Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 835893, 12 pages
doi:10.1155/2012/835893

Research Article

Stability of a Bi-Additive Functional Equation in Banach Modules Over a $C^\star$-Algebra

Won-Gil Park[1] and Jae-Hyeong Bae[2]

[1] *Department of Mathematics Education, College of Education, Mokwon University, Daejeon 302-729, Republic of Korea*
[2] *Graduate School of Education, Kyung Hee University, Yongin 446-701, Republic of Korea*

Correspondence should be addressed to Jae-Hyeong Bae, jhbae@khu.ac.kr

Received 6 April 2012; Accepted 30 May 2012

Academic Editor: Baodong Zheng

We solve the bi-additive functional equation $f(x+y,z-w)+f(x-y,z+w)=2f(x,z)-2f(y,w)$ and prove that every biadditive Borel function is bilinear. And we investigate the stability of a biadditive functional equation in Banach modules over a unital $C^\star$-algebra.

1. Introduction

In 1940, Ulam proposed the stability problem (see [1]).

Let G_1 be a group, and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, this problem was solved by Hyers [2] in the case of Banach space. Thereafter, many authors investigated solutions or stability of various functional equations (see [3–21]).

Let X and Y be real or complex vector spaces. In 1989, Aczél and Dhombres [22] proved that a mapping $g : X \to Y$ satisfies the quadratic functional equation

$$g(x+y)+g(x-y)=2g(x)+2g(y) \tag{1.1}$$

if and only if there exists a symmetric bi-additive mapping $S : X \times X \to Y$ such that $g(x) = S(x,x)$, where

$$S(x,y) := \frac{1}{4}\left[g(x+y)-g(x-y)\right] \tag{1.2}$$

for all $x, y \in X$. For a mapping $f : X \times X \to Y$, consider the bi-additive functional equation:

$$f(x+y, z-w) + f(x-y, z+w) = 2f(x,z) - 2f(y,w). \tag{1.3}$$

For a mapping $g : X \to Y$ satisfying (1.1), the Aczél's bi-additive mapping $S : X \times X \to Y$ given by (1.2) is a solution of (1.3).

In this paper, we find out the general solution of the bi-additive functional equation (1.3) and investigate the linearity of bi-additive Borel functions. And we investigate the stability of (1.3) in Banach modules over a unital C^*-algebra.

2. Solution of the bi-additive Functional Equation (1.3)

The general solution of the bi-additive functional equation (1.3) is as follows.

Theorem 2.1. *A mapping $f : X \times X \to Y$ satisfies* (1.3) *if and only if the mapping f is bi-additive.*

Proof. Assume that the mapping f satisfies (1.3). Letting $x = y = z = w = 0$ in (1.3), we gain $f(0,0) = 0$. Putting $w = z$ in (1.3), we get

$$f(x+y, 0) + f(x-y, 2z) = 2f(x,z) - 2f(y,z) \tag{2.1}$$

for all $x, y, z \in X$. Setting $y = x$ in (2.1), we have

$$f(x,0) = -f(0,z) \tag{2.2}$$

for all $x, z \in X$. Taking $z = 0$ (resp., $x = 0$) in the above equation, we obtain

$$f(x,0) = 0 \ (\text{resp., } f(0,z) = 0) \tag{2.3}$$

for all $x \in X$ (resp., for all $z \in X$). Letting $x = w = 0$ in (1.3) and using (2.3), we gain

$$f(-y, z) = -f(y,z) \tag{2.4}$$

for all $y, z \in X$. Putting $y = 0$ in (2.1) and using (2.3), we get

$$f(x, 2z) = 2f(x,z) \tag{2.5}$$

for all $x, z \in X$. Replacing y by $-y$ in (2.1) and using (2.3), (2.4), and (2.5) and the above equation, we see that $f(x+y, z) = f(x,z) + f(y,z)$ for all $x, y, z \in X$.

On the other hand, letting $y = x$ in (1.3) and using (2.3), we gain

$$f(2x, z-w) = 2f(x,z) - 2f(x,w) \tag{2.6}$$

for all $x, z, w \in X$. Putting $y = z = 0$ in (1.3) and using (2.3), we get

$$f(x, -w) = -f(x, w) \tag{2.7}$$

for all $x, w \in X$. Setting $w = 0$ in (2.6) and using (2.3), we have

$$f(2x, z) = 2f(x, z) \tag{2.8}$$

for all $x, z \in X$. Replacing w by $-w$ in (2.6) and using (2.7) and (2.8), we obtain that $f(x, z + w) = f(x, z) + f(x, w)$ for all $x, z, w \in X$.

The converse is trivial. □

The bi-additive functional equation (1.3) is related to the quadratic functional equation (1.1).

If $f : X \times X \to Y$ is a mapping satisfying (1.3) and $g : X \to Y$ is the mapping given by $g(x) := f(x, x)$ for all $x \in X$, then one can easily obtain that g satisfies (1.1).

Let $a \in \mathbb{R}$ and $g : X \to Y$ be a mapping satisfying (1.1). If $f : X \times X \to Y$ is the mapping given by $f(x, y) := (a/4)[g(x + y) - g(x - y)]$ for all $x, y \in X$, then one can easily prove that f satisfies (1.3). Furthermore, $g(x) = f(x, x)$ holds for all $x \in X$ if $a = 1$.

The following is a result on bi-additive Borel functions.

