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Research Article

The Painlevé Tests, Bäcklund Transformation and Bilinear Form for the KdV Equation with a Self-Consistent Source

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The Painlevé property and Bäcklund transformation for the KdV equation with a self-consistent source are presented. By testing the equation, it is shown that the equation has the Painlevé property. In order to further prove its integrality, we give its bilinear form and construct its bilinear Bäcklund transformation by the Hirota's bilinear operator. And then the soliton solution of the equation is obtained, based on the proposed bilinear form.

1. Introduction

It is well known that some nonlinear partial differential equations such as the soliton equations with self-consistent sources have important physical applications. In recent years, there are many ways for solving the soliton equations that can be used to the soliton equations with self-consistent sources as well. For example, the soliton solutions of some equations such as the KdV, AKNS, and nonlinear schrödinger equation with self-consistent sources are obtained through the inverse scattering method [1, 2]. In [3] a Darboux transformation, positon and negaton solutions to a Schrödinger self-consistent source equation are further constructed. Also, the binary Darboux transformations for the KdV hierarchies with self-consistent sources were presented in [4]. In addition to that, the Hirota bilinear method has been successfully used in the search for exact solutions of continuous and discrete systems, and also in the search for new integrable equations by testing for multisoliton solutions or Bäcklund transformation [5, 6]. Recently a bilinear Bäcklund transformation has been presented for a $(3 + 1)$-dimensional generalized KP equation. Meanwhile, two classes of exponential and rational traveling wave solutions with arbitrary wave numbers are computed by applying the proposed bilinear Bäcklund transformation (see [7] for details). It is a good reference for solving many high-dimensional soliton equations.

Besides, the Painlevé analysis is a powerful tool for identifying the integrability of a nonlinear system. A partial different equation has the Painlevé property when the solutions of the partial different equation are single-valued about the movable, singularity manifold [8]. The basic thought as follows: if the singularity manifold is determined by $\phi(z_1, z_2, \ldots, z_n) = 0$ and $u = u(z_1, z_2, \ldots, z_n)$ is a solution of the partial different equation, then we assume that

$$u = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j, \tag{1.1}$$

where

$$\phi = \phi(z_1, z_2, \ldots, z_n), \qquad u_j = u_j(z_1, z_2, \ldots, z_n), \qquad u_0 \neq 0, \tag{1.2}$$

are analytic functions of (z_j) in a neighborhood of the manifold $\phi = 0$ and α is an integer. Substitution of (1.1) into the partial different equation determines the values of α and defines the recursion relations for u_j, $j \neq 0, 1, 2, \cdots$. When the anatz (1.1) is correct, the pde is said to possess the Painlevé property and is conjectured to be integrable [9].

Motivated by the previous works, we focus our attention on the following nonlinear partial differential equations (PDEs) which is expressed by

$$\begin{gathered} u_t + u_{xxx} + 12uu_x = \left(\varphi^2\right)_x, \\ \varphi_{xx} + 2u\varphi = \lambda\varphi, \end{gathered} \tag{1.3}$$

where λ is an arbitrary constant. In fact, (1.3) is a reduced form of the KdV equation with a source by symmetry constraints [10, 11]. The main purpose of this paper is to demonstrate the connection between the Painlevé property and the Bäcklund transformation for (1.3). Moreover, we get the bilinear Bäcklund transformation and the exact solution for (1.3) by the Hirota bilinear method. Thus we further convince the integrability of the equation.

The paper is organized as follows. In Section 2, we investigate the Painlevé property for (1.3). By testing the equation it is shown that the equation has the Painlevé property. Furthermore, we obtain a Bäcklund transformation of (1.3). In Section 3, using the Hirota's bilinear operator, we obtain its bilinear form and construct its bilinear Bäcklund transformation. And then its one-soliton solution is obtained. Finally, conclusion is given in Section 4.

2. PainlEvÉ Test

As we know, the basic Painlevé test for ODEs consists of the following steps [12].

