3.11}$$

Thus,

$$\varphi_0 \le c_0\|T_0\| + c_1 T_0\varphi_0 \le c_0 + c_1 T_0 \varphi_0, \tag{3.12}$$

continuing this process, by induction, we obtain

$$\begin{aligned} \varphi_0 &\le c_0 + c_1 T_0\big(c_0 + c_1 T_0\varphi_0\big) \\ &= c_0 + c_0 c_1\|T_0\| + c_1^2 T_0^2 \varphi_0 \\ &\le \cdots \\ &\le c_0\Big(1 + c_1\|T_0\| + c_1^2\left\|T_0^2\right\| + \cdots + c_1^{n-1}\left\|T_0^{n-1}\right\| + c_1^n\left\|T_0^n\right\|\varphi_0\Big), \end{aligned} \tag{3.13}$$

which implies that

$$\varphi_0 \le c_0 \sum_{n=0}^{n-1} c_1^n \left\|T_0^n\right\| + c_1^n T_0^n \varphi_0. \tag{3.14}$$

By the Gelfand theorem [22], we have

$$r(T_0) = \lim_{n\to\infty} \sqrt[n]{\left\|T_0^n\right\|} = \frac{1}{\pi^2}. \tag{3.15}$$

Set $c_2 = (c_1 + \pi^2)/2 \in (c_1, \pi^2)$, then we have $1/c_2 > 1/\pi^2$. By (3.15), there exists an integer N_0, such that $\sqrt[n]{\|T_0^n\|} < 1/c_2$ as $n \geq N_0$, that is, $\|T_0^n\| < 1/c_2^n$, this means that $c_1^n\|T_0^n\| < (c_1/c_2)^n$. In view of series $\sum_{n=0}^{\infty}(c_1/c_2)^n$ converges, we know that $\sum_{n=0}^{\infty} c_1^n\|T_0^n\|$ also converges.

Denote $R_0 = c_0 \sum_{n=0}^{\infty} c_1^n\|T_0^n\|$. By (3.14), we get $\varphi_0(t) \leq R_0$, which implies that $\|u_0(t)\| \leq R_0$, hence $\|u_0\|_c \leq R_0$.

Take $R > R_0$, then we have $u - \lambda Au \neq \theta$, for all $\lambda \in (0,1]$, $u \in \partial\Omega$. Thus problem (1.1) has at least one solution. This proves the theorem. □

Remark 3.2. In [11], the nonlinear term $f(t,u,u')$ is bounded, if $f(t,u,u') = f(t,u)$, in our result, the nonlinear term $f(t,u)$ may no more than a linear growth.

Remark 3.3. In [11], if $f(t,u,u') = f(t,u)$, the growth restriction of L for (H_2) is $0 < L < 1/2$. However, in our result, L satisfies $0 < L < 4$.

Theorem 3.4. *Let E be a Banach space, and $f(t,x) \in C(I \times E, E)$. Assume that condition* (H_1) *and the following conditions hold:*

$(\mathrm{H}_2)'$ *for all $t \in I$, for any a bounded set D in E, there exists a constant $L > 0$ such that*

$$\alpha\big(f(t,D)\big) \leq L\alpha(D), \tag{3.16}$$

$(\mathrm{H}_3)'$ *there exist two positive numbers c_1 and L with $0 < c_1 < \pi^2$, $L < 2$.*

Then problem (1.1) *has at least one solution.*

Proof. Assume that the operator A is defined the same as in Theorem 3.1. We show that the operator A is condensing. In fact, for a bounded set $B \in C(I,E)$, there exists a countable subset $B_1 = \{u_n\}$, such that $\alpha(A(B)) \leq 2\alpha(A(B_1))$. However, on the other hand, we have