Theorem 2.2. *Let $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bi-additive Borel function; then it is bilinear, that is, it satisfies $\psi(s, t) = st\psi(1, 1)$ for all $s, t \in \mathbb{R}$.*

Proof. Since the function ψ is bi-additive, we gain

$$\psi(pu, qv) = pq\psi(u, v) \tag{2.9}$$

for all $p, q \in \mathbb{Q}$ and all $u, v \in \mathbb{R}$. Letting $p = v = 1$ in equality (2.9), we get

$$\psi(u, q) = q\psi(u, 1) \tag{2.10}$$

for all $q \in \mathbb{Q}$ and all $u \in \mathbb{R}$. Putting $u = v = 1$ in equality (2.9) again, we have

$$\psi(p, q) = pq\psi(1, 1) \tag{2.11}$$

for all $p, q \in \mathbb{Q}$. Note that the function $v \to \psi(u, v)$ is measurable for each fixed $u \in \mathbb{R}$ (see [23, Proposition 2.34]). Since the function $v \to \psi(u, v)$ is additive for each fixed $u \in \mathbb{R}$, by [24], it is continuous for each fixed $u \in \mathbb{R}$. By the same reasoning, the function $u \to \psi(u, v)$ is also continuous for each fixed $v \in \mathbb{R}$. Let $s, t \in \mathbb{R}$ be fixed. Since ψ is measurable, by [25, Theorem 7.14.26], for every $m \in \mathbb{N}$ there is a closed set $F_m \subset [s, s + 1]$ such that $\mu([s, s + 1] \setminus F_m) < 1/m$ and $\psi|_{F_m \times \mathbb{R}}$ is continuous. Since $\mu(F_m) \to 1$, one can choose $u_m \in F_m$ satisfying $u_m \to s$.

Take a sequence $\{q_n\}$ in $\mathbb{Q}$ converging to t. For each fixed $m \in \mathbb{N}$, take a sequence $\{p_n\}$ in $\mathbb{Q}$ converging to u_m. By equalities (2.10) and (2.11), we see that

$$\begin{aligned}\psi(u_m, t) &= \psi\left(u_m, \lim_{n\to\infty} q_n\right) = \lim_{n\to\infty} \psi(u_m, q_n) = \lim_{n\to\infty} q_n\psi(u_m, 1) = t\psi(u_m, 1)\\ &= t\psi\left(\lim_{n\to\infty} p_n, 1\right) = t\lim_{n\to\infty} \psi(p_n, 1) = t\lim_{n\to\infty} p_n\psi(1, 1) = tu_m\psi(1, 1)\end{aligned} \tag{2.12}$$

for all $m \in \mathbb{N}$. Hence we obtain that

$$\psi(s, t) = \psi\left(\lim_{m\to\infty} u_m, t\right) = \lim_{m\to\infty} \psi(u_m, t) = \lim_{m\to\infty} tu_m\psi(1, 1) = st\psi(1, 1), \tag{2.13}$$

as desired. □

3. Stability of the bi-additive Functional Equation (1.3)

From now on, let X be a normed space, Y a complete normed space, and $r \neq 2$ a nonnegative real number. In this section, we investigate the stability of the bi-additive functional equation (1.3).

Lemma 3.1. *Let $f : X \times X \to Y$ be a mapping such that*

$$\begin{aligned}&\|f(x+y, z-w) + f(x-y, z+w) - 2f(x, z) + 2f(y, w)\|\\ &\quad\le \begin{cases} 4\varepsilon, & (r = 0),\\ \varepsilon(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r), & (0 < r \neq 2)\end{cases}\end{aligned} \tag{3.1}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $F : X \times X \to Y$ satisfying (1.3) *such that*

$$\|f(x, y) - F(x, y)\| \le \begin{cases} 2\varepsilon + \|f(0, 0)\|, & (r = 0),\\ \dfrac{3\varepsilon}{4 - 2^r}(\|x\|^r + \|y\|^r), & (0 < r < 2),\\ \dfrac{3\varepsilon}{2^r - 4}(\|x\|^r + \|y\|^r), & (r > 2)\end{cases} \tag{3.2}$$

for all $x, y \in X$. The mapping F is given by

$$F(x, y) := \begin{cases} \lim_{j\to\infty} \dfrac{1}{4^j} f(2^j x, 2^j y), & (0 \le r < 2),\\ \lim_{j\to\infty} 4^j f\left(\dfrac{x}{2^j}, \dfrac{y}{2^j}\right), & (r > 2)\end{cases} \tag{3.3}$$

for all $x, y \in X$.

Proof. Consider the case $r \in (0,2)$. Letting $y = x$ and $w = -z$ in (3.1), we gain

$$\left\| f(2x,2z) + f(0,0) - 2f(x,z) + 2f(x,-z) \right\| \leq 2\varepsilon\left(\|x\|^r + \|z\|^r\right) \tag{3.4}$$

for all $x, z \in X$. Putting $x = z = 0$ in (3.4), we get $f(0,0) = 0$. Putting $x = z = 0$ in (3.1), we get

$$\left\| f(y,-w) + f(-y,w) + 2f(y,w) \right\| \leq \varepsilon\left(\left\|y\right\|^r + \|w\|^r\right) \tag{3.5}$$

for all $y, w \in X$. Replacing y by x and w by z in the above inequality, we have

$$\| f(x,-z) + f(-x,z) + 2f(x,z) \| \leq \varepsilon(\|x\|^r + \|z\|^r) \tag{3.6}$$

for all $x, z \in X$. Setting $y = -x$ and $w = z$ in (3.1), we obtain

$$\left\| f(2x,2z) - 2f(x,z) + 2f(-x,z) \right\| \leq 2\varepsilon\left(\|x\|^r + \|z\|^r\right) \tag{3.7}$$

for all $x, z \in X$. By (3.4) and (3.6), we gain

$$\left\| f(2x,2z) - 4f(x,z) + f(x,-z) - f(-x,z) \right\| \leq 3\varepsilon\left(\|x\|^r + \|z\|^r\right) \tag{3.8}$$