Step 1. Identify all possible dominant balances, that is, all singularities of form $u \sim u_0(z-z_0)^{\mu}$.

Step 2. If all exponents μ are integers, find the resonances where arbitrary constants can appear.

Step 3. If all resonances are integers, check the resonance conditions in each Laurent expansion.

Conclusion. If no obstruction is found in Steps 1–3 for every dominant balances, then the Painlevé test is satisfied.

The above series may be substituted into the PDEs. Now we apply the above steps to (1.3). We will further give all possible solutions with integer resonances but without further analysis of the last cases. The expansions about the singular manifold have the forms:

$$u = \sum_{j=0}^{\infty} a_j \phi^{j+\alpha}, \qquad \varphi = \sum_{j=0}^{\infty} b_j \phi^{j+\beta}. \tag{2.1}$$

To find the dominant balances, we are looking for leading order singular behaviour of the form

$$u \approx a_0 \phi^{\alpha}, \qquad \varphi \approx b_0 \phi^{\beta}. \tag{2.2}$$

And the derivatives of (2.2) are given by

$$u_t \approx a_0 \alpha \phi^{\alpha-1} \phi_t, \qquad u_x \approx a_0 \alpha \phi^{\alpha-1} \phi_x, \qquad u_{xxx} \approx a_0 \alpha (\alpha-1)(\alpha-2) \phi^{\alpha-3} \phi_x^3,$$
$$\varphi_x \approx b_0 \beta \phi^{\beta-1} \phi_x, \qquad \varphi_{xx} \approx b_0 \beta (\beta-1) \phi^{\beta-2} \phi_x^2. \tag{2.3}$$

Substituting (2.2)-(2.3) into (1.3), we get the following forms

$$a_0 \alpha (\alpha-1)(\alpha-2) \phi^{\alpha-3} \phi_x^3 + 12 a_0^2 \alpha \phi_x \phi^{2\alpha-1} = 2 b_0^2 \beta \phi_x \phi^{2\beta-1},$$
$$b_0 \beta (\beta-1) \phi^{\beta-2} \phi_x^2 + 2 a_0 b_0 \phi^{\alpha+\beta} = 0. \tag{2.4}$$

Calculating and simplifying equation (2.4), we get the dominant balances

$$u \approx -3\phi_x^2 \phi^{-2}, \qquad \varphi \approx \pm 6 \phi_x^2 \phi^{-2}, \tag{2.5}$$

where $\alpha = \beta = -2$, $a_0 = -3\phi_x^2, b_0 = \pm 6\phi_x^2$. So we complete the first step.

The second step in applying the Painlevé test is to find the resonances. To find the resonances numbers j, we substitute (2.1) into (1.3), and collecting terms of each order of ϕ, we obtain

ϕ^{-5}:

$$-24 a_0 \phi_x^3 - 24 a_0^2 \phi_x = -4 b_0^2 \phi_x. \tag{2.6}$$

By calculating equation (2.6), we get $b_0^2 = 6 a_0 (a_0 + \phi_x^2)$.

ϕ^{-4}:

$$\begin{gathered}18a_{0x}\phi_x^2 - 6a_1\phi_x^3 + 18a_0\phi_x\phi_{xx} + 12a_0a_{0x} - 36a_0a_1\phi_x = 2b_0b_{0x} - 6b_0b_1\phi_x,\\ 6b_0\phi_x^2 + 2a_0b_0 = 0.\end{gathered} \tag{2.7}$$

From (2.6) and (2.7), we have $a_0 = -3\phi_x^2$, $b_0 = \pm 6\phi_x^2$, $6\mathrm{b}_1 = \pm(15\phi_{xx} - 17a_1)$, where a_0 and b_0 are consistent with step one.

