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Review Article

Incomplete Bivariate Fibonacci and Lucas p-Polynomials

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We define the incomplete bivariate Fibonacci and Lucas p-polynomials. In the case $x = 1$, $y = 1$, we obtain the incomplete Fibonacci and Lucas p-numbers. If $x = 2$, $y = 1$, we have the incomplete Pell and Pell-Lucas p-numbers. On choosing $x = 1$, $y = 2$, we get the incomplete generalized Jacobsthal number and besides for $p = 1$ the incomplete generalized Jacobsthal-Lucas numbers. In the case $x = 1$, $y = 1$, $p = 1$, we have the incomplete Fibonacci and Lucas numbers. If $x = 1$, $y = 1$, $p = 1$, $k = \lfloor (n-1)/(p+1) \rfloor$, we obtain the Fibonacci and Lucas numbers. Also generating function and properties of the incomplete bivariate Fibonacci and Lucas p-polynomials are given.

1. Introduction

Djordjević introduced incomplete generalized Fibonacci and Lucas numbers using explicit formulas of generalized Fibonacci and Lucas numbers in [1]. In [2] incomplete Fibonacci and Lucas numbers are given as follows:

$$F_n(k) = \sum_{j=0}^{k} \binom{n-1-j}{j}, \quad 0 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor,$$
$$L_n(k) = \sum_{j=0}^{k} \frac{n}{n-j} \binom{n-j}{j}, \quad 0 \le k \le \left\lfloor \frac{n}{2} \right\rfloor, \tag{1.1}$$

where $n = 1, 2, 3, \ldots$. Note that for the case $k = \lfloor (n-1)/2 \rfloor$ incomplete Fibonacci numbers are reduced to Fibonacci numbers and for the case $k = \lfloor n/2 \rfloor$ incomplete Lucas numbers are reduced to Lucas numbers in [2]. Also the authors considered the generating functions of the incomplete Fibonacci and Lucas numbers in [3]. In [4] Djordjević and Srivastava defined incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers.

The generalized Fibonacci and Lucas p-numbers were studied in [5, 6]. Incomplete Fibonacci and Lucas p-numbers are defined by

$$\begin{aligned} F_p^k(n) &= \sum_{j=0}^{k} \binom{n-jp-1}{j}, \quad 0 \le k \le \left\lfloor \frac{n-1}{p+1} \right\rfloor, \\ L_p^k(n) &= \sum_{j=0}^{k} \frac{n}{n-jp} \binom{n-jp}{j}, \quad 0 \le k \le \left\lfloor \frac{n}{p+1} \right\rfloor, \end{aligned} \tag{1.2}$$

for $n \ge 1$ in [7]. In [8] the authors introduced incomplete Pell and Pell-Lucas p-numbers.

The generalized bivariate Fibonacci p-polynomials $F_{p,n}(x,y)$ and generalized bivariate Lucas p-polynomials $L_{p,n}(x,y)$ are defined the recursion for $p \ge 1$

$$F_{p,n}(x,y) = xF_{p,n-1}(x,y) + yF_{p,n-p-1}(x,y), \quad n > p, \tag{1.3}$$

with

$$F_{p,0}(x,y) = 0, \qquad F_{p,n}(x,y) = x^{n-1} \quad \text{for } n = 1,2,\ldots p, \tag{1.4}$$

and

$$L_{p,n}(x,y) = xL_{p,n-1}(x,y) + yL_{p,n-p-1}(x,y), \quad n > p, \tag{1.5}$$

with

$$L_{p,0}(x,y) = p+1, \qquad L_{p,n}(x,y) = x^{n} \quad \text{for } n = 1,2,\ldots p \tag{1.6}$$

in [5]. When $x = y = 1$, $F_{p,n}(1,1) = F_p(n)$. In [5], the authors obtained some relations for these polynomials sequences. In addition, in [5], the explicit formula of bivariate Fibonacci p-polynomials is