for all $x, z \in X$. By (3.4) and (3.7), we get

$$\left\| f(x,-z) - f(-x,z) \right\| \leq 2\varepsilon\left(\|x\|^r + \|z\|^r\right) \tag{3.9}$$

for all $x, z \in X$. By (3.4), (3.6), and (3.7), we have

$$\left\| f(2x,2z) - 4f(x,z) \right\| \leq 3\varepsilon\left(\|x\|^r + \|z\|^r\right) \tag{3.10}$$

for all $x, z \in X$. Replacing x by $2^j x$ and z by $2^j z$ and dividing 4^{j+1}, we obtain that

$$\left\| \frac{1}{4^j} f\left(2^j x, 2^j z\right) - \frac{1}{4^{j+1}} f\left(2^{j+1} x, 2^{j+1} z\right) \right\| \leq \frac{3\varepsilon \cdot 2^{rj}}{4^{j+1}} \left(\|x\|^r + \|z\|^r\right) \tag{3.11}$$

for all $x, z \in X$ and all $j = 0, 1, 2, \ldots$. For given integers l, m $(0 \leq l < m)$, we obtain that

$$\left\| \frac{1}{4^l} f\left(2^l x, 2^l z\right) - \frac{1}{4^m} f\left(2^m x, 2^m z\right) \right\| \leq \sum_{j=l}^{m-1} \frac{3\varepsilon \cdot 2^{rj}}{4^{j+1}} \left(\|x\|^r + \|z\|^r\right) \tag{3.12}$$

for all $x, z \in X$. By (3.12), the sequence $\{(1/4^j)f(2^jx, 2^jy)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{(1/4^j)f(2^jx, 2^jy)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by $F(x, y) := \lim_{j\to\infty}(1/4^j)f(2^jx, 2^jy)$ for all $x, y \in X$. By (3.1), we have

$$\left\| \frac{1}{4^j} f\left(2^j(x+y), 2^j(z-w)\right) + \frac{1}{4^j} f\left(2^j(x-y), 2^j(z+w)\right) - \frac{2}{4^j} f\left(2^jx, 2^jz\right) + \frac{2}{4^j} f\left(2^jy, 2^jw\right) \right\| \leq \varepsilon \frac{2^{rj}}{4^j}\left(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\right) \tag{3.13}$$

for all $x, y, z, w \in X$ and all $j = 0, 1, 2, \ldots$. Letting $j \to \infty$ in the above inequality, we see that F satisfies (1.3). Setting $l = 0$ and taking $m \to \infty$ in (3.12), one can obtain inequality (3.2). If $G : X \times X \to Y$ is another mapping satisfying (1.3) and (3.2), by Theorem 2.1, we obtain that

$$\begin{aligned} \|F(x,y) - G(x,y)\| &= \frac{1}{4^n}\|F(2^nx, 2^ny) - G(2^nx, 2^ny)\| \\ &\leq \frac{1}{4^n}\|F(2^nx, 2^ny) - f(2^nx, 2^ny)\| + \frac{1}{4^n}\|f(2^nx, 2^ny) - G(2^nx, 2^ny)\| \\ &\leq \frac{6\varepsilon \cdot 2^{n(r-2)}}{4-2^r}\left(\|x\|^r + \|y\|^r\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \end{aligned} \tag{3.14}$$

for all $x, y \in X$. Hence the mapping F is the unique bi-additive mapping satisfying (1.3), as desired.

The proof of the case $r \in \{0\} \cup (2, \infty)$ is similar to that of the case $r \in (0, 2)$. □

From now on, let A be a unital $C^\star$-algebra with a norm $|\cdot|$, and let ${}_A\mathcal{M}$ and ${}_A\mathcal{N}$ be left Banach A-modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Put $A_1 := \{a \in A \mid |a| = 1\}$.

A bi-additive mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (1.3) is called *A-quadratic* if $F(ax, ay) = a^2F(x, y)$ for all $a \in A$ and all $x, y \in {}_A\mathcal{M}$.

Theorem 3.2. *Let $f : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ be a mapping such that*

$$\left\| f(ax+ay, az-aw) + f(ax-ay, az+aw) - 2a^2f(x,z) + 2a^2f(y,w) \right\| \leq \begin{cases} 4\varepsilon, & (r = 0), \\ \varepsilon\left(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\right), & (0 < r \neq 2) \end{cases} \tag{3.15}$$

for all $a \in A_1$ and all $x, y, z, w \in {}_A\mathcal{M}$. If $f(tx, ty)$ is continuous in $t \in \mathbb{R}$ for each fixed $x, y \in {}_A\mathcal{M}$, then there exists a unique bi-additive A-quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (1.3) *and inequality* (3.2).

Proof. Consider the case $r \in (0, 2)$. By Lemma 3.1, it follows from the inequality of the statement for $a = 1$ that there exists a unique bi-additive mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (1.3) and inequality (3.2). Let $x_0, y_0 \in {}_A\mathcal{M}$ be fixed. And let $L : {}_A\mathcal{N} \to \mathbb{R}$ be any real continuous linear functional, that is, L is an arbitrary real functional element of the dual space of ${}_A\mathcal{N}$ restricted to the scalar field $\mathbb{R}$. For $n \in \mathbb{N}$, consider the functions $\psi_n : \mathbb{R} \to \mathbb{R}$ defined by $\psi_n(t) := (1/4^n)L[f(2^n tx_0, 2^n ty_0)]$ for all $t \in \mathbb{R}$. By the assumption that $f(tx, ty)$ is continuous in $t \in \mathbb{R}$ for each fixed $x, y \in {}_A\mathcal{M}$, the function ψ_n is continuous for all $n \in \mathbb{N}$. Note that $\psi_n(t) = (1/4^n)L[f(2^n tx_0, 2^n ty_0)] = L[(1/4^n)f(2^n tx_0, 2^n ty_0)]$ for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$. By the proof of Lemma 3.1, the sequence $\psi_n(t)$ is a Cauchy sequence for all $t \in \mathbb{R}$. Define a function $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(t) := \lim_{n\to\infty}\psi_n(t)$ for all $t \in \mathbb{R}$. Note that $\psi(t) = L[F(tx_0, ty_0)]$ for all $t \in \mathbb{R}$. Since F is bi-additive, we get