ϕ^{-3}:

$$\begin{aligned}&-a_0\phi_t - 3a_{0xx}\phi_x + 3a_{1x}\phi_x^2 - 3a_{0x}\phi_{xx} + 3a_1\phi_x\phi_{xx} - a_0\phi_{xxx} + 6a_0a_{1x}\\ &\quad + 6a_1a_{0x} - 6a_1^2\phi_x - 12a_0a_2\phi_x = b_0b_{1x} + b_1b_{0x} - b_1^2\phi_x - 2b_0b_2\phi_x,\\ &-2b_{0x}\phi_x + b_1\phi_x^2 - b_0\phi_{xx} + a_0b_1 + a_1b_0 = 0.\end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we get $a_1 = 3\phi_{xx}$, $b_1 = \mp 6\phi_{xx}$,

ϕ^{-2}:

$$\begin{aligned}&a_{0t} - a_1\phi_t + a_{0xxx} - 3a_{1xx}\phi_x - 3a_{1x}\phi_{xx} - a_1\phi_{xxx} + 12a_0a_{2x} + 12a_1a_{1x} + 12a_2a_{0x}\\ &\quad - 12a_0a_3\phi_x - 12a_1a_2\phi_x = 2b_0b_{2x} + 2b_1b_{1x} + 2b_2b_{0x} - 2b_0b_3\phi_x - 2b_1b_2\phi_x,\\ &b_{0xx} - 2b_{1x}\phi_x - b_1\phi_{xx} + 2a_0b_2 + 2a_1b_1 + 2a_2b_0 = \lambda b_0.\end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we get

$$a_2 = \frac{15\phi_{xx}^2 - 20\phi_x\phi_{xxx} - \phi_x\phi_t + 4\lambda\phi_x^2}{20\phi_x^2}, \qquad b_2 = \pm\frac{-15\phi_{xx}^2 + 20\phi_x\phi_{xxx} - \phi_x\phi_t - 6\lambda\phi_x^2}{10\phi_x^2}, \tag{2.10}$$

ϕ^{j-5}:

$$\begin{aligned}&a_{j-3,t} + (j-4)a_{j-2}\phi_t + a_{j-3,xxx} + 3(j-4)a_{j-2,xx}\phi_x + 3(j-3)(j-4)a_{j-1,x}\phi_x^2\\ &\quad + (j-2)(j-3)(j-4)a_j\phi_x^3 + 3(j-4)a_{j-2,x}\phi_{xx} + 3(j-3)(j-4)a_{j-1}\phi_x\phi_{xx}\\ &\quad + (j-4)a_{j-2}\phi_{xxx} + 12\{a_0a_{j-1,x} + a_1a_{j-2,x} + \cdots + a_{j-1}a_{0,x}\\ &\qquad\qquad + \phi_x[(j-2)a_0a_j + (j-3)a_1a_{j-1} + \cdots + 2a_{j-4}a_4\\ &\qquad\qquad\qquad + a_{j-3}a_3 - a_{j-1}a_1 - 2a_ja_0]\}\\ &= 2\{b_0b_{j-1,x} + b_1b_{j-2,x} + \cdots + b_{j-1}b_{0,x}\\ &\quad + \phi_x[(j-2)b_0b_j + (j-3)b_1b_{j-1} + \cdots + 2b_{j-4}b_4 + b_{j-3}b_3 - b_{j-1}b_1 - 2b_jb_0]\},\end{aligned} \tag{2.11}$$

ϕ^{j-4}:

$$\begin{aligned}&b_{j-2,xx}+2(j-3)b_{j-1,x}\phi_x+(j-2)(j-3)b_j\phi_x^2+(j-3)b_{j-1}\phi_{xx}\\&\quad+2a_0b_j+2a_1b_{j-1}+\cdots+a_jb_0=\lambda b_{j-2}.\end{aligned}\tag{2.12}$$