$$F_{p,n}(x,y) = \sum_{j=0}^{\lfloor (n-1)/(p+1) \rfloor} \binom{n-jp-1}{j} x^{n-j(p+1)-1} y^{j}, \quad n \ge 0,\ p \ge 1, \tag{1.7}$$

and the explicit formula of bivariate Lucas p-polynomials is

$$L_{p,n}(x,y) = \sum_{j=0}^{\lfloor n/(p+1) \rfloor} \frac{n}{n-jp} \binom{n-jp}{j} x^{n-j(p+1)} y^{j}, \quad n \ge 0,\ p \ge 1. \tag{1.8}$$

In this paper, we defined incomplete bivariate Fibonacci and Lucas p-polynomials. We generalize incomplete Fibonacci and Lucas numbers, incomplete generalized Fibonacci numbers, incomplete generalized Jacobsthal numbers, incomplete Fibonacci and Lucas p-numbers, incomplete Pell and Pell-Lucas p-numbers.

2. Incomplete Bivariate Fibonacci and Lucas p-Polynomials

Definition 2.1. For $p \geq 1$, $n \geq 1$, incomplete bivariate Fibonacci p-polynomials are defined as

$$F_{p,n}^{k}(x,y) = \sum_{j=0}^{k} \binom{n-jp-1}{j} x^{n-j(p+1)-1} y^{j}, \quad 0 \leq k \leq \left\lfloor \frac{n-1}{p+1} \right\rfloor. \tag{2.1}$$

For $x = 1,\ y = 1$, $F_{p,n}^{k}(x,y) = F_{p}^{k}(n)$, we get incomplete Fibonacci p-numbers [7].

If $x = 2,\ y = 1$, $F_{p,n}^{k}(x,y) = P_{p}^{k}(n)$, we obtained incomplete Pell p-numbers [8].

On choosing $x = 1,\ y = 2$, $F_{p,n}^{k}(x,y) = J_{n,p+1}^{k}$, we have incomplete generalized Jacobsthal numbers [4].

If $x = 1,\ y = 1,\ p = 1$, $F_{p,n}^{k}(x,y) = F_{n}(k)$, we get incomplete Fibonacci numbers [2].

For $x = 1,\ y = 1,\ p = 1,\ k = \lfloor (n-1)/(p+1) \rfloor$, $F_{p,n}^{k}(x,y) = F_{n}$, we obtained Fibonacci numbers [9].

Definition 2.2. For $p \geq 1$, $n \geq 1$, incomplete bivariate Lucas p-polynomials are defined as

$$L_{p,n}^{k}(x,y) = \sum_{j=0}^{k} \frac{n}{n-jp} \binom{n-jp}{j} x^{n-j(p+1)} y^{j}, \quad 0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor. \tag{2.2}$$

If $x = 1,\ y = 1$, $L_{p,n}^{k}(x,y) = L_{p}^{k}(n)$, we obtained incomplete Lucas p-numbers [7].

For $x = 2,\ y = 1$, $L_{p,n}^{k}(x,y) = Q_{p}^{k}(n)$, we have incomplete Pell-Lucas p-numbers [8].

On choosing $x = 1,\ y = 2, p = 1$, $L_{p,n}^{k}(x,y) = j_{n,p+1}^{k}$, we get incomplete generalized Jacobsthal-Lucas numbers [4].

If $x = 1,\ y = 1,\ p = 1$, $L_{p,n}^{k}(x,y) = L_{n}(k)$, we obtained incomplete Lucas numbers [2].

For $x = 1,\ y = 1,\ p = 1,\ k = \lfloor n/(p+1) \rfloor$, $L_{p,n}^{k}(x,y) = L_{n}$, we have Lucas numbers [9].

