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Research Article

An SLBRS Model with Vertical Transmission of Computer Virus over the Internet

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By incorporating an additional recovery compartment in the SLBS model, a new model, known as the SLBRS model, is proposed in this paper. The qualitative properties of this model are investigated. The result shows that the dynamic behavior of the model is determined by a threshold $\mathcal{R}^0$. Specially, virus-free equilibrium is globally asymptotically stable if $\mathcal{R}^0 \leq 1$, whereas the viral equilibrium is globally asymptotically stable if $\mathcal{R}^0 > 1$. Next, the sensitivity analysis of $\mathcal{R}^0$ to four system parameters is also analyzed. On this basis, a collection of strategies are advised for eradicating viruses spreading across the Internet effectively.

1. Introduction

Malicious computer virus programs form a great threat to information society by acquiring information and hard drives illegally, damaging motherboard, congesting network traffic, making devices out of control, and the like [1–5]. The popularization of the Internet further strengthens this threat; computer viruses can propagate through downloading files, opening email attachments, or even instant messaging [3, 4]. Although a considerable amount of work has been done to protect against the propagation of virus, the human effort to struggle against virus is still in its infancy [1–8].

Because of the development and unpredictability of computer virus, the creation of antivirus software, which aims to analyze the structures of viral codes, always lags behind the creation of virus programs. As a result, antivirus techniques are incompetent to effectively forecast the trend of evolution of virus [3, 4, 7, 8]. For the purpose of understanding the long-term behavior of viruses and posing effective strategies of controlling their spread across the Internet, and in view of that the propagation of computer virus across the Internet resembles that of biological virus across a population, it is appropriate to macroscopically establish and study computer virus propagation models by properly modifying their biological counterparts. Indeed, some classical epidemiological models were simply borrowed to establish computer virus models so as to characterize the way virus propagates [9–16], which share a common assumption that an infected computer in which the virus resides is in latency cannot infect other computers. Very recently, Yang et al. [17–20] proposed a new model, the SLBS model, by taking into account the fact that a computer immediately possesses infection ability once it is infected. This model, however, does not consider that those recovered computers can gain temporary immunity.

In this paper, we propose a new computer virus model, the SLBRS model, which incorporates an additional recovery compartment. In SLBRS model, all the computers connected to the Internet are partitioned into four compartments: uninfected computers having no immunity (S computers), infected computers that are latent (L computers), infected computers that are breaking out (B computers), and uninfected computers having temporary immunity (R computers) from which virus is removed after its outbreak. Because the incidence rate of SLBRS model is $f(S;L,B)$, not conventional $f(S,I)$ [11, 12], it is a challenge to research this new model. Specially, the global stability of the viral equilibrium for SLBRS model is mostly difficult to resolve by means of constructive Lyapunov function.

To establish our results, we apply a so-called geometric approach to global stability to obtain sufficient conditions for global stability of viral equilibrium, namely, equilibrium with all positive components. It is a generalization of the Poincare-Bendixson criterion for systems of ordinary differential equations, which first appeared in Smith [21] and was further developed in Li and Muldowney [22, 23] and Li [24, 25], the majority of applications refer to epidemic models, as SIR, SEIR, SEIS, SEIRS models (see, e.g., [26–31]), as well as other population dynamics context [32, 33]. It is proved that the dynamical behavior of the SLBRS model is fully determined by a threshold $\mathcal{R}^0$: the virus-free equilibrium is globally asymptotically stable if $\mathcal{R}^0 \leq 1$, whereas the viral equilibrium is globally asymptotically stable if $\mathcal{R}^0 > 1$. The theoretical results obtained imply some practical means of eradicating computer viruses distributed over the Internet.

The subsequent materials are organized in this pattern: Sections 2 formulates the SLBRS model and gives its basic reproduction number as well as virus-free and viral equilibria, respectively. Sections 3 examine the global stabilities of the virus-free and viral equilibria, respectively. By analyzing the dependence of $\mathcal{R}^0$ on the system parameters, Section 4 poses a set of effective strategies for controlling the spread of computer virus across the Internet. Finally, Section 5 makes a brief summary of this work.