Substituting $a_0=-3\phi_x^2,\ b_0=\pm6\phi_x^2$ into (2.11) and (2.12), the coefficients of a_j and b_j may be rearranged to give

$$\begin{aligned}&[(j-2)(j-3)-36](j-4)a_j\phi_x^3+F(a_0,a_1,\ldots,a_{j-1},\phi_t,\phi_x,\phi_{xx},\ldots)\\&\quad=\pm12(j-4)b_j\phi_x^3+G(b_0,b_1,\ldots,b_{j-1},\phi_t,\phi_x,\phi_{xx},\ldots),\end{aligned}\tag{2.13}$$

$$\begin{aligned}&[(j-2)(j-3)-6]b_j\phi_x^2\pm12a_j\phi_x^2\\&\quad+H(a_0,a_1,\ldots,a_{j-1},b_0,b_1,\cdots,b_{j-1},\phi_t,\phi_x,\phi_{xx},\ldots)=0,\end{aligned}\tag{2.14}$$

simplifying (2.13)-(2.14), we have

$$\begin{aligned}(j-4)&\left[j(j-5)\left(j^2-5j-30\right)+144\right]\left(j^2-5j-30\right)a_j\phi_x^3=-j(j-5)\left(j^2-5j-30\right)F\\&+j(j-5)\left(j^2-5j-30\right)G\mp12(j-4)\left(j^2-5j-30\right)\phi_xH,\end{aligned}\tag{2.15}$$

$$(j-4)\left[j(j-5)\left(j^2-5j-30\right)+144\right]b_j\phi_x^3=-(j^2-5j-30)(j-4)\phi_xH\mp12G\pm12F.\tag{2.16}$$

There, it is found that the resonance occurs at $j=4$, so the second step is completed.

For the last step, we will check the resonance conditions. So we need to find the orders in the expansion (2.1) where arbitrary constants may appear:

ϕ^{-1}:

$$\begin{aligned}&a_{1t}+a_{1xxx}+12\left[a_0a_{3x}+a_1a_{2x}+a_2a_{1x}+a_3a_{0x}+\phi_x(2a_0a_4+a_1a_3-a_3a_1-2a_4a_0)\right]\\&\quad=2\left[b_0b_{3x}+b_1b_{2x}+b_2b_{1x}+b_3b_{0x}+\phi_x(2b_0b_4+b_1b_3-b_3b_1-2b_4b_0)\right],\\&b_{1xx}+2a_0b_3+2a_1b_2+2a_2b_1+2a_3b_0=\lambda b_1.\end{aligned}\tag{2.17}$$

From (2.17), we know a_4 and b_4 are both arbitrary. Thus (1.3) possess the Painlevé property.

We now specialize (2.1) by setting the resonance functions $a_4=b_4=0$. Furthermore, we require $a_3=b_3=0$, it is easily demonstrated that $a_j=0,b_j=0,j\geq3$ from the recursion relations.

If a_2 and b_2 satisfy

$$\begin{aligned}a_{2t}+a_{2xxx}+12a_2a_{2x}&=\left(b_2^2\right)_x,\\b_{2xx}+2a_2b_2&=\lambda b_2,\end{aligned}\tag{2.18}$$

we obtain a Bäcklund transformation of (1.3):

$$\begin{aligned} u &= 3(\ln \phi)_{xx} + \tilde{u}, \qquad \tilde{u} \equiv a_2, \\ \varphi &= -6(\ln \phi)_{xx} + \tilde{\varphi}, \qquad \tilde{\varphi} \equiv b_2, \end{aligned} \tag{2.19}$$

where we consider the case of $b_0 = 6\phi_x^2$, $b_1 = -6\phi_{xx}$ moreover, u, φ and $\tilde{u}, \tilde{\varphi}$ satisfy (1.3) and

$$\begin{aligned} &3\tilde{u}(\ln \phi)_{xx} + \tilde{\varphi}(\ln \phi)_{xx} = \theta(t), \\ &\frac{\phi_t}{\phi_x} = 5\lambda + \left(\frac{15\phi_{xx}^2}{\phi_x^2} - \frac{20\phi_{xxx}}{\phi_x} \right). \end{aligned} \tag{2.20}$$

Many studies [9, 13] show that a new solution can usually be obtained from a given solution of an equation if the so-called Bäcklund transformation for the equation is found. Therefore, it is worth to find the Bäcklund transformation of an equation. In the next section, we will give the bilinear Bäcklund transformation of (1.3).