Proposition 2.3. *The incomplete bivariate Fibonacci p-polynomials satisfy the following recurrence relation:*

$$F_{p,n}^{k+1}(x,y) = xF_{p,n-1}^{k+1}(x,y) + yF_{p,n-p-1}^{k}(x,y), \quad 0 \leq k \leq \frac{n-p-3}{p+1}. \tag{2.3}$$

Proof. Using (2.1), we obtain

$$\begin{aligned}
&xF^{k+1}_{p,n-1}(x,y)+yF^{k}_{p,n-p-1}(x,y)\\
&=x\sum_{j=0}^{k+1}\binom{n-jp-2}{j}x^{n-j(p+1)-2}y^{j}+y\sum_{j=0}^{k}\binom{n-p-pj-2}{j}x^{n-p-j(p+1)-2}y^{j}\\
&=\sum_{j=0}^{k+1}\binom{n-jp-2}{j}x^{n-j(p+1)-1}y^{j}+\sum_{j=0}^{k}\binom{n-p-pj-2}{j}x^{n-p-j(p+1)-2}y^{j+1}\\
&=\sum_{j=0}^{k+1}\binom{n-jp-2}{j}x^{n-j(p+1)-1}y^{j}+\sum_{j=1}^{k+1}\binom{n-pj-2}{j-1}x^{n-j(p+1)-1}y^{j}\\
&=\sum_{j=0}^{k+1}\left[\binom{n-jp-2}{j}+\binom{n-pj-2}{j-1}\right]x^{n-j(p+1)-1}y^{j}-\binom{n-2}{-1}x^{n-1}\\
&=\sum_{j=0}^{k+1}\binom{n-jp-1}{j}x^{n-j(p+1)-1}y^{j}-0\\
&=F^{k+1}_{p,n}(x,y).
\end{aligned} \tag{2.4}$$

□

Taking $x = y = 1$ in (2.3), we could obtain a formula for incomplete Fibonacci p-numbers (see [7, Proposition 3]). Taking $x = y = p = 1$ in (2.3), we could obtain a formula for incomplete Fibonacci numbers (see [2, Proposition 1]).

Proposition 2.4. *The nonhomogeneous recurrence relation of incomplete bivariate Fibonacci p-polynomials is*

$$F^{k}_{p,n}(x,y)=xF^{k}_{p,n-1}(x,y)+yF^{k}_{p,n-p-1}(x,y)-\binom{n-p(k+1)-2}{k}x^{n-p(k+1)-k-2}y^{k+1}. \tag{2.5}$$

Proof. It is easy to obtain from (2.1) and (2.3). □

Proposition 2.5. *For $0 \le k \le (n-h-p-1)/(p+1)$, one has*

$$\sum_{j=0}^{h}\binom{h}{j}y^{h-j}x^{j}F^{k+j}_{p,n+p(j-1)}(x,y)=F^{k+h}_{p,n+(p+1)h-p}(x,y). \tag{2.6}$$

Proof. Equation (2.6) clearly holds for $h = 0$. Suppose that the equation holds for $h > 0$. We show that the equation holds for $(h+1)$. We have

$$\begin{aligned}
&\sum_{j=0}^{h+1}\binom{h+1}{j} y^{h+1-j}x^{j}F^{k+j}_{p,n+p(j-1)}(x,y)\\
&= \sum_{j=0}^{h+1}\left[\binom{h}{j}+\binom{h}{j-1}\right] y^{h+1-j}x^{j}F^{k+j}_{p,n+p(j-1)}(x,y)\\
&= \sum_{j=0}^{h+1}\binom{h}{j} y^{h+1-j}x^{j}F^{k+j}_{p,n+p(j-1)}(x,y) + \sum_{j=0}^{h+1}\binom{h}{j-1} y^{h+1-j}x^{j}F^{k+j}_{p,n+p(j-1)}(x,y)\\
&= yF^{k+h}_{p,n+(p+1)h-p}(x,y) + \binom{h}{h+1}x^{h+1}F^{k+h+1}_{p,n+ph}(x,y)\\
&\quad + \sum_{j=-1}^{h}\binom{h}{j} y^{h-j}x^{j+1}F^{k+j+1}_{p,n+pj}(x,y)\\
&= yF^{k+h}_{p,n+(p+1)h-p}(x,y) + x\sum_{j=0}^{h}\binom{h}{j} y^{h-j}x^{j}F^{k+j+1}_{p,n+pj}(x,y)\\
&\quad + \binom{h}{-1} y^{h+1}F^{k}_{p,n-p}(x,y)\\
&= yF^{k+h}_{p,n+(p+1)h-p}(x,y) + xF^{k+h+1}_{p,n+(p+1)h}(x,y)\\
&= F^{k+h+1}_{p,n+(p+1)h+1}(x,y).
\end{aligned} \tag{2.7}$$

