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Research Article

Lyapunov-Type Inequalities for the Quasilinear Difference Systems

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We establish several Lyapunov-type inequalities for quasilinear difference systems, which generalize or improve all related existing ones. Applying these results, we also obtain some lower bounds for the first eigencurve in the generalized spectra.

1. Introduction

In 1964, Atkinson [1] investigated the following boundary value problem:

$$\Delta(r(n)\Delta u(n)) = \lambda q(n)u(n+1) \tag{1.1}$$

with Dirichlet boundary condition:

$$u(a) = u(b) = 0, \qquad u(n) \not\equiv 0, \quad \forall n \in \mathbb{Z}[a,b], \tag{1.2}$$

and he proved that boundary value problem (1.1) with (1.2) has exactly $b-a-1$ real and simple eigenvalues, which can be arranged in the increasing order

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{b-a-1}, \tag{1.3}$$

where $a, b \in \mathbb{Z}$ with $a \leq b-2$, $\lambda \in \mathbb{R}$, $r(n) > 0$ and $q(n) > 0$ for all $n \in \mathbb{Z}$. Here and in the sequel, $\mathbb{Z}[a,b] = \{a, a+1, a+2, \ldots, b-1, b\}$.

In 1983, Cheng [2] proved that if the second-order difference equation

$$\Delta^2 u(n) + q(n)u(n+1) = 0 \tag{1.4}$$

has a real solution $u(n)$ such that

$$u(a) = u(b) = 0, \qquad u(n) \not\equiv 0, \quad n \in \mathbb{Z}[a,b], \tag{1.5}$$

then one has the following inequality

$$\mathfrak{F}(b-a)\sum_{n=a}^{b-2} q(n) \geq 4, \tag{1.6}$$

where $q(n) \geq 0$ for all $n \in \mathbb{Z}$, and

$$\mathfrak{F}(m) = \begin{cases} \dfrac{m^2-1}{m}, & \text{if } m-1 \text{ is even}, \\ m, & \text{if } m-1 \text{ is odd}, \end{cases} \tag{1.7}$$

and the constant 4 in (1.6) cannot be replaced by a larger number. Inequality (1.6) is a discrete analogy of the following so-called Lyapunov inequality:

$$(b-a)\int_a^b |q(t)|dt > 4, \tag{1.8}$$

if Hill's equation

$$u''(t) + q(t)u(t) = 0 \tag{1.9}$$

has a real solution $u(t)$ such that

$$u(a) = u(b) = 0, \qquad u(t) \neq 0, \quad \forall t \in (a,b), \tag{1.10}$$

where $q(t)$ is a real-valued continuous function defined on $\mathbb{R}$, $a, b \in \mathbb{R}$ with $a < b$. Equation (1.8) was first established by Liapounoff [3] in 1907.

In 2008, Ünal et al. [4] established the following Lyapunov-type inequality:

$$\left(\sum_{n=a}^{b-1} \frac{1}{[r(n)]^{1/(p-1)}}\right)^{1-(1/p)} \left(\sum_{n=a}^{b-2} q^+(n)\right)^{1/p} \geq 2, \tag{1.11}$$

if the following second-order half-linear difference equation:

$$\Delta\Big(r(n)|\Delta u(n)|^{p-2}\Delta u(n)\Big) + q(n)|u(n+1)|^{p-2}u(n+1) = 0 \tag{1.12}$$

has a solution $u(n)$ satisfying

$$u(a) = u(b) = 0, \qquad u(n) \not\equiv 0, \quad \forall n \in \mathbb{Z}[a, b], \tag{1.13}$$

where and in the sequel $q^+(n) = \max\{q(n), 0\}$.

Applying inequality (1.11) to (1.4) (i.e., (1.12) with $p = 2$, $r(n) = 1$, and $q(n) \ge 0$), we can obtain the following Lyapunov-type inequality:

$$(b - a)\sum_{n=a}^{b-2} q(n) \ge 4, \tag{1.14}$$

which was also obtained in [5]. When $b - a - 1$ is odd, (1.14) is the same as (1.6). However, (1.14) is worse than (1.6) when $b - a - 1$ is even. For more discrete cases and continuous cases for Lyapunov-type inequalities, we refer the reader to [5–18].

For a single p-Laplacian equation (1.12), there are many papers which deal with various dynamics behavior of its solutions in the literatures. However, we are not aware of similar works for p-Laplacian systems. We consider here the following quasilinear difference system of resonant type

$$\begin{aligned} -\Delta\left(r_1(n)|\Delta u(n)|^{p_1-2}\Delta u(n)\right) &= f_1(n)|u(n+1)|^{\alpha_1-2}|v(n+1)|^{\alpha_2}u(n+1), \\ -\Delta\left(r_2(n)|\Delta v(n)|^{p_2-2}\Delta v(n)\right) &= f_2(n)|u(n+1)|^{\beta_1}|v(n+1)|^{\beta_2-2}v(n+1), \end{aligned} \tag{1.15}$$

and the quasilinear difference system involving the $(p_1, p_2, \ldots, p_n)$-Laplacian

$$\begin{aligned} -\Delta\left(r_1(n)|\Delta u_1(n)|^{p_1-2}\Delta u_1(n)\right) &= f_1(n)|u_1(n+1)|^{\alpha_1-2}|u_2(n+1)|^{\alpha_2}\cdots|u_m(n+1)|^{\alpha_m}u_1(n+1), \\ -\Delta\left(r_2(n)|\Delta u_2(n)|^{p_2-2}\Delta u_2(n)\right) &= f_2(n)|u_1(n+1)|^{\alpha_1}|u_2(n+1)|^{\alpha_2-2}\cdots|u_m(n+1)|^{\alpha_m}u_2(n+1), \\ &\vdots \\ -\Delta\left(r_m(n)|\Delta u_m(n)|^{p_m-2}\Delta u_m(n)\right) &= f_m(n)|u_1(n+1)|^{\alpha_1}|u_2(n+1)|^{\alpha_2}\cdots|u_m(n+1)|^{\alpha_m-2}u_m(n+1). \end{aligned} \tag{1.16}$$