3. Bilinear Form

As we know, when you want to use Hirota method, the first thing you need to do is to rewrite the equation under consideration as the bilinear form [14]. This can be achieved for (1.3) by the following dependent variable transformation:

$$u = (\ln f)_{xx}, \qquad \varphi = \frac{g}{f}. \tag{3.1}$$

Equation (1.3) can be written into bilinear forms

$$\begin{aligned} \left(D_x D_t + D_x^4\right) f \cdot f &= 2g^2, \\ D_x^2 f \cdot g &= \lambda f g, \end{aligned} \tag{3.2}$$

where D is the well-known Hirota bilinear operator

$$D_x^m D_t^n a \cdot b = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)^m \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} \right)^n a(x_1, t_1) b(x_2, t_2) \Big|_{\substack{x_1 = x_2, \\ t_1 = t_2.}} . \tag{3.3}$$

Now we will give the bilinear Bäcklund transformation of (1.3).

Theorem 3.1. *Suppose that* (f, g) *is a solution of* (3.2), *then* (f', g'), *satisfying the following relations:*

$$D_x g \cdot f' - \mu f g' = 0, \tag{3.4}$$

$$D_x f \cdot g' - \mu g f' = 0, \tag{3.5}$$

$$\left(D_x^2 - \nu\right) f \cdot f' = 0, \tag{3.6}$$

$$\left(D_t + D_x^3 + 3\nu D_x\right) f \cdot f' + \frac{1}{\mu} g g' = 0, \tag{3.7}$$

is another solution of (3.2), *where* μ *and* ν *are arbitrary constants and* $\mu \neq 0$.

Proof. We consider the following:

$$\begin{aligned} P_1 &= \left(D_x D_t + D_x^4\right) f' \cdot f' - 2g'^2, \\ P_2 &= D_x^2 f' \cdot g' - \lambda f' g'. \end{aligned} \tag{3.8}$$

We will show that (3.4)–(3.7) imply $P_1 = 0$ and $P_2 = 0$. We first work on the case of P_1. We will use various bilinear identities which, for convenience, are presented in the appendix:

$$\begin{aligned} -P_1 f f &= \left[\left(D_x D_t + D_x^4\right) f \cdot f - 2g^2\right] f' f' - \left[\left(D_x D_t + D_x^4\right) f' \cdot f' - 2g'^2\right] f f \\ &\overset{\text{(A.1)(A.2)}}{=} 2D_x (D_t f \cdot f') \cdot f f' + 2D_x \left[\left(D_x^3 f \cdot f'\right) \cdot f f' + 3\left(D_x^2 f \cdot f'\right) \cdot (D_x f' \cdot f)\right] \\ &\quad + 2g' g' f f - 2 g g f' f' \\ &\overset{(3.6)}{=} 2D_x (D_t f \cdot f') \cdot f f' + 2D_x \left[\left(D_x^3 f \cdot f'\right) \cdot f f' + 3\nu f f' \cdot (D_x f' \cdot f)\right] \\ &\quad + 2 f g' f g' - 2 g f' g f' \\ &\overset{(3.4)(3.5)}{=} 2D_x \left[\left(D_t + D_x^3 + 3\nu D_x\right) f \cdot f'\right] \cdot f f' + 2\frac{1}{\mu} \left[(D_x g \cdot f') f g' - (D_x f \cdot g') g f'\right] \\ &\overset{(A.3)}{=} 2D_x \left[\left(D_t + D_x^3 + 3\nu D_x\right) f \cdot f'\right] \cdot f f' + 2\frac{1}{\mu} D_x g g' \cdot f f' \\ &= 2D_x \left[\left(D_t + D_x^3 + 3\nu D_x\right) f \cdot f' + \frac{1}{\mu} g g'\right] \cdot f f' \\ &\overset{(3.7)}{\equiv} 0. \end{aligned} \tag{3.9}$$