□

Proposition 2.6. *For $n \geq k(p+1)+p+2$,*

$$\sum_{j=0}^{h-1}\frac{y}{x^{j}}F^{k}_{p,n-p+j}(x,y) = \frac{1}{x^{h-1}}F^{k+1}_{p,n+h}(x,y) - xF^{k+1}_{p,n}(x,y). \tag{2.8}$$

Proof. Equation (2.8) can be easily proved by using (2.3) and induction on h. □

We have the following proposition in which the relationship between the incomplete bivariate Fibonacci and Lucas p-polynomials is preserved as found in [5] before.

Proposition 2.7. *One has*

$$L^{k}_{p,n}(x,y) = F^{k}_{p,n+1}(x,y) + pyF^{k-1}_{p,n-p}(x,y), \quad 0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor. \tag{2.9}$$

Proof. By (2.1), rewrite the right-hand side of (2.9) as

$$\begin{aligned}
F_{p,n+1}^{k}(x,y) + pyF_{p,n-p}^{k-1}(x,y) &= \sum_{j=0}^{k}\binom{n-jp}{j}x^{n-j(p+1)}y^{j} + py\sum_{j=0}^{k-1}\binom{n-p-jp-1}{j}x^{n-p-j(p+1)-1}y^{j} \\
&= \sum_{j=0}^{k}\binom{n-jp}{j}x^{n-j(p+1)}y^{j} + py\sum_{j=1}^{k}\binom{n-jp-1}{j-1}x^{n-j(p+1)}y^{j-1} \\
&= \sum_{j=0}^{k}\left[\binom{n-jp}{j} + \binom{n-jp-1}{j-1}\right]x^{n-j(p+1)}y^{j} - \binom{n-1}{-1}x^{n} \\
&= \sum_{j=0}^{k}\frac{n}{n-jp}\binom{n-jp}{j}x^{n-j(p+1)}y^{j} \\
&= L_{p,n}^{k}(x,y).
\end{aligned} \tag{2.10}$$

□

Proposition 2.8. *The incomplete bivariate Lucas p-polynomials satisfy the following recurrence relation:*

$$L_{p,n}^{k+1}(x,y) = xL_{p,n-1}^{k+1}(x,y) + yL_{p,n-p-1}^{k}(x,y), \quad 0 \le k \le \frac{n-p-2}{p+1}. \tag{2.11}$$

Proof. We write by using (2.3) and (2.9)

$$\begin{aligned}
L_{p,n}^{k+1}(x,y) &= F_{p,n+1}^{k+1}(x,y) + pyF_{p,n-p}^{k}(x,y) \\
&= xF_{p,n}^{k+1}(x,y) + yF_{p,n-p}^{k}(x,y) + py\left[xF_{p,n-p-1}^{k}(x,y) + yF_{p,n-2p-1}^{k-1}(x,y)\right] \\
&= x\left[F_{p,n}^{k+1}(x,y) + pyF_{p,n-p-1}^{k}(x,y)\right] + y\left[F_{p,n-p}^{k}(x,y) + pyF_{p,n-2p-1}^{k-1}(x,y)\right] \\
&= xL_{p,n-1}^{k+1}(x,y) + yL_{p,n-p-1}^{k}(x,y).
\end{aligned} \tag{2.12}$$