For the sake of convenience, we give the following hypotheses (H1) and (H2) for system (1.15) and hypothesis (H3) for system (1.16):

(H1) $r_1(n)$, $r_2(n)$, $f_1(n)$ and $f_2(n)$ are real-valued functions and $r_1(n) > 0$ and $r_2(n) > 0$ for all $n \in \mathbb{Z}$;

(H2) $1 < p_1$, $p_2 < \infty$, α_1, α_2, β_1, $\beta_2 > 0$ satisfy $\alpha_1/p_1 + \alpha_2/p_2 = 1$ and $\beta_1/p_1 + \beta_2/p_2 = 1$;

(H3) $r_i(n)$ and $f_i(n)$ are real-valued functions and $r_i(n) > 0$ for $i = 1, 2, \ldots, m$. Furthermore, $1 < p_i < \infty$ and $\alpha_i > 0$ satisfy $\sum_{i=1}^{m}(\alpha_i/p_i) = 1$.

System (1.15) and (1.16) are the discrete analogies of the following two quasilinear differential systems:

$$-\left(r_1(t)\left|u'(t)\right|^{p-2}u'(t)\right)' = f_1(t)|u(t)|^{\alpha_1-2}|v(t)|^{\alpha_2}u(t),$$
$$-\left(r_2(t)\left|v'(t)\right|^{q-2}v'(t)\right)' = f_2(t)|u(t)|^{\beta_1}|v(t)|^{\beta_2-2}v(t), \tag{1.17}$$

$$-\left(r_1(t)\left|u_1'(t)\right|^{p_1-2}u_1'(t)\right)' = f_1(t)|u_1(t)|^{\alpha_1-2}|u_2(t)|^{\alpha_2}\cdots|u_n(t)|^{\alpha_n}u_1(t),$$
$$-\left(r_2(t)\left|u_2'(t)\right|^{p_2-2}u_2'(t)\right)' = f_2(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2-2}\cdots|u_n(t)|^{\alpha_n}u_2(t),$$
$$\vdots \tag{1.18}$$
$$-\left(r_n(t)\left|u_n'(t)\right|^{p_n-2}u_n'(t)\right)' = f_n(t)|u_1(t)|^{\alpha_1}|u_2(t)|^{\alpha_2}\cdots|u_n(t)|^{\alpha_n-2}u_n(t),$$

respectively. Recently, Nápoli and Pinasco [19], Cakmak and Tiryaki [20, 21], and Tang and He [22] established some Lyapunov-type inequalities for systems (1.17) and (1.18). Motivated by the above-mentioned papers, the purpose of this paper is to establish some Lyapunov-type inequalities for systems (1.15) and (1.16). As a byproduct, we derive a better Lyapunov-type inequality than (1.11)

$$\sum_{n=a}^{b-2}\left[\frac{\left(\sum_{s=a}^{n}[r(s)]^{1/(1-p)}\right)^{p-1}\left(\sum_{s=n+1}^{b-1}[r(s)]^{1/(1-p)}\right)^{p-1}}{\left(\sum_{s=a}^{n}[r(s)]^{1/(1-p)}\right)^{p-1}+\left(\sum_{s=n+1}^{b-1}[r(s)]^{1/(1-p)}\right)^{p-1}}\,q^{+}(n)\right]\geq 1 \tag{1.19}$$

for the second-order half-linear difference equation (1.12). In particular, (1.19) produces a new Lyapunov-type inequality

$$\frac{1}{b-a}\sum_{n=a}^{b-2}(n+1-a)(b-n-1)q^{+}(n)\geq 1 \tag{1.20}$$

for Hill's equation (1.4) when $p = 2$ and $r(t) = 1$. It is easy to see that (1.20) is better than (1.6).

This paper is organized as follows. Section 2 gives some Lyapunov-type inequalities for system (1.15), and Lyapunov-type inequalities for system (1.16) are established in Section 3. In Section 4, we apply our Lyapunov-type inequalities to obtain lower bounds for the first eigencurve in the generalized spectra.

2. Lyapunov-Type Inequalities for System (1.15)

In this section, we establish some Lyapunov-type inequalities for system (1.15).

Denote

$$\zeta_1(n) := \left(\sum_{\tau=a}^{n}[r_1(\tau)]^{1/(1-p_1)}\right)^{p_1-1}, \qquad \eta_1(n) := \left(\sum_{\tau=n+1}^{b-1}[r_1(\tau)]^{1/(1-p_1)}\right)^{p_1-1}, \tag{2.1}$$

$$\zeta_2(n) := \left(\sum_{\tau=a}^{n}[r_2(\tau)]^{1/(1-p_2)}\right)^{p_2-1}, \qquad \eta_2(n) := \left(\sum_{\tau=n+1}^{b-1}[r_2(\tau)]^{1/(1-p_2)}\right)^{p_2-1}. \tag{2.2}$$

Theorem 2.1. *Let $a, b \in \mathbb{Z}$ with $a \le b-2$. Suppose that hypotheses (H1) and (H2) are satisfied. If system* (1.15) *has a solution $(u(n), v(n))$ satisfying the boundary value conditions:*

$$u(a) = u(b) = 0 = v(a) = v(b), \qquad u(n) \not\equiv 0, \quad v(n) \not\equiv 0, \quad \forall n \in \mathbb{Z}[a,b], \tag{2.3}$$

then one has the following inequality:

$$\begin{aligned}&\left(\sum_{n=a}^{b-2}\frac{\zeta_1(n)\eta_1(n)}{\zeta_1(n)+\eta_1(n)}f_1^+(n)\right)^{\alpha_1\beta_1/p_1^2}\left(\sum_{n=a}^{b-2}\frac{\zeta_1(n)\eta_1(n)}{\zeta_1(n)+\eta_1(n)}f_2^+(n)\right)^{\beta_1\alpha_2/p_1p_2}\\&\times\left(\sum_{n=a}^{b-2}\frac{\zeta_2(n)\eta_2(n)}{\zeta_2(n)+\eta_2(n)}f_1^+(n)\right)^{\beta_1\alpha_2/p_1p_2}\left(\sum_{n=a}^{b-2}\frac{\zeta_2(n)\eta_2(n)}{\zeta_2(n)+\eta_2(n)}f_2^+(n)\right)^{\alpha_2\beta_2/p_2^2}\ge 1,\end{aligned} \tag{2.4}$$

where and in the sequel $f_i^+(n) = \max\{f_i(n), 0\}$ for $i = 1,2$.