2. SLBRS Model

In this paper, all computers connected to the Internet are partitioned into four compartments: uninfected computers having no immunity (S computers), infected computers that are latent (L computers), infected computers that are breaking out (B computers), and uninfected computers having temporary immunity (R computers). Let $S(t)$, $L(t)$, $B(t)$, and $R(t)$ denote their corresponding percentages of all computers at time t, respectively. The basic assumptions in concern with our model are presented below.

(A1) All newly connected computers are all virus free.

(A2) External computers are connected to the Internet at positive constant rate μ. Also, internal computers are disconnected from the Internet at the same rate μ.

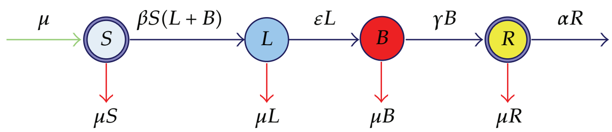


Figure 1: State transition diagram for the SLBRS model.

(A3) Each virus-free computer gets contact with an infected computer at a bilinear incidence rate $\beta S(L+B)$, where β is positive constant.

(A4) Latent computers break out at nonnegative constant rate ε.

(A5) Breaking-out computers are cured at nonnegative constant rate γ.

(A6) Recovered computers become susceptibly virus-free again at nonnegative constant rate α.

According to the above assumptions, the transfer diagram is depicted in Figure 1.

Therefor, the corresponding computer virus model is of the form:

$$\begin{aligned}\dot{S} &= \mu - \beta S(L+B) + \alpha R - \mu S,\\ \dot{L} &= \beta S(L+B) - \varepsilon L - \mu L,\\ \dot{B} &= \varepsilon L - \gamma B - \mu B,\\ \dot{R} &= \gamma B - \alpha R - \mu R.\end{aligned} \tag{2.1}$$

Because $S + L + B + R = 1$, system (2.1) simplifies to the following three-dimensional subsystem:

$$\begin{aligned}\dot{S} &= \mu(1-S) - \beta S(L+B) + \alpha(1-L-B-S),\\ \dot{L} &= \beta S(L+B) - (\varepsilon+\mu)L,\\ \dot{B} &= \varepsilon L - (\gamma+\mu)B,\end{aligned} \tag{2.2}$$

on the closed, positively invariant set $\Omega = \{(S, L, B) : S \geq 0, L \geq 0, B \geq 0, S + L + B \leq 1\}$, with initial conditions $S(0) \geq 0$, $L(0) \geq 0$, $B(0) \geq 0$.

Hence, we consider only solutions with initial conditions inside the region Ω, in which the usual existence, uniqueness of solutions, and continuation results hold.

Clearly, the system (2.2) always has the virus-free equilibrium:

$$E_0 = (S_0, L_0, B_0) = (1, 0, 0). \tag{2.3}$$

As one of the most useful threshold parameters mathematically characterizing the spread of virus, the basic reproduction number, $\mathcal{R}^0$, is defined as the expected number of secondary cases produced by a single (typical) infection in a population of susceptible computers [24]. Let $X = (L, B)$, it follows from system (2.1) that

$$\frac{dX}{dt} = \mathcal{F} - \mathcal{V}, \tag{2.4}$$

where

$$\mathcal{F} = \begin{bmatrix} \beta S(L+B) \\ 0 \end{bmatrix}, \qquad \mathcal{V} = \begin{bmatrix} (\mu+\varepsilon)L \\ -\varepsilon L + (\mu+\gamma)B \end{bmatrix}. \tag{2.5}$$

Let F be the Jacobian matrix of $\mathcal{F}$ at E_0, and V the Jacobian matrix of $\mathcal{V}$ at E_0, respectively, then we get

$$F = \begin{bmatrix} \beta & \beta \\ 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} (\mu+\varepsilon) & 0 \\ -\varepsilon & \mu+\gamma \end{bmatrix}. \tag{2.6}$$

The spectral radius $\mathcal{R}^0$ of the matrix $K = FV^{-1}$ is exactly the basic reproduction number of the model, that is, $\mathcal{R}^0$. Hence

$$\mathcal{R}^0 = \beta \frac{\mu+\gamma+\varepsilon}{(\mu+\varepsilon)(\mu+\gamma)}. \tag{2.7}$$