□

Proposition 2.9. *The nonhomogeneous recurrence relation of incomplete bivariate Lucas p-polynomials is*

$$\begin{aligned}
L_{p,n}^{k}(x,y) &= xL_{p,n-1}^{k}(x,y) + yL_{p,n-p-1}^{k}(x,y) \\
&\quad - \frac{n-p-1}{n-p(k+1)-1}\binom{n-p(k+1)-1}{k}x^{n-(p+1)(k+1)}y^{k+1}.
\end{aligned} \tag{2.13}$$

Proof. The proof can be done by using (2.2) and (2.11). □

Proposition 2.10. *For* $0 \leq k \leq (n-p-h)/(p+1)$, *one has*

$$\sum_{j=0}^{h} \binom{h}{j} x^j y^{h-j} L_{p,n+p(j-1)}^{k+j}(x,y) = L_{p,n+(p+1)h-p}^{k+h}(x,y). \quad (2.14)$$

Proof. Proof is similar to the proof of Proposition 2.5. □

Proposition 2.11. *For* $n \geq (k+1)(p+1)$, *one has*

$$\sum_{j=0}^{h-1} \frac{y}{x^j} L_{p,n-p+j}^{k}(x,y) = \frac{1}{x^{h-1}} L_{p,n+h}^{k+1}(x,y) - x L_{p,n}^{k+1}(x,y). \quad (2.15)$$

Proof. Proof is obtained immediately by using (2.11) and induction h. □

Proposition 2.12. *One has*

$$\sum_{k=0}^{\lfloor n/(p+1)\rfloor} L_{p,n}^{k}(x,y) = \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1 \right) L_{p,n}(x,y) + \frac{n}{p+1} \left[x F_{p,n}(x,y) - L_{p,n}(x,y) \right]. \quad (2.16)$$

Proof. We can write from (2.2)

$$\begin{aligned}
\sum_{k=0}^{\lfloor n/(p+1)\rfloor} L_{p,n}^{k}(x,y) = {} & L_{p,n}^{0}(x,y) + L_{p,n}^{1}(x,y) + L_{p,n}^{2}(x,y) + \cdots + L_{p,n}^{\lfloor n/(p+1)\rfloor}(x,y) \\
& = \frac{n}{n}\binom{n}{0} x^n + \left[\frac{n}{n}\binom{n}{0} x^n + \frac{n}{n-p}\binom{n-p}{1} x^{n-(p+1)} y \right] \\
& + \left[\frac{n}{n}\binom{n}{0} x^n + \frac{n}{n-p}\binom{n-p}{1} x^{n-(p+1)} y + \frac{n}{n-2p}\binom{n-2p}{2} x^{n-2(p+1)} y^2 \right] + \cdots \\
& + \left[\frac{n}{n}\binom{n}{0} x^n + \cdots + \frac{n}{n - \lfloor n/(p+1) \rfloor p} \right. \\
& \left. \times \binom{n - \left\lfloor \frac{n}{p+1} \right\rfloor p}{\left\lfloor \frac{n}{p+1} \right\rfloor} x^{n-(p+1)\lfloor n/(p+1)\rfloor} y^{\lfloor n/(p+1)\rfloor} \right]
\end{aligned}$$