Proof. By (1.15) and (2.3), we obtain

$$\sum_{n=a}^{b-1}r_1(n)|\Delta u(n)|^{p_1} = \sum_{n=a}^{b-2}f_1(n)|u(n+1)|^{\alpha_1}|v(n+1)|^{\alpha_2}, \tag{2.5}$$

$$\sum_{n=a}^{b-1}r_2(n)|\Delta v(n)|^{p_2} = \sum_{n=a}^{b-2}f_2(n)|u(n+1)|^{\beta_1}|v(n+1)|^{\beta_2}. \tag{2.6}$$

It follows from (2.1), (2.3), and the Hölder inequality that

$$\begin{aligned}|u(n+1)|^{p_1} &= \left|\sum_{\tau=a}^{n}\Delta u(\tau)\right|^{p_1}\\&\le\left(\sum_{\tau=a}^{n}|\Delta u(\tau)|\right)^{p_1}\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sum_{\tau=a}^{n}[r_1(\tau)]^{1/(1-p_1)}\right)^{p_1-1}\sum_{\tau=a}^{n}r_1(\tau)|\Delta u(\tau)|^{p_1}\\
&= \zeta_1(n)\sum_{\tau=a}^{n}r_1(\tau)|\Delta u(\tau)|^{p_1}, \quad a\leq n\leq b-1,
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
|u(n+1)|^{p_1} &= \left|\sum_{\tau=n+1}^{b-1}\Delta u(\tau)\right|^{p_1}\\
&\leq \left(\sum_{\tau=n+1}^{b-1}|\Delta u(\tau)|\right)^{p_1}\\
&\leq \left(\sum_{\tau=n+1}^{b-1}[r_1(\tau)]^{1/(1-p_1)}\right)^{p_1-1}\sum_{\tau=n+1}^{b-1}r_1(\tau)|\Delta u(\tau)|^{p_1}\\
&= \eta_1(n)\sum_{\tau=n+1}^{b-1}r_1(\tau)|\Delta u(\tau)|^{p_1}, \quad a\leq n\leq b-1.
\end{aligned}
\tag{2.8}
$$

From (2.7) and (2.8), we have

$$
|u(n+1)|^{p_1} \leq \frac{\zeta_1(n)\eta_1(n)}{\zeta_1(n)+\eta_1(n)}\sum_{\tau=a}^{b-1}r_1(\tau)|\Delta u(\tau)|^{p_1}, \quad a\leq n\leq b-1. \tag{2.9}
$$

Now, it follows from (2.3), (2.5), (2.9), (H2), and the Hölder inequality that

$$
\begin{aligned}
\sum_{n=a}^{b-2}f_1^{+}(n)|u(n+1)|^{p_1} &\leq \sum_{n=a}^{b-2}\left[\frac{\zeta_1(n)\eta_1(n)}{\zeta_1(n)+\eta_1(n)}f_1^{+}(n)\right]\sum_{n=a}^{b-1}r_1(n)|\Delta u(n)|^{p_1}\\
&= M_{11}\sum_{n=a}^{b-2}f_1(n)|u(n+1)|^{\alpha_1}|v(n+1)|^{\alpha_2}\\
&\leq M_{11}\sum_{n=a}^{b-2}f_1^{+}(n)|u(n+1)|^{\alpha_1}|v(n+1)|^{\alpha_2}\\
&\leq M_{11}\left(\sum_{n=a}^{b-2}f_1^{+}(n)|u(n+1)|^{p_1}\right)^{\alpha_1/p_1}\left(\sum_{n=a}^{b-2}f_1^{+}(n)|v(n+1)|^{p_2}\right)^{\alpha_2/p_2},
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
\sum_{n=a}^{b-2}f_2^{+}(n)|u(n+1)|^{p_1} &\leq \sum_{n=a}^{b-2}\left[\frac{\zeta_1(n)\eta_1(n)}{\zeta_1(n)+\eta_1(n)}f_2^{+}(n)\right]\sum_{n=a}^{b-1}r_1(n)|\Delta u(n)|^{p_1}\\
&= M_{12}\sum_{n=a}^{b-2}f_1(n)|u(n+1)|^{\alpha_1}|v(n+1)|^{\alpha_2}
\end{aligned}
$$

$$\begin{aligned}
&\leq M_{12}\sum_{n=a}^{b-2} f_1^+(n)|u(n+1)|^{\alpha_1}|v(n+1)|^{\alpha_2}\\
&\leq M_{12}\left(\sum_{n=a}^{b-2} f_1^+(n)|u(n+1)|^{p_1}\right)^{\alpha_1/p_1}\left(\sum_{n=a}^{b-2} f_1^+(n)|v(n+1)|^{p_2}\right)^{\alpha_2/p_2},
\end{aligned} \tag{2.11}$$

where

$$M_{11}=\sum_{n=a}^{b-2}\left[\frac{\zeta_1(n)\eta_1(n)}{\zeta_1(n)+\eta_1(n)}f_1^+(n)\right],\qquad M_{12}=\sum_{n=a}^{b-2}\left[\frac{\zeta_1(n)\eta_1(n)}{\zeta_1(n)+\eta_1(n)}f_2^+(n)\right]. \tag{2.12}$$

Similar to the proof of (2.9), from (2.2) and (2.3), we have

$$|v(n+1)|^{p_2}\leq\frac{\zeta_2(n)\eta_2(n)}{\zeta_2(n)+\eta_2(n)}\sum_{\tau=a}^{b-1}r_2(\tau)|\Delta v(\tau)|^{p_2},\quad a\leq n\leq b-1. \tag{2.13}$$