Further, system (2.2) also has an interior equilibrium called viral equilibrium given by

$$E_* = (S_*, L_*, B_*), \tag{2.8}$$

where

$$\begin{gathered} S_* = \frac{(\mu+\gamma)(\mu+\varepsilon)}{\beta(\mu+\gamma+\varepsilon)} = \frac{1}{\mathcal{R}^0}, \\ L_* = K^*(\mu+\gamma)(\mu+\alpha), \\ B_* = K^*\varepsilon(\mu+\alpha), \end{gathered} \tag{2.9}$$

and $K^* = ((\mu+\varepsilon)(\mu+\gamma)(\mathcal{R}^0-1))/(\beta(\mu+\gamma+\varepsilon)[\gamma(\varepsilon+\alpha+\mu)+(\mu+\varepsilon)(\mu+\gamma)])$. $E_* = (S_*, L_*, B_*) \in \Omega' = \Omega - \{E_0\}$. It is clear that the viral equilibrium exists if and only if $\mathcal{R}^0 > 1$.

In the following sections, we will study the dynamical behavior of the system (2.2).

3. Global Stability

In this section, we study the global stabilities of virus-free and viral equilibria, respectively. First, we have the following.

Theorem 3.1. (a) *If* $\mathcal{R}^0 \leq 1$*, then* E_0 *is the only equilibrium and it is globally stable in* Ω*.* (b) *If* $\mathcal{R}^0 > 1$*, then* E_0 *is unstable and there exists a unique vrial equilibrium* E_**. Furthermore, all solutions starting in* Ω *and sufficiently close to* E_0 *move away from* E_0 *if* $\mathcal{R}^0 > 1$.

Proof. Consider a Lyapunov function:

$$V(L,B) = \frac{1}{2}\left[L^2 + \frac{(\mu+\varepsilon)(\mu+\gamma)}{\varepsilon(\mu+\gamma+\varepsilon)}B^2\right]. \tag{3.1}$$

Clearly, V is positively definite. By calculation, we get

$$\begin{aligned}\left.\frac{dv}{dt}\right|_{(2)} &= \beta SL^2 + \beta SLB - (\mu+\varepsilon)L^2 + \frac{(\mu+\varepsilon)(\mu+\gamma)}{\varepsilon(\mu+\gamma+\varepsilon)}[\varepsilon L - (\mu+\alpha)]B \\ &= \left[\beta S - \frac{(\mu+\varepsilon)(\mu+\gamma)}{\mu+\gamma+\varepsilon}\right](L+B)L - \frac{(\mu+\varepsilon)\varepsilon}{\mu+\gamma+\varepsilon}\left(L - \frac{\mu+\gamma}{\varepsilon}B\right)^2.\end{aligned} \tag{3.2}$$

Because $\mathcal{R}^0 \leq 1$, that is, $\beta(\mu+\gamma+\varepsilon) < (\mu+\varepsilon)(\mu+\gamma)$. It can be seen that $dV/dt \leq 0$ holds for $(S,L,B) \in \Omega$. Furthermore, $dV/dt = 0$ if and only if $(L,B) = (0,0)$. Besides, $V(L,B) \to \infty$ as $L \to \infty$ or $B \to \infty$. It follows from LaSalle invariance principle [34] that E_0 is globally asymptotically stable with respect to Ω if $\mathcal{R}^0 \leq 1$. This proves claim (a).

We linearize the system (2.2) at E_0, giving the Jacobian matrix:

$$J_{E_0} = \begin{bmatrix}\beta-\mu-\varepsilon & \beta & 0 \\ \varepsilon & -\mu-\gamma & 0 \\ 0 & \gamma & -\mu-\alpha\end{bmatrix}. \tag{3.3}$$

The corresponding eigenvalues of J_{E_0} are

$$\begin{aligned}\lambda_1 &= -\mu-\alpha, \\ \lambda_2 &= \frac{1}{2}\left(-\gamma-\varepsilon-2\mu+\beta+\sqrt{\beta^2+2\beta(\varepsilon+\gamma)+(\varepsilon-\gamma)^2}\right), \\ \lambda_3 &= \frac{1}{2}\left(-\gamma-\varepsilon-2\mu+\beta-\sqrt{\beta^2+2\beta(\varepsilon+\gamma)+(\varepsilon-\gamma)^2}\right).\end{aligned} \tag{3.4}$$