$$\begin{aligned}
\psi(s+t) + \psi(s-t) &= L\left(F\left[(s+t)x_0, (s+t)y_0\right]\right) + L\left(F\left[(s-t)x_0, (s-t)y_0\right]\right) \\
&= L\left(F\left[(s+t)x_0, (s+t)y_0\right] + F\left[(s-t)x_0, (s-t)y_0\right]\right) \\
&= L\left[F(sx_0 + tx_0, sy_0 + ty_0) + F(sx_0 - tx_0, sy_0 - ty_0)\right] \\
&= L\left[2F(sx_0, sy_0) + 2F(tx_0, ty_0)\right] \\
&= 2L\left[F(sx_0, sy_0)\right] + 2L\left[F(tx_0, ty_0)\right] = 2\psi(s) + 2\psi(t)
\end{aligned} \tag{3.16}$$

for all $s, t \in \mathbb{R}$. Since ψ is the pointwise limit of continuous functions, it is a Borel function. Thus the function ψ as a measurable quadratic function is continuous (see [26]) so has the form $\psi(t) = t^2\psi(1)$ for all $t \in \mathbb{R}$. Hence we have

$$L\left[F(tx_0, ty_0)\right] = \psi(t) = t^2\psi(1) = t^2 L\left[F(x_0, y_0)\right] = L\left[t^2 F(x_0, y_0)\right] \tag{3.17}$$

for all $t \in \mathbb{R}$. Since L is any continuous linear functional, the bi-additive mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfies $F(tx_0, ty_0) = t^2 F(x_0, y_0)$ for all $t \in \mathbb{R}$. Therefore we obtain

$$F(tx, ty) = t^2 F(x, y) \tag{3.18}$$

for all $t \in \mathbb{R}$ and all $x, y \in {}_A\mathcal{M}$. Let j be an arbitrary positive integer. Replacing x and z by $2^j x$ and $2^j z$, respectively, and letting $y = w = 0$ in inequality (3.15), we gain

$$\left\| f\left(2^j ax, 2^j az\right) - a^2 f\left(2^j x, 2^j z\right) + a^2 f(0,0) \right\| \le 2^{rj-1}\varepsilon\left(\|x\|^r + \|z\|^r\right) \tag{3.19}$$

for all $a \in A_1$ and all $x, z \in {}_A\mathcal{M}$. Note that there is a constant $K > 0$ such that the condition

$$\|av\| \le K|a|\|v\| \tag{3.20}$$

for each $a \in A$ and each $v \in {}_A\mathcal{N}$ (see [27, Definition 12]). For all $a \in A_1$ and all $x, y \in {}_A\mathcal{M}$, we get

$$\frac{1}{4^j}\left\| f\left(2^j ax, 2^j ay\right) - a^2 f\left(2^j x, 2^j y\right) \right\| \le 2^{(r-2)j-1}\varepsilon\left(\|x\|^r + \|z\|^r\right) + \frac{K|a|^2}{4^j}\|f(0,0)\| \longrightarrow 0 \tag{3.21}$$

as $j \to \infty$. Hence we have

$$F(ax, ay) = \lim_{j\to\infty} \frac{1}{4^j} f\left(2^j ax, 2^j ay\right) = a^2 \lim_{j\to\infty} \frac{1}{4^j} f\left(2^j x, 2^j y\right) = a^2 F(x, y) \tag{3.22}$$

for all $a \in A_1$ and all $x, y \in {}_A\mathcal{M}$. Since $F(ax, ay) = a^2 F(x, y)$ for each $a \in A_1$, by (3.18), we obtain

$$F(ax, ay) = F\left(|a|\frac{a}{|a|}x, |a|\frac{a}{|a|}y\right) = |a|^2 F\left(\frac{a}{|a|}x, \frac{a}{|a|}y\right) = a^2 F(x, y) \tag{3.23}$$

for all nonzero $a \in A$ and all $x, y \in {}_A\mathcal{M}$. By (3.18), we get $F(0x, 0y) = 0^2 F(x, y)$ for all $x, y \in {}_A\mathcal{M}$. Therefore the bi-additive mapping F is the unique A-quadratic mapping satisfying the inequality (3.2).

The proof of the case $r \in \{0\} \cup (2, \infty)$ is similar to that of the case $r \in (0, 2)$. □

We obtain the Hyers-Ulam stability of (1.3) as a corollary of Theorem 3.2.

Corollary 3.3. *Let E be a complex normed space and $f : E \times E \to \mathbb{C}$ a function such that*

$$\left\| f(\lambda x + \lambda y, \lambda z - \lambda w) + f(\lambda x - \lambda y, \lambda z + \lambda w) - 2\lambda^2 f(x, z) + 2\lambda^2 f(y, w) \right\| \leq \varepsilon \tag{3.24}$$

for all $\lambda \in \mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, z, w \in E$. If $f(tx, ty)$ is continuous in $t \in \mathbb{R}$ for each fixed $x, y \in E$, then there exists a unique bi-additive $\mathbb{C}$-quadratic mapping $F : E \times E \to \mathbb{C}$ satisfying (1.3) *such that $\|f(x, y) - F(x, y)\| \leq \varepsilon/2 + \|f(0, 0)\|$ for all $x, y \in E$.*

Put $A_{in} := \{a \in A \mid a \text{ is invertible in } A\}$, $A_{sa} := \{a \in A \mid a^\star = a\}$, $A^+ := \{a \in A_{sa} \mid Sp(a) \subset [0, \infty)\}$, and $A_1^+ := A_1 \cap A^+$.