It follows from (2.3), (2.6), (2.13), (H2), and the Hölder inequality that

$$\begin{aligned}
\sum_{n=a}^{b-2} f_1^+(n)|v(n+1)|^{p_2}&\leq\sum_{n=a}^{b-2}\left[\frac{\zeta_2(n)\eta_2(n)}{\zeta_2(n)+\eta_2(n)}f_1^+(n)\right]\sum_{n=a}^{b-1}r_2(n)|\Delta v(n)|^{p_2}\\
&=M_{21}\sum_{n=a}^{b-2} f_2(n)|u(n+1)|^{\beta_1}|v(n+1)|^{\beta_2}\\
&\leq M_{21}\sum_{n=a}^{b-2} f_2^+(n)|u(n+1)|^{\beta_1}|v(n+1)|^{\beta_2}\\
&\leq M_{21}\left(\sum_{n=a}^{b-2} f_2^+(n)|u(n+1)|^{p_1}\right)^{\beta_1/p_1}\left(\sum_{n=a}^{b-2} f_2^+(n)|v(n+1)|^{p_2}\right)^{\beta_2/p_2},\\
\sum_{n=a}^{b-2} f_2^+(n)|v(n+1)|^{p_2}&\leq\sum_{n=a}^{b-2}\left[\frac{\zeta_2(n)\eta_2(n)}{\zeta_2(n)+\eta_2(n)}f_2^+(n)\right]\sum_{n=a}^{b-1}r_2(n)|\Delta v(n)|^{p_2}\\
&=M_{22}\sum_{n=a}^{b-2} f_2(n)|u(n+1)|^{\beta_1}|v(n+1)|^{\beta_2}\\
&\leq M_{22}\sum_{n=a}^{b-2} f_2^+(n)|u(n+1)|^{\beta_1}|v(n+1)|^{\beta_2}\\
&\leq M_{22}\left(\sum_{n=a}^{b-2} f_2^+(n)|u(n+1)|^{p_1}\right)^{\beta_1/p_1}\left(\sum_{n=a}^{b-2} f_2^+(n)|v(n+1)|^{p_2}\right)^{\beta_2/p_2},
\end{aligned} \tag{2.14}$$

where

$$M_{21}=\sum_{n=a}^{b-2}\left[\frac{\zeta_2(n)\eta_2(n)}{\zeta_2(n)+\eta_2(n)}f_1^+(n)\right],\qquad M_{22}=\sum_{n=a}^{b-2}\left[\frac{\zeta_2(n)\eta_2(n)}{\zeta_2(n)+\eta_2(n)}f_2^+(n)\right]. \tag{2.15}$$

Next, we prove that

$$\sum_{n=a}^{b-2} f_1^+(n)|u(n+1)|^{p_1} > 0. \tag{2.16}$$

If (2.16) is not true, then

$$\sum_{n=a}^{b-2} f_1^+(n)|u(n+1)|^{p_1} = 0. \tag{2.17}$$

From (2.5) and (2.17), we have

$$0 \le \sum_{n=a}^{b-1} r_1(n)|\Delta u(n)|^{p_1} = \sum_{n=a}^{b-2} f_1(n)|u(n+1)|^{\alpha_1}|v(n+1)|^{\alpha_2} \le \sum_{n=a}^{b-2} f_1^+(n)|u(n+1)|^{\alpha_1}|v(n+1)|^{\alpha_2} = 0. \tag{2.18}$$

It follows from (H1) that

$$\Delta u(n) \equiv 0, \quad a \le n \le b-1. \tag{2.19}$$

Combining (2.7) with (2.19), we obtain that $u(n) \equiv 0$ for $a \le n \le b$, which contradicts (2.3). Therefore, (2.16) holds. Similarly, we have

$$\sum_{n=a}^{b-2} f_2^+(n)|u(n+1)|^{p_1} > 0, \qquad \sum_{n=a}^{b-2} f_1^+(n)|v(n+1)|^{p_2} > 0, \qquad \sum_{n=a}^{b-2} f_2^+(n)|v(n+1)|^{p_2} > 0. \tag{2.20}$$

From (2.10), (2.11), (2.14), (2.16), (2.20), and (H2), we have

$$M_{11}^{\alpha_1\beta_1/p_1^2} M_{12}^{\beta_1\alpha_2/p_1p_2} M_{21}^{\beta_1\alpha_2/p_1p_2} M_{22}^{\alpha_2\beta_2/p_2^2} \ge 1. \tag{2.21}$$

It follows from (2.12), (2.15), and (2.21) that (2.4) holds. □

Corollary 2.2. *Let $a, b \in \mathbb{Z}$ with $a \le b-2$. Suppose that hypothesis (H1) and (H2) are satisfied. If system* (1.15) *has a solution* $(u(n), v(n))$ *satisfying* (2.3), *then one has the following inequality:*

$$\left(\sum_{n=a}^{b-2} f_1^+(n)\left[\zeta_1(n)\eta_1(n)\right]^{1/2}\right)^{\alpha_1\beta_1/p_1^2} \left(\sum_{n=a}^{b-2} f_2^+(n)\left[\zeta_1(n)\eta_1(n)\right]^{1/2}\right)^{\beta_1\alpha_2/p_1p_2}$$
$$\times \left(\sum_{n=a}^{b-2} f_1^+(n)\left[\zeta_2(n)\eta_2(n)\right]^{1/2}\right)^{\beta_1\alpha_2/p_1p_2} \left(\sum_{n=a}^{b-2} f_2^+(n)\left[\zeta_2(n)\eta_2(n)\right]^{1/2}\right)^{\alpha_2\beta_2/p_2^2} \ge 2^{(p_2\beta_1+p_1\alpha_2)/p_1p_2}. \tag{2.22}$$