Obviously $\lambda_1 < 0$ and $\lambda_2 > 0$ if $\mathcal{R}^0 > 1$ which implies $\beta(\mu+\gamma+\varepsilon) > \mu^2+\mu\gamma+\mu\varepsilon+\varepsilon\gamma$. By the stability theory [35], E_0 is unstable and there exists a unique vrial equilibrium E_* if $\mathcal{R}^0 > 1$. This proves claim (b). □

Secondly, we study the global stability of viral equilibrium, E_*. So we have the following.

Lemma 3.2. *The viral equilibrium* E_* *is locally asymptotically stable with respect to* Ω' *if* $\mathcal{R}^0 > 1$.

Proof. The Jacobian matrix of the linearized system of system (2.2) evaluated at E_* is

$$J_{E_*} = \begin{bmatrix} -\beta(L_* + B_* - S_*) - \mu - \varepsilon & -\beta(L_* + B_* - S_*) & -\beta(L_* + B_*) \\ \varepsilon & -\mu - \gamma & 0 \\ 0 & \gamma & -\mu - \alpha \end{bmatrix}. \tag{3.5}$$

The characteristic equation of J_{E_*} is $p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$, where

$$\begin{aligned} a_2 &= \beta\left(L_* + B_* + \frac{S_* B_*}{L_*}\right) + \varepsilon\frac{L_*}{B_*} + \gamma\frac{B_*}{R_*}, \\ a_1 &= \frac{\varepsilon\gamma L_* + \beta\varepsilon(L_* + B_*)^2}{B_*} + \beta\gamma\frac{B_* L_*^2 + B_*^2 L_* + S_* B_*^2}{R_* L_*}, \\ a_1 &= \beta\gamma\varepsilon\frac{(R_* + L_* + B_*)(L_* + B_*)}{R_*}. \end{aligned} \tag{3.6}$$

We have

$$\begin{aligned} a_2 a_1 - a_0 = {} & \beta^2\varepsilon\left[\frac{(L_* + B_*)^3}{R_*} + \frac{S_*(L_* + B_*)^2}{L_*}\right] + \beta\varepsilon\gamma\left[\frac{L_*^2 + B_* L_* + S_* B_*}{R_*} + \frac{L_*^2}{B_*}\right] \\ & + \beta^2\gamma\left[\frac{\left(L_*^2 + B_* L_* + S_* B_*\right)\left(B_* L_*^2 + B_*^2 + B_*^2 L_* + S_* B_*^2\right)}{B_* L_*^2}\right] \\ & + \beta\varepsilon^2\frac{L_*^2(L_* + 2B_*)}{B_*^2} + \beta\gamma^2\frac{B_*\left(B_* L_*^2 + B_*^2 L_* + S_* B_*^2\right)}{R_* L_*} \\ & + \varepsilon\gamma^2\frac{L_*}{R_*} + \beta\varepsilon\mu B_* + \beta\varepsilon\gamma S_* + \varepsilon^2\gamma. \end{aligned} \tag{3.7}$$

Clearly, $a_2 > 0$, $a_1 > 0$, $a_2 > 0$, and $(a_2 a_1 - a_0) > 0$ if $\mathcal{R}^0 > 1$. By Routh-Hurwitz criterion, E_* is locally asymptotically stable. □

It is obvious that $M = \{(S, 0, 0) : 0 \leq S \leq 1\}$ is the maximum invariant set on the boundary of Ω. By Theorem 3.1, all orbits started from the interior of Ω will not get away from M if $\mathcal{R}^0 > 1$. Furthermore, the stable set of M, $M_S = \{P \in \Omega : \omega(P) \subseteq M\}$, is equal to M and on the boundary of Ω. Then, we can derive the following proposition.