$$
\begin{aligned}
&= \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1\right)\frac{n}{n}\binom{n}{0}x^n + \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1 - 1\right)\frac{n}{n-p}\binom{n-p}{1}x^{n-(p+1)}y \\
&\quad + \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1 - 2\right)\frac{n}{n-2p}\binom{n-2p}{1}x^{n-2(p+1)}y^2 + \cdots \\
&\quad + \frac{n}{n - \lfloor n/(p+1) \rfloor p}\binom{n - \left\lfloor \frac{n}{p+1} \right\rfloor p}{\left\lfloor \frac{n}{p+1} \right\rfloor}x^{n-(p+1)\lfloor n/(p+1) \rfloor}y^{\lfloor n/(p+1) \rfloor} \\
&= \sum_{j=0}^{\lfloor n/(p+1) \rfloor}\left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1 - j\right)\frac{n}{n-jp}\binom{n-jp}{j}x^{n-j(p+1)}y^j \\
&= \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1\right)\sum_{j=0}^{\lfloor n/(p+1) \rfloor}\frac{n}{n-jp}\binom{n-jp}{j}x^{n-j(p+1)}y^j \\
&\quad - \sum_{j=0}^{\lfloor n/(p+1) \rfloor} j\frac{n}{n-jp}\binom{n-jp}{j}x^{n-j(p+1)}y^j.
\end{aligned}
\tag{2.17}
$$

Equation (2.17) is calculated using the formula $L_{p,n}(x,y)$ and $\partial L_{p,n}(x,y)/\partial x = nF_{p,n}(x,y)$ [5]

$$
\begin{aligned}
\sum_{k=0}^{\lfloor n/(p+1) \rfloor} L_{p,n}^k(x,y) &= \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1\right)\sum_{j=0}^{\lfloor n/(p+1) \rfloor}\frac{n}{n-jp}\binom{n-jp}{j}x^{n-j(p+1)}y^j \\
&\quad + \frac{nF_{p,n}(x,y) - nx^{-1}L_{p,n}(x,y)}{(p+1)x^{-1}} \\
&= \left(\left\lfloor \frac{n}{p+1} \right\rfloor + 1\right)L_{p,n}(x,y) + \frac{n}{p+1}\left[xF_{p,n}(x,y) - L_{p,n}(x,y)\right].
\end{aligned}
\tag{2.18}
$$

□

Then we have the following conclusion.

Conclusion 1. When $x = y = p = 1$ in (2.16), we obtain

$$
\sum_{k=0}^{\lfloor n/2 \rfloor} L_n(k) = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)L_n + \frac{n}{2}(F_n - L_n)
\tag{2.19}
$$

which is Proposition 11 in [2].

3. Generating Functions of the Incomplete Bivariate Fibonacci and Lucas p-Polynomials

Lemma 3.1 (see [3]). *Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the following nonhomogeneous recurrence relation:*

$$s_n = xs_{n-1} + ys_{n-p-1} + r_n, \quad n > p, \tag{3.1}$$

where $\{r_n\}$ is a given complex sequence. Then the generating function $S_p^k(x,y;t)$ of the sequence $\{s_n\}$ is

$$S_p^k(x,y;t) = \left[s_0 - r_0 + \sum_{i=1}^{p}(s_i - xs_{i-1} - r_i)t^i + G(t)\right]\left[1 - xt - yt^{p+1}\right]^{-1}, \tag{3.2}$$

where $G(t)$ denotes the generating function of $\{r_n\}$.

Theorem 3.2. *The generating function of the incomplete bivariate Fibonacci p-polynomials is*

$$\begin{aligned} R_p^k(x,y;t) = t^{k(p+1)+1}&\left[F_{p,k(p+1)+1}(x,y) + \sum_{i=1}^{p} t^i\left(F_{p,k(p+1)+1+i}(x,y) - xF_{p,k(p+1)+i}(x,y)\right)\right. \\ &\left.+\frac{y^{k+1}t^{p+1}}{(1-xt)^{k+1}}\right]\left[1 - xt - yt^{p+1}\right]^{-1}. \end{aligned} \tag{3.3}$$