Next we come to the second part of the proof:

$$\begin{aligned}
-P_2 fg &= \left(D_x^2 f\cdot g-\lambda fg\right)f'g'-\left(D_x^2 f'\cdot g'-\lambda f'g'\right)fg\\
&\overset{(A.4)}{=} D_x\left[\left(D_x f\cdot g'\right)\cdot f'g+fg'\cdot\left(D_x f'\cdot g\right)\right]\\
&\overset{(3.4)(3.5)}{=} D_x\left(\mu g f'\cdot f'g-fg'\cdot\mu f g'\right)\overset{(A.5)}{\equiv}0.
\end{aligned} \tag{3.10}$$

Thus we have completed the proof of Theorem 3.1. □

We will show that our Bäcklund transformation (3.4)–(3.7) supplies us with a Lax representation for (1.3). Suppose

$$v=(\ln f)_{xx},\ \ g=\varphi f,\ \ f'=Xf,\ \ g'=Yg, \tag{3.11}$$

then the variables f, g, f' and g' can be eliminated from (3.4)–(3.7). The elimination results in

$$X_x=\frac{\varphi_x}{\varphi}X-\mu Y, \tag{3.12}$$

$$Y_x=-\frac{\varphi_x}{\varphi}Y-\mu X, \tag{3.13}$$

$$X_{xx}=(\nu-2\omega)X, \tag{3.14}$$

$$X_t=-X_{xxx}-3(2\omega+\nu)X_x+\frac{1}{\mu}\varphi^2 Y. \tag{3.15}$$

Therefore, we have the following.

Theorem 3.2. *The compatibility condition of* (3.12)–(3.15) *is* (1.3). *In fact, using the compatibility conditions* $X_{xxt}=X_{txx}$, *one can obtain* (1.3) *where* ν *and* ω *satisfy* $\nu-2\omega=\lambda$.

Finally we will give the soliton solution of the equation (1.3) by the standard perturbation method:

$$u=\left[\ln\left(1+e^{2\xi}\right)\right]_{xx},\qquad \varphi=\frac{\sqrt{2k\beta(t)}e^{\xi}}{1+e^{2\xi}}, \tag{3.16}$$

where $\xi=kx-4k^3t+\int_0^t\beta(s)ds+c$, c, k are arbirtary constants and $\lambda=k^2$.

4. Conclusion

In this paper, we investigate the Painlevé property for the KdV equation with a self-consistent source. By tests to the equation, it is shown that only the principal balance of the equation has the Painlevé property. While noninteger resonances are allowed with the weak extension of the Painlevé test [12]. We obtain the two different Bäcklund transformations. And then the soliton solution for (1.3) is given.

Appendix

In this appendix, we list the relevant bilinear identities, which can be proved directly. Here a, b, c, and d are arbitrary functions of the independent variables x and t :

$$(D_x D_t a \cdot a)b^2 - a^2(D_x D_t b \cdot b) = 2D_x(D_t a \cdot b) \cdot ba, \tag{A.1}$$

$$\left(D_x^4 a \cdot a\right)b^2 - a^2\left(D_x^4 b \cdot b\right) = 2D_x\left[\left(D_x^3 a \cdot b\right) \cdot ab + 3\left(D_x^2 a \cdot b\right) \cdot (D_x b \cdot a)\right], \tag{A.2}$$

$$(D_x a \cdot b)cd - ab(D_x c \cdot d) = D_x(ad) \cdot (cb), \tag{A.3}$$

$$\left(D_x^2 a \cdot b\right)cd - ab\left(D_x^2 c \cdot d\right) = D_x[(D_x a \cdot d) \cdot cb + ad \cdot (D_x c \cdot b)], \tag{A.4}$$

$$D_x a \cdot a = 0. \tag{A.5}$$

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