Proposition 3.3. *If $\mathcal{R}^0 > 1$, the system* (2.2) *is uniformly persistent.*

System (2.2) is said to be uniformly persistent in Ω, or rather there exists constant c such that

$$\liminf_{t\to\infty} S(t) > c, \qquad \liminf_{t\to\infty} L(t) > c, \qquad \liminf_{t\to\infty} B(t) > c, \tag{3.8}$$

$$\liminf_{t\to\infty} [1 - S(t) - L(t) - B(t)] > c. \tag{3.9}$$

provided $(S(0), L(0), B(0)) \in \Omega'$ [9, 36–38]. Here, c is independent of initial conditions.

The uniform persistence of (2.2) in the bounded set Ω is equivalent to the existence of a compact $K \in \Omega'$ that is absorbing for (2.2), namely, each compact set $K \in \Omega'$ satisfies $x(t, x_0) \subset K$ for sufficiently large t, where $x(t, x_0)$ denotes the solution of (2.2) such that $x(0, x_0) = x_0$ [38]. We state our main result in the following theorem. Appendix A outlines a general mathematical framework for providing global stability, which will be used in the following to prove the Theorem 3.4.

Theorem 3.4. *E_* is globally stable in Ω' if $\mathcal{R}^0 > 1$.*

Proof. From the Appendix A, we know that system (2.2) satisfies the following assumptions.

(H_1) There exists a compact absorbing set $K \in \Omega'$.

(H_2) System (2.2) has a unique equilibrium $E_* \in \Omega'$.

Let $X = (S, L, B)$ and $F(X)$ denote the vector field of system (2.2), the Jacobian matrix $J = \partial F/\partial X$ associated with a general solution $x(t)$ of system (2.2) is

$$J = \begin{bmatrix} -\beta(L+B) - (\mu+\alpha) & -\beta S - \alpha & -\beta S - \alpha \\ \beta(L+B) & \beta S - (\mu+\varepsilon) & \beta S \\ 0 & \varepsilon & -(\mu+\gamma) \end{bmatrix}, \tag{3.10}$$

and its second additive compound Jacobian matrix $J^{[2]}$ [39, 40] is

$$J^{[2]} = \begin{bmatrix} \begin{matrix} -\beta(L+B-S) \\ -(2\mu+\varepsilon+\alpha) \end{matrix} & \beta S & \beta S + \alpha \\ \varepsilon & \begin{matrix} -\beta(L+B) \\ -(2\mu+\varepsilon+\gamma) \end{matrix} & -\beta S - \alpha \\ 0 & \beta(L+B) & \beta S - (2\mu+\varepsilon+\gamma) \end{bmatrix}. \tag{3.11}$$

We consider the matrix-valued function $P(X) = P(S, L, B)$ as

$$P(S,L,B) = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & (1-a_2)\dfrac{L}{B} & 0 \\ 0 & a_2\dfrac{L}{B} & \dfrac{L}{B} \end{bmatrix}, \tag{3.12}$$

where

$$1 < a_1 < 1 + \frac{\alpha(1-c) + 2\beta c}{\alpha+\beta}, \tag{3.13}$$

and $0 < a_2 < 1$. Here, c is the uniform persistence constant in (3.8). Then,

$$P_f P^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{\dot{L}}{L} - \dfrac{\dot{B}}{B} & 0 \\ 0 & 0 & \dfrac{\dot{L}}{L} - \dfrac{\dot{B}}{B} \end{bmatrix}, \tag{3.14}$$

and the matrix $B = P_f P^{-1} + PJ^{[2]}P^{-1}$ in (A.4) can be written in block form:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{3.15}$$

where

$$\begin{aligned}
B_{11} &= \left[-\beta(L+B-S) - (2\mu+\varepsilon+\alpha)\right], \\
B_{12} &= \left[\frac{a_1\beta SB - a_1a_2(\beta S+\alpha)B}{L(1-a_2)} \quad a_1(\beta S+\alpha)\frac{B}{L}\right], \\
B_{21} &= \begin{bmatrix} \dfrac{1-a_2}{a_1}\dfrac{L}{B}\varepsilon \\ \dfrac{a_2}{a_1}\dfrac{L}{B}\varepsilon \end{bmatrix}, \\
B_{22} &= \begin{bmatrix} \Im - \beta(L+B) - (2\mu+\gamma+\alpha) + a_2(\beta S+\alpha) & (a_2-1)(\beta S+\alpha) \\ \beta(L+B) - a_2(\beta S+\alpha) + \dfrac{a_2}{1-a_2}\varepsilon & \Im - a_2(\beta S+\alpha) + \beta S - (2\mu+\gamma+\varepsilon) \end{bmatrix},
\end{aligned} \tag{3.16}$$

where $\Im$ donates $\dot{L}/L - \dot{B}/B$.