A unital $C^\star$-algebra A is said to have *real rank* 0 (see [28]) if the invertible self-adjoint elements are dense in A_{sa}.

For any element $a \in A$, $a = a_1 + ia_2$, where $a_1 := (a + a^\star)/2$ and $a_2 := (a - a^\star)/2i$ are self-adjoint elements, furthermore, $a = a_1^+ - a_1^- + ia_2^+ - ia_2^-$, where a_1^+, a_1^-, a_2^+, and a_2^- are positive elements (see [27, Lemma 38.8]).

Theorem 3.4. *Let A be of real rank* 0*, and let $f : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ be a mapping such that*

$$\begin{aligned} &\| f(ax + ay, bz - bw) + f(ax - ay, bz + bw) - 2abf(x, z) + 2ab(y, w) \| \\ &\quad \leq \begin{cases} 4\varepsilon, & (r = 0), \\ \varepsilon\left(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r\right), & (0 < r \neq 2) \end{cases} \end{aligned} \tag{3.25}$$

for all $a, b \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y, z, w \in {}_A\mathcal{M}$. For each fixed $x, y \in {}_A\mathcal{M}$, let the sequence $\{(1/4^j) f(2^j ax, 2^j by)\}$ converge uniformly on $A_1 \times A_1$. If $f(ax, by)$ is continuous in $(a, b) \in (A_1 \cup \mathbb{R})^2$ for each fixed $x, y \in {}_A\mathcal{M}$, then there exists a unique bi-additive A-quadratic mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (1.3) *and inequality* (3.2) *such that $F(ax, by) = abF(x, y)$ for all $a, b \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{M}$.*

Proof. Consider the case $r \in (0,2)$. By Lemma 3.1, there exists a unique bi-additive mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (1.3) and inequality (3.2) on ${}_A\mathcal{M} \times {}_A\mathcal{M}$. Let $x_0, y_0 \in {}_A\mathcal{M}$ be fixed. And let L be an arbitrary real functional element of the dual space of ${}_A\mathcal{N}$ restricted to the scalar field $\mathbb{R}$. For $n \in \mathbb{N}$, consider the functions $\varphi_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\varphi_n(s,t) := (1/4^n)L[f(2^n s x_0, 2^n t y_0)]$ for all $s,t \in \mathbb{R}$. By the assumption that $f(ax,by)$ is continuous in $(a,b) \in (A_1 \cup \mathbb{R})^2$ for each fixed $x,y \in {}_A\mathcal{M}$, the function φ_n is continuous for all $n \in \mathbb{N}$. Note that $\varphi_n(s,t) = (1/4^n)L[f(2^n s x_0, 2^n t y_0)] = L[(1/4^n)f(2^n s x_0, 2^n t y_0)]$ for all $n \in \mathbb{N}$ and all $s,t \in \mathbb{R}$. By the proof of Lemma 3.1, the sequence $\varphi_n(s,t)$ is a Cauchy sequence for all $s,t \in \mathbb{R}$. Define a function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\varphi(s,t) := \lim_{n\to\infty}\varphi_n(s,t)$ for all $s,t \in \mathbb{R}$. Note that $\varphi(s,t) = L[F(sx_0, ty_0)]$ for all $s,t \in \mathbb{R}$. Since the mapping F is bi-additive, we have

$$\begin{aligned}
&\varphi(s_1+s_2, t_1-t_2) + \varphi(s_1-s_2, t_1+t_2)\\
&\quad = L\left(F\left[(s_1+s_2)x_0, (t_1-t_2)y_0\right]\right) + L\left(F\left[(s_1-s_2)x_0, (t_1+t_2)y_0\right]\right)\\
&\quad = L\left(F\left[(s_1+s_2)x_0, (t_1-t_2)y_0\right] + F\left[(s_1-s_2)x_0, (t_1+t_2)y_0\right]\right)\\
&\quad = L\left[F\left(s_1x_0+s_2x_0, t_1y_0-t_2y_0\right) + F\left(s_1x_0-s_2x_0, t_1y_0+t_2y_0\right)\right]\\
&\quad = L\left[2F\left(s_1x_0, t_1y_0\right) - 2F\left(s_2x_0, t_2y_0\right)\right] = 2L\left[F\left(s_1x_0, t_1y_0\right)\right] - 2L\left[F\left(s_2x_0, t_2y_0\right)\right]\\
&\quad = 2\varphi(s_1,t_1) - 2\varphi(s_2,t_2)
\end{aligned} \tag{3.26}$$

for all $s_1, s_2, t_1, t_2 \in \mathbb{R}$. Since φ is the pointwise limit of continuous functions, it is a Borel function. By Theorem 2.2, we gain $\varphi(s,t) = st\varphi(1,1)$ for all $s,t \in \mathbb{R}$. Hence we get

$$L\left[F\left(sx_0, ty_0\right)\right] = \varphi(s,t) = st\varphi(1,1) = stL\left[F\left(x_0,y_0\right)\right] = L\left[stF\left(x_0,y_0\right)\right] \tag{3.27}$$

for all $s,t \in \mathbb{R}$. Since L is any continuous linear functional, the bi-additive mapping $F : {}_A\mathcal{M} \times {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfies $F(sx_0, ty_0) = stF(x_0,y_0)$ for all $s,t \in \mathbb{R}$. Therefore we obtain

$$F(sx,ty) = stF(x,y) \tag{3.28}$$

for all $s,t \in \mathbb{R}$ and all $x,y \in {}_A\mathcal{M}$. Let j be an arbitrary positive integer. Replacing x and z by $2^j x$ and $2^j z$, respectively, and letting $y = w = 0$ in inequality (3.25), we get