$$\alpha(A(B_1)) = \max_{t\in I} \alpha(A(B_1)(t)). \tag{3.17}$$

By Lemma 2.11 and $(\mathrm{H}_2)'$, we obtain

$$\begin{aligned}
\alpha(A(B_1)) &= \alpha\left(\left\{\int_0^1 G_0(t,s)f(s,u_n(s))ds \mid n \in \mathbb{N}\right\}\right) \\
&\leq 2\int_0^1 G_0(t,s)\alpha\big(\{f(s,u_n(s)) \mid n \in \mathbb{N}\}\big)ds \\
&\leq 2\int_0^1 G_0(t,s)L\alpha(B_1(s))ds \\
&\leq 2L\int_0^1 G_0(t,s)ds \cdot \alpha(B_1) \\
&\leq \frac{1}{4}L\alpha(B_1).
\end{aligned} \tag{3.18}$$

Thus,

$$\alpha(A(B)) \leq \frac{1}{2} L\alpha(B). \tag{3.19}$$

By $(\mathrm{H}_3)'$, we get $0 < L/2 < 1$, therefore A is condensing. By using the same arguments of Theorem 3.1, we can obtain the conclusion of Theorem 3.4. The detailed proof is omitted here. The proof is achieved. □

Next, we establish a uniqueness of solution for the problem (1.1).

Theorem 3.5. *Let E be a Banach space. Suppose that $f(t, x) \in C(I \times E, E)$ and that there exists a constant L with $0 < L < \pi^2$ such that*

$$\left\| f(t, u_2) - f(t, u_1) \right\| \leq L \|u_2 - u_1\|, \quad \forall u_1, u_2 \in E. \tag{3.20}$$

Then problem (1.1) *has a unique solution.*

Proof. Assume that operator A is defined the same as in Theorem 3.1, and the fixed point of A is a solution of the problem (1.1).

We will prove that for sufficiently large n the operator A^n is a contraction operator. Indeed, by the definition of A and (3.20), we have the estimate

$$\begin{aligned}
\|(A^n u_2)(t) - (A^n u_1)(t)\| &= \left\| \int_0^1 G_0(t, s) \left[f\left(s, \left(A^{n-1} u_2\right)(s)\right) - f\left(s, \left(A^{n-1} u_1\right)(s)\right) \right] ds \right\| \\
&\leq L \int_0^1 G_0(t, s) \left\| \left(A^{n-1} u_2\right)(s) - \left(A^{n-1} u_1\right)(s) \right\| ds \\
&= L T_0 \left(\left\| \left(A^{n-1} u_2\right)(s) - \left(A^{n-1} u_1\right)(s) \right\| \right).
\end{aligned} \tag{3.21}$$

By induction, we have

$$\begin{aligned}
\|(A^n u_2)(t) - (A^n u_1)(t)\| &\leq L^n \cdot T_0^n \cdot \|u_2(s) - u_1(s)\| \\
&\leq L^n \cdot \left\| T_0^n \right\| \cdot \|u_2 - u_1\|_c.
\end{aligned} \tag{3.22}$$

Thus,

$$\|A^n u_2 - A^n u_1\|_c \leq L^n \cdot \left\| T_0^n \right\| \cdot \|u_2 - u_1\|_c. \tag{3.23}$$

Moreover, we can choose n to be sufficiently large such that $\sqrt[n]{\|T_0^n\|}$ tends to $r(T_0) = 1/\pi^2$.

Further, take $L_1 \in (L, \pi^2)$, there exists an integer N_0, such that $\|T_0\| \leq 1/L_1^n$ as $n \geq N_0$. By (3.23), we obtain

$$\begin{aligned} \|A^n u_2 - A^n u_1\|_c &\leq L^n \cdot \left\|T_0^n\right\| \cdot \|u_2 - u_1\|_c \\ &\leq \left(\frac{L}{L_1}\right)^n \cdot \|u_2 - u_1\|_c, \end{aligned} \tag{3.24}$$

which implies that A^n is a contraction mapping by $L/L_1 < 1$. By the contraction mapping principle, we conclude that there exists a unique fixed point for A, this proves that problem (1.1) has a unique solution. This completes the proof. □

Remark 3.6. By the direct application of the Banach contraction mapping principle, the conclusion of Theorem 3.5 holds true under the condition $0 < L < 8$. However, we require the condition $0 < L < \pi^2$, here π^2 is optimum.