Let $z = (u, v, w)$ denote the vectors in $\mathbb{R}^3$, we select a norm in $\mathbb{R}^3$ as

$$|(u, v, w)| = \max\{|u|, |v| + |w|\}, \tag{3.17}$$

and let μ denote the Lozinskii measure with respect to this norm.

As described in [41], we have the estimate:

$$\mu(B) \le \sup\{g_1, g_2\} = \sup\{\mu_1(B_{11}) + |B_{12}|, \mu_1(B_{22}) + |B_{21}|\}. \tag{3.18}$$

Here, $|B_{21}|$, $|B_{12}|$ are matrix norms with respect to the L^1 vector norm, and μ_1 denotes the Lozinskii measure with respect to the L^1 norm.

We thus obtain

$$\begin{aligned}
g_1 &= \mu_1(B_{11}) + |B_{12}| \\
&= -\beta(L+B-S) - (2\mu+\varepsilon+\alpha) \\
&\quad + \max\left\{a_1\beta S\frac{B}{L} - \alpha\frac{a_1a_2}{1-a_2}\frac{B}{L}, a_1(\beta S+\alpha)\frac{B}{L}\right\},
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
g_2 &= |B_{21}| + \mu_1(B_{22}) \\
&= \frac{\dot{L}}{L} - \frac{\dot{B}}{B} + \frac{\varepsilon}{a_1}\frac{B}{L} - (2\mu+\gamma+\alpha) \\
&\quad + \max\left\{\frac{a_2\varepsilon}{1-a_2}, 2(1-a_2)(\beta S+\alpha) - \varepsilon\right\}.
\end{aligned} \tag{3.20}$$

Rewriting (2.2), we have

$$\frac{\dot{L}}{L} = \beta S + \beta S \frac{B}{L} - (\mu + \varepsilon), \tag{3.21}$$

$$\frac{\dot{B}}{B} = \varepsilon \frac{L}{B} - (\mu + \gamma). \tag{3.22}$$

The uniform persistence constant c in (3.8) can be adjusted so that there exists $T > 0$ independent of $x(0) \in K$ the compact absorbing set, such that

$$S(t) \geq c, \quad L(t) \geq c, \quad B(t) \geq c \quad \text{for } t > T. \tag{3.23}$$

Substituting (3.21) into (3.19) and (3.22) into (3.20) and using (3.23) and our choice of a_1, we obtain, for $t > T$,

$$\begin{aligned} g_1 &= \mu_1(B_{11}) + |B_{12}| \\ &= \frac{\dot{L}}{L} - \mu - \alpha - \beta(L+B) - \beta S \frac{B}{L} + a_1(\beta S + \alpha)\frac{B}{L} \\ &\leq \frac{\dot{L}}{L} - \mu, \\ g_2 &\leq \frac{\dot{L}}{L} - \mu - \alpha + \frac{a_2 \varepsilon}{1 - a_2} \\ &\leq \frac{\dot{L}}{L} - \mu. \end{aligned} \tag{3.24}$$

Therefore, $\mu(B) \leq \dot{L}/L - \mu$ for $t > T$ by (3.18) and (3.24). Along each solution $x(t, x_0)$ to (2.2) such that $x_0 \in K$ and for $t > T$, we thus have

$$\frac{1}{t}\int_0^t \mu(B)ds \leq \frac{1}{t}\int_0^T \mu(B)ds + \frac{1}{t}\log\frac{L(t)}{L(T)} - \mu, \tag{3.25}$$

which implies that

$$\overline{q}_2 = \limsup_{t \to \infty} \sup_{x_0 \in \Omega} \frac{1}{t}\int_0^t \mu(B(x, x_0))ds \leq -\frac{\mu}{2} < 0, \tag{3.26}$$