$$\left\| f\left(2^j ax, 2^j bz\right) - abf\left(2^j x, 2^j z\right) + abf(0,0)\right\| \le 2^{rj-1}\varepsilon\left(\|x\|^r + \|z\|^r\right) \tag{3.29}$$

for all $a,b \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x,z \in {}_A\mathcal{M}$. By inequality (3.20) and the above inequality, for all $a,b \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x,z \in {}_A\mathcal{M}$, we have

$$\begin{aligned}
&\frac{1}{4^j}\left\| f\left(2^j ax, 2^j bz\right) - abf\left(2^j x, 2^j z\right)\right\|\\
&\quad \le 2^{(r-2)j-1}\varepsilon\left(\|x\|^r + \|z\|^r\right) + \frac{K|a||b|}{4^j}\left\| f(0,0)\right\| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{3.30}$$

Hence we obtain that

$$F(ax,by) = \lim_{j\to\infty}\frac{1}{4^j}f\left(2^j ax, 2^j by\right) = ab\lim_{j\to\infty}\frac{1}{4^j}f\left(2^j x, 2^j y\right) = abF(x,y) \tag{3.31}$$

for all $a,b \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x,y \in {}_A\mathcal{M}$. Let $c,d \in A_1^+ \setminus A_{in}$. Since $A_{in} \cap A_{sa}$ is dense in A_{sa}, there exists two sequences $\{c_j\}$ and $\{d_j\}$ in $A_{in} \cap A_{sa}$ such that $c_j \to c$ and $d_j \to d$ as $j \to \infty$. Put $p_j := (1/|c_j|)c_j$ and $q_j := (1/|d_j|)d_j$ for all $j \in \mathbb{N}$. Then $p_j \to c$ and $q_j \to d$ as $j \to \infty$. Set $a_j := \sqrt{p_j{}^\star p_j}$ and $b_j := \sqrt{q_j{}^\star q_j}$ for all $j \in \mathbb{N}$. Then $a_j \to c$ and $b_j \to d$ as $j \to \infty$ and $a_j, b_j \in A_1^+ \cap A_{in}$. Since $\{(1/4^j)f(2^j ax, 2^j by)\}$ is uniformly converges on $A_1 \times A_1$ for each $x,y \in {}_A\mathcal{M}$ and $f(ax,by)$ is continuous in $a,b \in A_1$ for each $x,y \in {}_A\mathcal{M}$, we see that $F(ax,by)$ is also continuous in $a,b \in A_1$ for each $x,y \in {}_A\mathcal{M}$. In fact, we gain

$$\begin{aligned}\lim_{(a,b)\to(c,d)} F(ax,by) &= \lim_{(a,b)\to(c,d)}\lim_{j\to\infty}\frac{1}{4^j}f\left(2^j ax, 2^j by\right)\\ &= \lim_{j\to\infty}\lim_{(a,b)\to(c,d)}\frac{1}{4^j}f\left(2^j ax, 2^j by\right) = \lim_{j\to\infty}\frac{1}{4^j}f\left(2^j cx, 2^j dy\right) = F(cx,dy)\end{aligned} \tag{3.32}$$

for all $x,y \in {}_A\mathcal{M}$. Thus we get

$$\lim_{j\to\infty} F(a_j x, b_j y) = F\left(\lim_{j\to\infty} a_j x, \lim_{j\to\infty} b_j y\right) = F(cx,dy) \tag{3.33}$$

for all $x,y \in {}_A\mathcal{M}$. By equality (3.31), we have

$$\begin{aligned}\left\|F(a_j x, b_j y) - cdF(x,y)\right\| &= \left\|a_j b_j F(x,y) - cdF(x,y)\right\|\\ &\longrightarrow \left\|cdF(x,y) - cdF(x,y)\right\| = 0\end{aligned} \tag{3.34}$$

as $j \to \infty$ for all $x,y \in {}_A\mathcal{M}$. By equality (3.33) and the above convergence, we see that

$$\left\|F(cx,dy) - cdF(x,y)\right\| \le \left\|F(cx,dy) - F(a_j x, b_j y)\right\| + \left\|F(a_j x, b_j y) - cdF(x,y)\right\| \longrightarrow 0 \tag{3.35}$$

as $j \to \infty$ for all $x,y \in {}_A\mathcal{M}$. By equality (3.31) and the above convergence, we obtain

$$F(ax,by) = abF(x,y) \tag{3.36}$$

for all $a, b \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{M}$. Since the mapping F is bi-additive, we see that

$$\begin{aligned} F(ax, ay) &= F\left(a_1^+x - a_1^-x + ia_2^+x - ia_2^-x, a_1^+y - a_1^-y + ia_2^+y - ia_2^-y\right) \\ &= F\left(a_1^+x, a_1^+y\right) - F\left(a_1^+x, a_1^-y\right) + F\left(a_1^+x, ia_2^+y\right) - F\left(a_1^+x, ia_2^-y\right) \\ &\quad - F\left(a_1^-x, a_1^+y\right) + F\left(a_1^-x, a_1^-y\right) - F\left(a_1^-x, ia_2^+y\right) + F\left(a_1^-x, ia_2^-y\right) \\ &\quad + F\left(ia_2^+x, a_1^+y\right) - F\left(ia_2^+x, a_1^-y\right) + F\left(ia_2^+x, ia_2^+y\right) - F\left(ia_2^+x, ia_2^-y\right) \\ &\quad - F\left(ia_2^-x, a_1^+y\right) + F\left(ia_2^-x, a_1^-y\right) - F\left(ia_2^-x, ia_2^+y\right) + F\left(ia_2^-x, ia_2^-y\right) \end{aligned} \tag{3.37}$$