Proof. Since

$$\zeta_i(n)+\eta_i(n)\geq 2\left[\zeta_i(n)\eta_i(n)\right]^{1/2},\qquad i=1,2, \tag{2.23}$$

it follows from (2.4) and (H2) that (2.22) holds. □

Corollary 2.3. *Let $a,b\in\mathbb{Z}$ with $a\leq b-2$. Suppose that hypotheses (H1) and (H2) are satisfied. If system* (1.15) *has a solution* $(u(n),v(n))$ *satisfying* (2.3)*, then one has the following inequality:*

$$\begin{aligned}&\left(\sum_{n=a}^{b-1}[r_1(n)]^{1/(1-p_1)}\right)^{\beta_1(p_1-1)/p_1}\left(\sum_{n=a}^{b-1}[r_2(n)]^{1/(1-p_2)}\right)^{\alpha_2(p_2-1)/p_2}\\&\times\left(\sum_{n=a}^{b-2}f_1^{+}(n)\right)^{\beta_1/p_1}\left(\sum_{n=a}^{b-2}f_2^{+}(n)\right)^{\alpha_2/p_2}\geq 2^{\beta_1+\alpha_2}.\end{aligned} \tag{2.24}$$

Proof. Since

$$\begin{aligned}\left[\zeta_1(n)\eta_1(n)\right]^{1/2}&=\left(\sum_{\tau=a}^{n}[r_1(\tau)]^{1/(1-p_1)}\sum_{\tau=n+1}^{b-1}[r_1(\tau)]^{1/(1-p_1)}\right)^{(p_1-1)/2}\\&\leq\frac{1}{2^{p_1-1}}\left(\sum_{\tau=a}^{b-1}[r_1(\tau)]^{1/(1-p_1)}\right)^{p_1-1},\\\left[\zeta_2(n)\eta_2(n)\right]^{1/2}&=\left(\sum_{\tau=a}^{n}[r_2(\tau)]^{1/(1-p_2)}\sum_{\tau=n+1}^{b-1}[r_2(\tau)]^{1/(1-p_2)}\right)^{(p_2-1)/2}\\&\leq\frac{1}{2^{p_2-1}}\left(\sum_{\tau=a}^{b-1}[r_2(\tau)]^{1/(1-p_2)}\right)^{p_2-1},\end{aligned} \tag{2.25}$$

it follows from (2.22) and (H2) that (2.24) holds. □

When $\alpha_1=\beta_2=p_1=p_2=p$, $\alpha_2=\beta_1=0$, $r_1(t)=r_2(t)=r(t)$, and $f_1(t)=f_2(t)=q(t)$, system (1.15) reduces to the second-order half-linear difference equation (1.12). Hence, we can directly derive the following Lyapunov-type inequality for (1.12) from (2.10) and (2.16).

Theorem 2.4. *Let $a,b\in\mathbb{Z}$ with $a\leq b-2$. Suppose that $p>1$ and $r(n)>0$. If* (1.12) *has a solution* $u(n)$ *satisfying* (1.13)*, then one has the following inequality:*

$$\sum_{n=a}^{b-2}\left[\frac{\left(\sum_{\tau=a}^{n}[r(\tau)]^{1/(1-p)}\right)^{p-1}\left(\sum_{\tau=n+1}^{b-1}[r(\tau)]^{1/(1-p)}\right)^{p-1}}{\left(\sum_{\tau=a}^{n}[r(\tau)]^{1/(1-p)}\right)^{p-1}+\left(\sum_{\tau=n+1}^{b-1}[r(\tau)]^{1/(1-p)}\right)^{p-1}}\,q^{+}(n)\right]\geq 1. \tag{2.26}$$

Since

$$\left(\sum_{\tau=a}^{n}[r(\tau)]^{1/(1-p)}\right)^{p-1}+\left(\sum_{\tau=n+1}^{b-1}[r(\tau)]^{1/(1-p)}\right)^{p-1}\geq 2\left(\sum_{\tau=a}^{n}[r(\tau)]^{1/(1-p)}\sum_{\tau=n+1}^{b-1}[r(\tau)]^{1/(1-p)}\right)^{(p-1)/2}, \tag{2.27}$$

it follows from Theorem 2.4 that the following corollary holds.

Corollary 2.5. *Let $a,b\in\mathbb{Z}$ with $a\leq b-2$. Suppose that $p>1$ and $r(n)>0$. If* (1.12) *has a solution $u(n)$ satisfying* (1.13)*, then one has the following inequality:*

$$\sum_{n=a}^{b-2}\left[q^{+}(n)\left(\sum_{\tau=a}^{n}[r(\tau)]^{1/(1-p)}\sum_{\tau=n+1}^{b-1}[r(\tau)]^{1/(1-p)}\right)^{(p-1)/2}\right]\geq 2. \tag{2.28}$$

Remark 2.6. It is easy to see that Lyapunov-type inequalities (2.26) and (2.28) are better than (1.11).

3. Lyapunov-Type Inequalities for System (1.16)

In this section, we establish some Lyapunov-type inequalities for system (1.16). Denote

$$\zeta_i(n):=\left(\sum_{\tau=a}^{n}[r_i(\tau)]^{1/(1-p_i)}\right)^{p_i-1},\quad i=1,2,\ldots,m, \tag{3.1}$$

$$\eta_i(n):=\left(\sum_{\tau=n+1}^{b-1}[r_i(\tau)]^{1/(1-p_i)}\right)^{p_i-1},\quad i=1,2,\ldots,m. \tag{3.2}$$

Theorem 3.1. *Let $a,b\in\mathbb{Z}$ with $a\leq b-2$. Suppose that hypothesis (H3) is satisfied. If system* (1.16) *has a solution $(u_1(n),u_2(n),\ldots,u_m(n))$ satisfying the boundary value conditions:*

$$u_i(a)=u_i(b)=0,\qquad u_i(n)\not\equiv 0,\quad \forall n\in\mathbb{Z}[a,b],\ i=1,2,\ldots,m, \tag{3.3}$$

then one has the following inequality:

$$\prod_{i=1}^{m}\prod_{j=1}^{m}\left(\sum_{n=a}^{b-2}\left[\frac{\zeta_i(n)\eta_i(n)}{\zeta_i(n)+\eta_i(n)}f_j^{+}(n)\right]\right)^{\alpha_i\alpha_j/p_ip_j}\geq 1. \tag{3.4}$$