Proof. From (2.1) and (2.5), $F_{p,n}^k(x,y) = 0$ for $0 \le n < k(p+1)+1$,

$$\begin{aligned} F_{p,k(p+1)+1}^k(x,y) &= F_{p,k(p+1)+1}(x,y), \\ F_{p,k(p+1)+2}^k(x,y) &= F_{p,k(p+1)+2}(x,y), \\ &\vdots \\ F_{p,k(p+1)+p+1}^k(x,y) &= F_{p,k(p+1)+p+1}(x,y), \end{aligned} \tag{3.4}$$

and for $n \ge k(p+1)+p+2$

$$F_{p,n}^k(x,y) = xF_{p,n-1}^k(x,y) + yF_{p,n-p-1}^k(x,y) - \binom{n-p(k+1)-2}{n-k(p+1)-p-2} x^{n-p(k+1)-k-2}y^{k+1}. \tag{3.5}$$

Now let

$$s_0 = F_{p,k(p+1)+1}^k(x,y), \qquad s_1 = F_{p,k(p+1)+2}^k(x,y), \ldots, \qquad s_p = F_{p,k(p+1)+p+1}^k(x,y), \tag{3.6}$$

and

$$s_n = F^k_{p,n+k(p+1)+1}(x,y). \tag{3.7}$$

Also

$$r_0 = r_1 = \cdots = r_p = 0, \qquad r_n = \binom{n+k-p-1}{n-p-1} x^{n-p-1} y^{k+1}. \tag{3.8}$$

We obtained that $G(t) = y^{k+1}t^{p+1}/(1-xt)^{k+1}$ is the generating function of the sequence $\{r_n\}$. From Lemma 3.1, we get that the generating function $S^k_p(x,y;t)$ of sequence $\{s_n\}$ is

$$\begin{aligned} S^k_p(x,y;t) = \Bigg[& F^k_{p,k(p+1)+1}(x,y) + \sum_{i=1}^{p} t^i \left(F^k_{p,k(p+1)+1+i}(x,y) - xF^k_{p,k(p+1)+i}(x,y) \right) \\ & + \frac{y^{k+1}t^{p+1}}{(1-xt)^{k+1}} \Bigg] \left[1 - xt - yt^{p+1}\right]^{-1}. \end{aligned} \tag{3.9}$$

Therefore,

$$R^k_p(x,y;t) = t^{k(p+1)+1} S^k_p(x,y;t). \tag{3.10}$$

□

Theorem 3.3. *The generating function of the incomplete bivariate Lucas p-polynomials is*

$$\begin{aligned} W^k_p(x,y;t) = t^{k(p+1)} \Bigg[& L_{p,k(p+1)}(x,y) + \sum_{i=1}^{p} t^i \left(L_{p,k(p+1)+i}(x,y) - xL_{p,k(p+1)+i-1}(x,y) \right) \\ & + \frac{t^{p+1}y^{k+1}\left[p(1-xt)+1\right]}{(1-xt)^{k+1}} \Bigg] \left[1 - xt - yt^{p+1}\right]^{-1}. \end{aligned} \tag{3.11}$$

Proof. From (2.9) and (3.3),

$$\begin{aligned} W^k_p(x,y;t) &= \sum_{n=0}^{\infty} L^k_{p,n}(x,y) t^n \\ &= \sum_{n=0}^{\infty} \left[F^k_{p,n+1}(x,y) + pyF^{k-1}_{p,n-p}(x,y) \right] t^n \\ &= \sum_{n=0}^{\infty} F^k_{p,n+1}(x,y) t^n + py \sum_{n=0}^{\infty} F^{k-1}_{p,n-p}(x,y) t^n \\ &= t^{-1} R^k_p(x,y;t) + pyt^p R^{k-1}_p(x,y;t). \end{aligned} \tag{3.12}$$

For the general case in Theorems 3.2 and 3.3, we find the generating functions of some special numbers by the special cases $x,\ y,\ p$. For example, $x = y = 1$ in (3.3) we obtain the generating function of incomplete Fibonacci p-numbers. □

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