The following theorem is concerned with the existence of positive solutions for problem (1.1).

Theorem 3.7. *Let E be an ordered Banach space, K be a normal cone with positive elements. Suppose that $f(t,x) \in C(I \times E, E)$ satisfy the following conditions:*

(P_1) *there exists a constant c with $0 < c < \pi^2$, and $h_0 \in C(I, K)$, such that*

$$\theta \leq f(t,x) \leq cx + h_0(t), \quad \forall x \geq \theta, \tag{3.25}$$

(P_2) *for any a bounded set D in E, there exists a constant L with $0 < L < 8$ such that*

$$\alpha\big(f(I \times D)\big) \leq L\alpha(D). \tag{3.26}$$

Then problem (1.1) has at least one positive solution.

Proof. Consider the linear boundary value problem

$$-u'' - cu = h_0, \qquad u(0) = u(1) = \theta. \tag{3.27}$$

By Lemma 2.3, the linear boundary value problem (3.27) has unique a positive solution $u_0 \in C(I, K)$.

Set $D = [\theta, u_0] \subset C(I, E)$, then D is bounded and convex closed set in $C(I, E)$. For any $u \in D$, by (P_1), we have

$$\theta \leq f(t, u(t)) \leq cu(t) + h_0(t) \leq cu_0(t) + h_0(t). \tag{3.28}$$

Multiply by $G_0(t,s)$ and integrate from 0 to 1, we obtain

$$\theta \le Au(t) \le \int_0^1 G_0(t,s)[cu_0(s)+h_0(s)]ds = u_0(t), \tag{3.29}$$

that is, $Au \in D$, so $A(D) \subset D$. By the proof of Theorem 3.1, it follows that A is equicontinuous. Thus, by Lemma 2.12, we know that $\Omega_0 = \overline{\mathrm{co}}(A(D))$ is equicontinuous.

Next we show that $A : \Omega_0 \to \Omega_0$ is condensing. For any $B \subset \Omega_0$, then B is bounded and equicontinuous, therefore $A(B) \subset \Omega_0$ is bounded and equicontinuous. By Lemma 2.13, we have

$$\alpha(A(B)) = \max_{t\in I} \alpha(A(B(t))). \tag{3.30}$$

However, on the other hand, we have

$$\alpha(A(B(t))) = \alpha\left(\left\{\int_0^1 G_0(t,s)f(s,u(s))ds \mid u \in B\right\}\right). \tag{3.31}$$

Thus, For any $u \in B$, $t \in I$, we acquire

$$\begin{aligned} Au(t) &= \int_0^1 G_0(t,s)f(s,u(s))ds \\ &\in \left(\int_0^1 G_0(t,s)ds\right)\cdot \overline{\mathrm{co}}\{f(s,u(s)) \mid s \in I\} \\ &\subset \left(\int_0^1 G_0(t,s)ds\right)\cdot \overline{\mathrm{co}}\{f(I\times B(I))\}. \end{aligned} \tag{3.32}$$

Hence,

$$A(B)(t) \subset \left(\int_0^1 G_0(t,s)ds\right)\cdot \overline{\mathrm{co}}\{f(I\times B(I))\}. \tag{3.33}$$

Further, we obtain

$$\begin{aligned} \alpha(\mathrm{A}(B)(t)) &\le \left(\int_0^1 G_0(t,s)ds\right)\cdot \alpha\big(f(I\times B(I))\big) \\ &\le \frac{L}{2}t(1-t)\cdot \alpha(B(I)) \\ &\le \frac{L}{8}\cdot \alpha(B). \end{aligned} \tag{3.34}$$

This means that

$$\alpha(A(B)) \leq \frac{L}{8} \cdot \alpha(B). \tag{3.35}$$

By (P_2), we have $0 < L/8 < 1$, so A is condensing. Applying Lemma 2.14, we conclude that A has a fixed point which is a solution of problem (1.1). The proof of the theorem is completed. □