Proof. By (1.16), (H3), and (3.3), we obtain

$$\sum_{n=a}^{b-1}r_i(n)|\Delta u_i(n)|^{p_i}=\sum_{n=a}^{b-2}\left[f_i(n)\prod_{k=1}^{m}|u_k(n+1)|^{\alpha_k}\right],\quad i=1,2,\ldots,m. \tag{3.5}$$

It follows from (3.1), (3.3), and the Hölder inequality that

$$|u_i(n+1)|^{p_i} = \left|\sum_{\tau=a}^{n}\Delta u_i(\tau)\right|^{p_i} \le \left(\sum_{\tau=a}^{n}[r_i(\tau)]^{1/(1-p_i)}\right)^{p_i-1}\sum_{\tau=a}^{n}r_i(\tau)|\Delta u_i(\tau)|^{p_i} = \zeta_i(n)\sum_{\tau=a}^{n}r_i(\tau)|\Delta u_i(\tau)|^{p_i}, \quad a \le n \le b-1,\ i = 1,2,\dots,m. \tag{3.6}$$

Similarly, it follows from (3.2), (3.3), and the Hölder inequality that

$$|u_i(n+1)|^{p_i} = \left|\sum_{\tau=n+1}^{b-1}\Delta u_i(\tau)\right|^{p_i} \le \left(\sum_{\tau=n+1}^{b-1}[r_i(\tau)]^{1/(1-p_i)}\right)^{p_i-1}\sum_{\tau=n+1}^{b-1}r_i(\tau)|\Delta u_i(\tau)|^{p_i} = \eta_i(n)\sum_{\tau=n+1}^{b-1}r_i(\tau)|\Delta u_i(\tau)|^{p_i}, \quad a \le n \le b-1,\ i = 1,2,\dots,m. \tag{3.7}$$

From (3.6) and (3.7), we have

$$|u_i(n+1)|^{p_i} \le \frac{\zeta_i(n)\eta_i(n)}{\zeta_i(n)+\eta_i(n)}\sum_{\tau=a}^{b-1}r_i(\tau)|\Delta u_i(\tau)|^{p_i}, \quad a \le n \le b-1,\ i = 1,2,\dots,m. \tag{3.8}$$

Now, it follows from (3.3), (3.5), (3.8), (H3), and the generalized Hölder inequality that

$$\begin{aligned}\sum_{n=a}^{b-2}f_j^{+}(n)|u_i(n+1)|^{p_i} &\le \sum_{n=a}^{b-2}\left[\frac{\zeta_i(n)\eta_i(n)}{\zeta_i(n)+\eta_i(n)}f_j^{+}(n)\right]\sum_{\tau=a}^{b-1}r_i(\tau)|\Delta u_i(\tau)|^{p_i}\\ &= M_{ij}\sum_{n=a}^{b-2}\left[f_i(n)\prod_{k=1}^{m}|u_k(n+1)|^{\alpha_k}\right]\\ &\le M_{ij}\sum_{n=a}^{b-2}\left[f_i^{+}(n)\prod_{k=1}^{m}|u_k(n+1)|^{\alpha_k}\right]\\ &\le M_{ij}\prod_{k=1}^{m}\left[\sum_{n=a}^{b-2}f_i^{+}(n)|u_k(n+1)|^{p_k}\right]^{\alpha_k/p_k},\end{aligned} \tag{3.9}$$

where

$$M_{ij} = \sum_{n=a}^{b-2}\left[\frac{\zeta_i(n)\eta_i(n)}{\zeta_i(n)+\eta_i(n)}f_j^{+}(n)\right], \quad i,j = 1,2,\dots,m. \tag{3.10}$$

Next, we prove that

$$\sum_{n=a}^{b-2} f_i^+(n)|u_k(n+1)|^{p_k} > 0, \quad i,k = 1,2,\dots,m. \tag{3.11}$$

If (3.11) is not true, then there exists $i_0, k_0 \in \{1,2,\dots,m\}$ such that

$$\sum_{n=a}^{b-2} f_{i_0}^+(n)|u_{k_0}(n+1)|^{p_{k_0}} = 0. \tag{3.12}$$

From (3.5), (3.12), and the generalized Hölder inequality, we have

$$0 \le \sum_{n=a}^{b-1} r_{i_0}(n)|\Delta u_{i_0}(n)|^{p_{i_0}} = \sum_{n=a}^{b-2} f_{i_0}(n)\prod_{k=1}^{m}|u_k(n+1)|^{\alpha_k} \le \prod_{k=1}^{m}\left[\sum_{n=a}^{b-2} f_{i_0}^+(n)|u_k(n+1)|^{p_k}\right]^{\alpha_k/p_k} = 0. \tag{3.13}$$

It follows from the fact that $r_{i_0}(n) > 0$ that

$$\Delta u_{i_0}(n) \equiv 0, \quad a \le n \le b-1. \tag{3.14}$$

Combining (3.6) with (3.14), we obtain that $u_{i_0}(n) \equiv 0$ for $a \le n \le b$, which contradicts (3.3). Therefore, (3.11) holds. From (3.9), (3.11), and (H3), we have

$$\prod_{i=1}^{m}\prod_{j=1}^{m} M_{ij}^{\alpha_i\alpha_j/p_ip_j} \ge 1. \tag{3.15}$$

It follows from (3.10) and (3.15) that (3.4) holds. □

Corollary 3.2. *Let $a, b \in \mathbb{Z}$ with $a \le b-2$. Suppose that hypothesis (H3) is satisfied. If system* (1.16) *has a solution $(u_1(n), u_2(n), \dots, u_m(n))$ satisfying* (3.3)*, then one has the following inequality:*

$$\prod_{i=1}^{m}\prod_{j=1}^{m}\left(\sum_{n=a}^{b-2} f_j^+(n)\left[\zeta_i(n)\eta_i(n)\right]^{1/2}\right)^{\alpha_i\alpha_j/p_ip_j} \ge 2. \tag{3.16}$$