for all $a \in A$ and all $x, y \in {}_A\mathcal{M}$. By (3.28) and equality (3.36), we have

$$F(px, qy) = F\left(|p|\frac{p}{|p|}x, |q|\frac{q}{|q|}y\right) = |p||q|F\left(\frac{p}{|p|}x, \frac{q}{|q|}y\right) = pqF(x, y) \tag{3.38}$$

for all $p, q \in \{a_1^+, a_1^-, a_2^+, a_2^-\}$ and all $x, y \in_A \mathcal{M}$. Note that $a_1^+a_1^- = a_1^-a_1^+ = a_2^+a_2^- = a_2^-a_2^+ = 0$. Hence we obtain that

$$\begin{aligned} F(ax, ay) &= \left(a_2^+\right)^2F(x, y) + ia_1^+a_2^+F(x, y) - ia_1^+a_2^-F(x, y) + \left(a_1^-\right)^2F(x, y) \\ &\quad - ia_1^-a_2^+F(x, y) + ia_1^-a_2^-F(x, y) + ia_2^+a_1^+F(x, y) - ia_2^+a_1^-F(x, y) \\ &\quad - \left(a_2^+\right)^2F(x, y) - ia_2^-a_1^+F(x, y) + ia_2^-a_1^-F(x, y) - \left(a_2^-\right)^2F(x, y) \\ &= \Big[\left(a_1^+\right)^2 + ia_1^+a_2^+ - ia_1^+a_2^- + \left(a_1^-\right)^2 - ia_1^-a_2^+ + ia_1^-a_2^- \\ &\qquad + ia_2^+a_1^+ - ia_2^+a_1^- - \left(a_2^+\right)^2 - ia_2^-a_1^+ + ia_2^-a_1^- - \left(a_2^-\right)^2\Big]F(x, y) \\ &= \left(a_1^+ - a_1^- + ia_2^+ - ia_2^-\right)^2F(x, y) = a^2F(x, y) \end{aligned} \tag{3.39}$$

for all $a \in A$ and all $x, y \in {}_A\mathcal{M}$.

The proof of the case $r \in \{0\} \cup (2, \infty)$ is similar to that of the case $r \in (0, 2)$. □

Acknowledgment

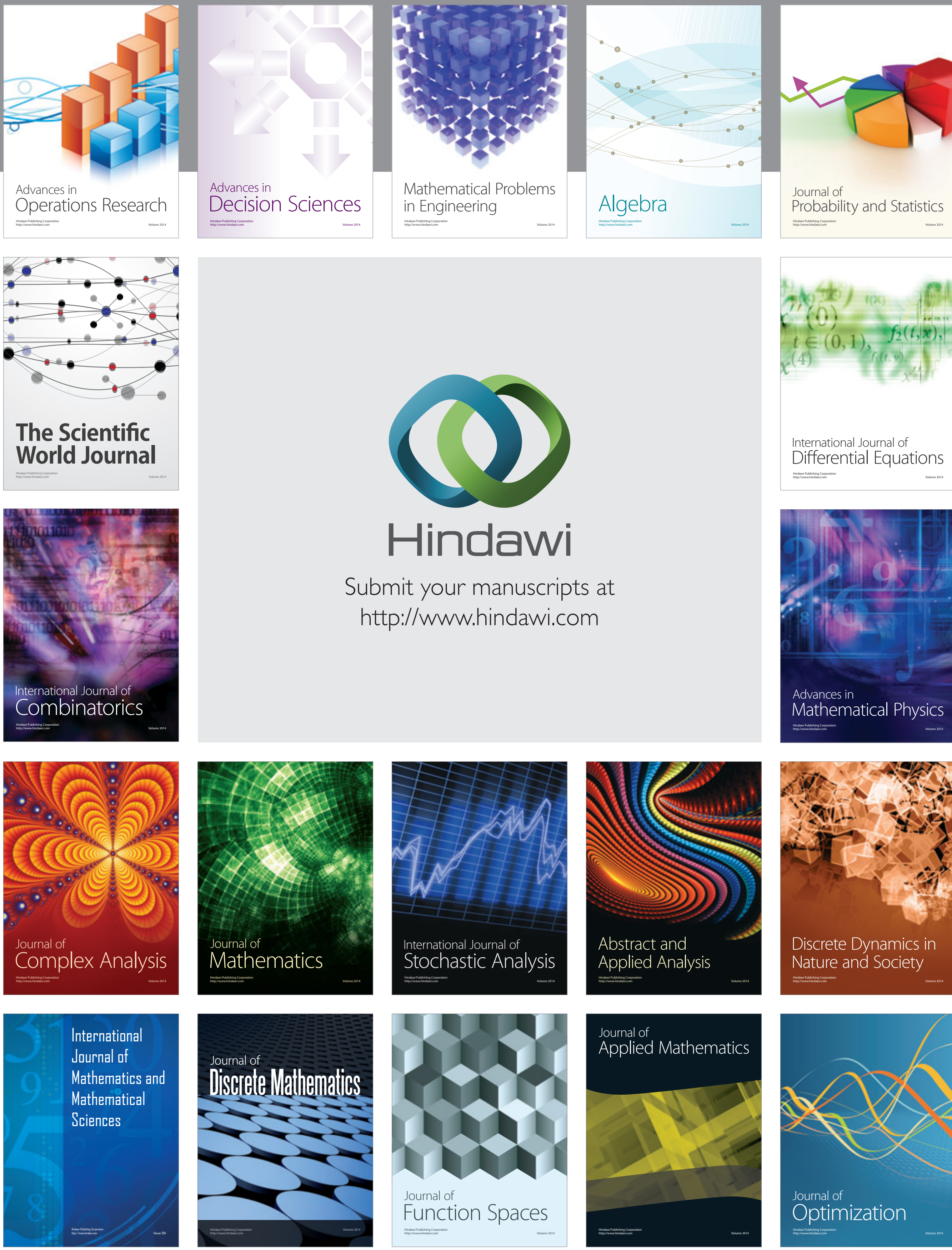
This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant no. 2012003499).