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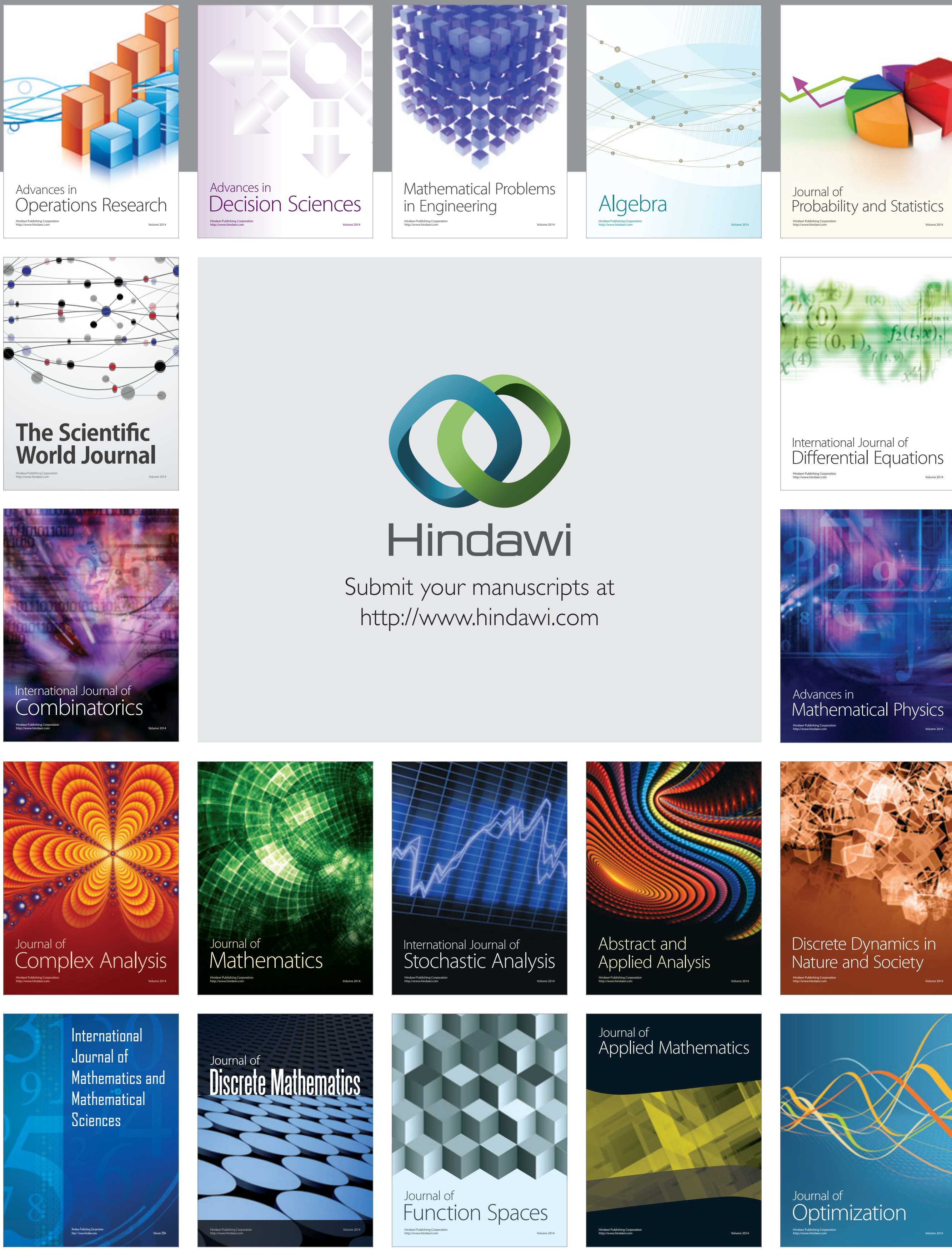

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