Proof. Since

$$\zeta_i(n) + \eta_i(n) \ge 2\left[\zeta_i(n)\eta_i(n)\right]^{1/2}, \quad i = 1,2,\dots,m, \tag{3.17}$$

it follows from (3.4) and (H3) that (3.16) holds. □

Corollary 3.3. *Let $a, b \in \mathbb{Z}$ with $a \le b-2$. Suppose that hypothesis (H3) is satisfied. If system* (1.16) *has a solution* $(u_1(n), u_2(n), \ldots, u_m(n))$ *satisfying* (3.3)*, then one has the following inequality*

$$\prod_{i=1}^{m}\left(\sum_{n=a}^{b-1}[r_i(n)]^{1/(1-p_i)}\right)^{\alpha_i(p_i-1)/p_i}\prod_{j=1}^{m}\left(\sum_{n=a}^{b-2}f_j^{+}(n)\right)^{\alpha_j/p_j} \ge 2^{\mathcal{A}}, \tag{3.18}$$

where $\mathcal{A} = \sum_{i=1}^{m}\alpha_i$.

Proof. Since

$$\begin{aligned}\left[\zeta_i(n)\eta_i(n)\right]^{1/2} &= \left(\sum_{\tau=a}^{n}[r_i(n)]^{1/(1-p_i)}\sum_{\tau=n+1}^{b-1}[r_i(\tau)]^{1/(1-p_i)}\right)^{(p_i-1)/2} \\ &\le \frac{1}{2^{p_i-1}}\left(\sum_{\tau=a}^{b-1}[r_i(\tau)]^{1/(1-p_i)}\right)^{p_i-1}, \quad i = 1,2,\ldots,m,\end{aligned} \tag{3.19}$$

it follows from (3.16) and (H3) that (3.18) holds. □

4. Some Applications

In this section, we apply our Lyapunov-type inequalities to obtain lower bounds for the first eigencurve in the generalized spectra.

Let $a, b \in \mathbb{Z}$ with $a \le b-2$. We consider here a quasilinear difference system of the form:

$$\begin{aligned}&-\Delta\left(|\Delta u_1(n)|^{p_1-2}\Delta u_1(n)\right) \\ &\quad = \lambda_1\alpha_1 q(n)|u_1(n+1)|^{\alpha_1-2}|u_2(n+1)|^{\alpha_2}\cdots|u_m(n+1)|^{\alpha_m}u_1(n+1), \\ &-\Delta\left(|\Delta u_2(n)|^{p_2-2}\Delta u_2(n)\right) \\ &\quad = \lambda_2\alpha_2 q(n)|u_1(n+1)|^{\alpha_1}|u_2(n+1)|^{\alpha_2-2}\cdots|u_m(n+1)|^{\alpha_m}u_2(n+1), \\ &\quad\vdots \\ &-\Delta\left(|\Delta u_m(n)|^{p_m-2}\Delta u_m(n)\right) \\ &\quad = \lambda_m\alpha_m q(n)|u_1(n+1)|^{\alpha_1}|u_2(n+1)|^{\alpha_2}\cdots|u_m(n+1)|^{\alpha_m-2}u_m(n+1),\end{aligned} \tag{4.1}$$

where $q(n) > 0$, $\lambda_i \in \mathbb{R}$, p_i and α_i are the same as those in (1.16), and u_i satisfies Dirichlet boundary conditions:

$$u_i(a) = u_i(b) = 0, \qquad u_i(n) > 0, \quad \forall n \in \mathbb{Z}[a+1, b-1],\ i = 1,2,\ldots,m. \tag{4.2}$$

We define the generalized spectrum S of a nonlinear difference system as the set of vector $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m$ such that the eigenvalue problem (4.1) with (4.2) admits a nontrivial solution.

Eigenvalue problem or boundary value problem (4.1) with (4.2) is a generalization of the following p-Laplacian difference equation

$$-\Delta\left(|\Delta u(n)|^{p-2}\Delta u(n)\right) = \lambda p q(n)|u(n+1)|^{p-2}u(n+1), \tag{4.3}$$

with Dirichlet boundary condition:

$$u(a) = 0 = u(b), \qquad u(n) > 0, \quad \forall n \in \mathbb{Z}[a+1, b-1], \tag{4.4}$$

where $p > 1$, $\lambda \in \mathbb{R}$, and $q(n) > 0$. When $p = 2$, Atkinson [1, Theorems 4.3.1 and 4.3.5] investigated the existence of eigenvalues for (4.3) with (4.4), see also [23].

Let $f_i(n) = \lambda_i \alpha_i q(n)$ and $r_i(n) = 1$ for $i = 1, 2, \ldots, m$. Then we can apply Theorem 3.1 to boundary value problem (4.1) with (4.2) and obtain a lower bound for the first eigencurve in the generalized spectra.

Theorem 4.1. *Let $a, b \in \mathbb{Z}$ with $a \le b-2$. Assume that $1 < p_i < \infty$, $\alpha_i > 0$ satisfy $\sum_{i=1}^{m}(\alpha_i/p_i) = 1$, and that $q(n) > 0$ for all $n \in \mathbb{Z}$. Then there exists a function $h(\lambda_1, \ldots, \lambda_{m-1})$ such that $\lambda_m \ge h(\lambda_1, \ldots, \lambda_{m-1})$ for every generalized eigenvalue $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ of boundary value problem* (4.1) *with* (4.2), *where $h(\lambda_1, \ldots, \lambda_{m-1})$ is given by:*

$$\begin{aligned} &h(\lambda_1, \ldots, \lambda_{m-1}) \\ &\quad = \frac{1}{\alpha_m}\left[\prod_{j=1}^{m-1}(\lambda_j\alpha_j)^{\alpha_j/p_j}\prod_{i=1}^{m}\left(\sum_{n=a}^{b-2}\frac{[(n-a+1)(b-n-1)]^{p_i-1}}{(n-a+1)^{p_i-1}+(b-n-1)^{p_i-1}}q(n)\right)^{\alpha_i/p_i}\right]^{-(p_m/\alpha_m)}. \end{aligned} \tag{4.5}$$