References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, vol. 8 of *Interscience Tracts in Pure and Applied Mathematics*, Interscience Publishers, New York, NY, USA, 1968.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] J.-H. Bae and W.-G. Park, "On stability of a functional equation with n variables," *Nonlinear Analysis*, vol. 64, no. 4, pp. 856–868, 2006.
[4] J.-H. Bae and W.-G. Park, "On a cubic equation and a Jensen-quadratic equation," *Abstract and Applied Analysis*, vol. 2007, Article ID 45179, 10 pages, 2007.
[5] M. E. Gordji, A. Ebadian, and N. Ghobadipour, "A fixed point method for perturbation of bimultipliers and Jordan bimultipliers in C^*-ternary algebras," *Journal of Mathematical Physics*, vol. 51, 10 pages, 2010.
[6] M. E. Gordji and A. Fazeli, "Stability and superstability of $*$-bihomomorphisms on C^*-ternary algebras," *Journal of Concrete and Applicable Mathematics*, vol. 10, pp. 245–258, 2012.
[7] M. E. Gordji, R. Khodabakhsh, and H. Khodaei, "On approximate n-ary derivations," *International Journal of Geometric Methods in Modern Physics*, vol. 8, no. 3, pp. 485–500, 2011.
[8] K.-W. Jun and H.-M. Kim, "Remarks on the stability of additive functional equation," *Bulletin of the Korean Mathematical Society*, vol. 38, no. 4, pp. 679–687, 2001.
[9] K.-W. Jun and H.-M. Kim, "On the Hyers-Ulam stability of a generalized quadratic and additive functional equation," *Bulletin of the Korean Mathematical Society*, vol. 42, no. 1, pp. 133–148, 2005.
[10] S.-M. Jung, "On the Hyers-Ulam-Rassias stability of the equation $f(x^2 - y^2 + rxy) = f(x^2) - f(y^2) + rf(xy)$," *Bulletin of the Korean Mathematical Society*, vol. 33, no. 4, pp. 513–519, 1996.
[11] S.-M. Jung, T.-S. Kim, and K.-S. Lee, "A fixed point approach to the stability of quadratic functional equation," *Bulletin of the Korean Mathematical Society*, vol. 43, no. 3, pp. 531–541, 2006.
[12] M. B. Moghimi, A. Najati, and C. Park, "A fixed point approach to the stability of a quadratic functional equation in C^*-algebras," *Advances in Difference Equations*, vol. 2009, Article ID 256165, 10 pages, 2009.
[13] A. Najati and C. Park, "On the stability of an n-dimensional functional equation originating from quadratic forms," *Taiwanese Journal of Mathematics*, vol. 12, no. 7, pp. 1609–1624, 2008.
[14] A. Najati, C. Park, and J. R. Lee, "Homomorphisms and derivations in C^*-ternary algebras," *Abstract and Applied Analysis*, vol. 2009, Article ID 612392, 16 pages, 2009.
[15] A. Najati and T. M. Rassias, "Stability of a mixed functional equation in several variables on Banach modules," *Nonlinear Analysis*, vol. 72, no. 3-4, pp. 1755–1767, 2010.
[16] C. Park and J. S. An, "Isomorphisms in quasi-Banach algebras," *Bulletin of the Korean Mathematical Society*, vol. 45, no. 1, pp. 111–118, 2008.
[17] C. Park, S.-K. Hong, and M.-J. Kim, "Jensen type quadratic-quadratic mapping in Banach spaces," *Bulletin of the Korean Mathematical Society*, vol. 43, no. 4, pp. 703–709, 2006.
[18] W.-G. Park and J.-H. Bae, "On a Cauchy-Jensen functional equation and its stability," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 1, pp. 634–643, 2006.
[19] W.-G. Park and J.-H. Bae, "A multidimensional functional equation having quadratic forms as solutions," *Journal of Inequalities and Applications*, vol. 2007, Article ID 24716, 8 pages, 2007.
[20] W.-G. Park and J.-H. Bae, "A functional equation originating from elliptic curves," *Abstract and Applied Analysis*, vol. 2008, Article ID 135237, 10 pages, 2008.
[21] W.-G. Park and J.-H. Bae, "Stability of a 2-dimensional functional equation in a class of vector variable functions," *Journal of Inequalities and Applications*, vol. 2010, Article ID 167042, 12 pages, 2010.
[22] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, UK, 1989.
[23] G. B. Folland, *Real Analysis*, Pure and Applied Mathematics, John Wiley & Sons, New York, NY, USA, 2nd edition, 1999.
[24] M. Fréchet, "Pri la Funkcia Ekvacio $f(x+y) = f(x) + f(y)$," *L'Enseignement Mathématique*, vol. 15, pp. 390–393, 1913.
[25] V. I. Bogachev, *Measure Theory*, vol. 2, Springer, Berlin, Germany, 2007.
[26] S. Kurepa, "On the quadratic functional," *Académie Serbe des Sciences, Publications de l'Institut Mathématique*, vol. 13, pp. 57–72, 1959.
[27] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, Berlin, Germany, 1973.
[28] K. R. Davidson, *C^*-Algebras by Example*, vol. 6 of *Fields Institute Monographs*, American Mathematical Society, Providence, RI, USA, 1996.



Advances in Operations Research
Advances in Decision Sciences
Mathematical Problems in Engineering
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
International Journal of Stochastic Analysis
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
Journal of Applied Mathematics
Journal of Optimization