Proof. For the eigenvalue $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, (4.1) with (4.2) has a nontrivial solution $(u_1(n), u_2(n), \ldots, u_m(n))$. That is system (1.16) with $r_i(n) = 1$ and $f_i(n) = \lambda_i\alpha_i q(n)$ has a solution $(u_1(n), u_2(n), \ldots, u_m(n))$ satisfying (3.3), it follows from (3.4) that $f_i(n) = \lambda_i\alpha_i q(n) > 0$, for all $n \in \mathbb{Z}$, $i = 1, 2, \ldots, m$, and that

$$\begin{aligned} 1 &\le \prod_{i=1}^{m}\prod_{j=1}^{m}\left(\sum_{n=a}^{b-2}\frac{\zeta_i(n)\eta_i(n)}{\zeta_i(n)+\eta_i(n)}f_j^{+}(n)\right)^{\alpha_i\alpha_j/p_ip_j} \\ &= \prod_{j=1}^{m}(\lambda_j\alpha_j)^{\alpha_j/p_j}\prod_{i=1}^{m}\left(\sum_{n=a}^{b-2}\frac{\zeta_i(n)\eta_i(n)}{\zeta_i(n)+\eta_i(n)}q(n)\right)^{\alpha_i/p_i} \\ &= \prod_{j=1}^{m}(\lambda_j\alpha_j)^{\alpha_j/p_j}\prod_{i=1}^{m}\left(\sum_{n=a}^{b-2}\frac{[(n-a+1)(b-n-1)]^{p_i-1}}{(n-a+1)^{p_i-1}+(b-n-1)^{p_i-1}}q(n)\right)^{\alpha_i/p_i}. \end{aligned} \tag{4.6}$$

Hence, we have

$$\lambda_m \ge \frac{1}{\alpha_m}\left[\prod_{j=1}^{m-1}(\lambda_j\alpha_j)^{\alpha_j/p_j}\prod_{i=1}^{m}\left(\sum_{n=a}^{b-2}\frac{[(n-a+1)(b-n-1)]^{p_i-1}}{(n-a+1)^{p_i-1}+(b-n-1)^{p_i-1}}q(n)\right)^{\alpha_i/p_i}\right]^{-(p_m/\alpha_m)}. \tag{4.7}$$

This completes the proof of Theorem 4.1. □

When $m = 2$, boundary value problem (4.1) with (4.2) reduces to the simpler form:

$$\begin{aligned} -\Delta\left(|\Delta u_1(n)|^{p_1-2}\Delta u_1(n)\right) &= \lambda_1\alpha_1 q(n)|u_1(n+1)|^{\alpha_1-2}|u_2(n+1)|^{\alpha_2}u_1(n+1),\\ -\Delta\left(|\Delta u_2(n)|^{p_2-2}\Delta u_2(n)\right) &= \lambda_2\alpha_2 q(n)|u_1(n+1)|^{\alpha_1}|u_2(n+1)|^{\alpha_2-2}u_2(n+1), \end{aligned} \tag{4.8}$$

with Dirichlet boundary conditions:

$$u_i(a) = u_i(b) = 0, \qquad u_i(n) > 0, \quad \forall n \in \mathbb{Z}[a+1, b-1],\ i = 1, 2, \tag{4.9}$$

where $1 < p_1,\ p_2 < \infty$, $\alpha_1, \alpha_2 > 0$ satisfy $\alpha_1/p_1 + \alpha_2/p_2 = 1$, and $q(n) > 0$ for all $n \in \mathbb{Z}$.

Applying Theorem 4.1 to system (4.8) with (4.9) and system (4.3) with (4.4), respectively, we have the following two corollaries immediately.

Corollary 4.2. *Let $a, b \in \mathbb{Z}$ with $a \leq b - 2$. Assume that $1 < p_1,\ p_2 < \infty$, $\alpha_1, \alpha_2 > 0$ satisfy $\alpha_1/p_1 + \alpha_2/p_2 = 1$, and that $q(n) > 0$ for all $n \in \mathbb{Z}$. Then there exists a function $h(\lambda_1)$ such that $\lambda_2 \geq h(\lambda_1)$ for every generalized eigenvalue (λ_1, λ_2) of system* (4.8) *with* (4.9)*, where $h(\lambda_1)$ is given by:*

$$h(\lambda_1) = \frac{1/\alpha_2}{\left(\lambda_1\alpha_1 \sum_{n=a}^{b-2}\left([\mathcal{X}\mathcal{Y}]^{p_1-1}/(\mathcal{X}^{p_1-1}+\mathcal{Y}^{p_1-1})\right)q(n)\right)^{(p_2/\alpha_2)-1} \sum_{n=a}^{b-2}\left([\mathcal{X}\mathcal{Y}]^{p_2-1}/(\mathcal{X}^{p_2-1}+\mathcal{Y}^{p_2-1})\right)q(n)}, \tag{4.10}$$

where $\mathcal{X}$ denote $(n - a + 1)$ and $\mathcal{Y}$ denote $(b - n - 1)$.

Corollary 4.3. *Let $a, b \in \mathbb{Z}$ with $a \leq b - 2$. Assume that $p > 1$ and $q(n) > 0$ for all $n \in \mathbb{Z}$. Then for every eigenvalue λ of system* (4.3) *with* (4.4)*, one has*

$$\lambda \geq \frac{1}{p}\left[\sum_{n=a}^{b-2}\frac{[(n-a+1)(b-n-1)]^{p-1}}{(n-a+1)^{p-1}+(b-n-1)^{p-1}}q(n)\right]^{-1}. \tag{4.11}$$

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