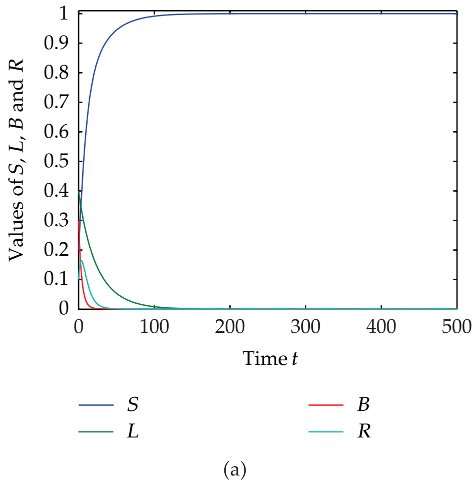
from (3.18), proving Theorem 3.4. □

Consider system (2.2) with parameter values shown in Table 1.

For $\mathcal{R}^0 \leq 1$, by Theorem 3.1, the virus-free equilibrium of system (2.2) is globally asymptotically stable. Figures 2(a)-2(b) demonstrate how $S(t)$, $L(t)$, $B(t)$, and $R(t)$ evolve with the elapse of time in the case of $\mathcal{R}^0 = 0.2908$, and 0.9944 in Table 1, respectively.

Table 1: Typical $\mathcal{R}^0$ and corresponding parameter values for system (2.2).

$\mathcal{R}^0$	β	ε	γ	α	μ	$S(0)$	$L(0)$	$B(0)$
0.2908	0.015	0.002	0.2	0.1	0.05	0.2	0.4	0.3
0.9944	0.05	0.001	0.02	0.02	0.05	0.2	0.4	0.3
1.1607	0.025	0.03	0.025	0.02	0.01	0.5	0.2	0.2
4.2273	0.045	0.001	0.02	0.02	0.01	0.5	0.2	0.2
9.9440	0.15	0.05	0.02	0.01	0.001	0.5	0.2	0.2
109.0909	0.2	0.002	0.005	0.01	0.0005	0.5	0.2	0.2



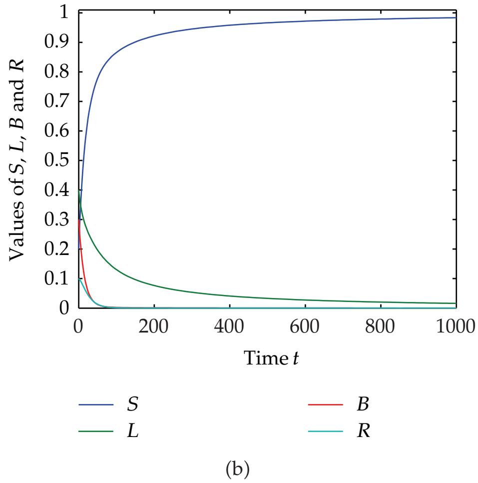


Figure 2: (a) Evolutions of S, L, B, and R in the case of $\mathcal{R}^0 = 0.2908$ and (b) evolutions of S, L, B, and R in the case of $\mathcal{R}^0 = 0.9944$.

For $\mathcal{R}^0 > 1$, by Theorem 3.4, the viral equilibrium of system (2.2) is globally asymptotically stable. Figures 3(a)–3(d) demonstrate how $S(t)$, $L(t)$, $B(t)$ and $R(t)$ evolve with the elapse of time in the case of $\mathcal{R}^0 = 1.1607, 4.2273, 9.9440$ and 109.0909 in Table 1, respectively.

4. Discussions

As was indicated in the previous section, it is critical to take various actions to control the system parameters so that $\mathcal{R}^0$ is remarkably below one. This section is intended to propose some effective measures for achieving this goal.

For our purpose, it is instructive to examine the sensitivities of $\mathcal{R}^0$ to four system parameters: β, ε, γ, and μ, respectively. Following Arriola and Hyman [42], the normalized forward sensitivity indices with respect to β, ε, γ, and μ are calculated, respectively, as follows:

$$\frac{\partial \mathcal{R}^0/\mathcal{R}^0}{\partial \beta/\beta} = \frac{\beta}{\mathcal{R}^0}\frac{\mu+\gamma+\varepsilon}{(\mu+\gamma)(\mu+\varepsilon)} = 1 > 0,$$

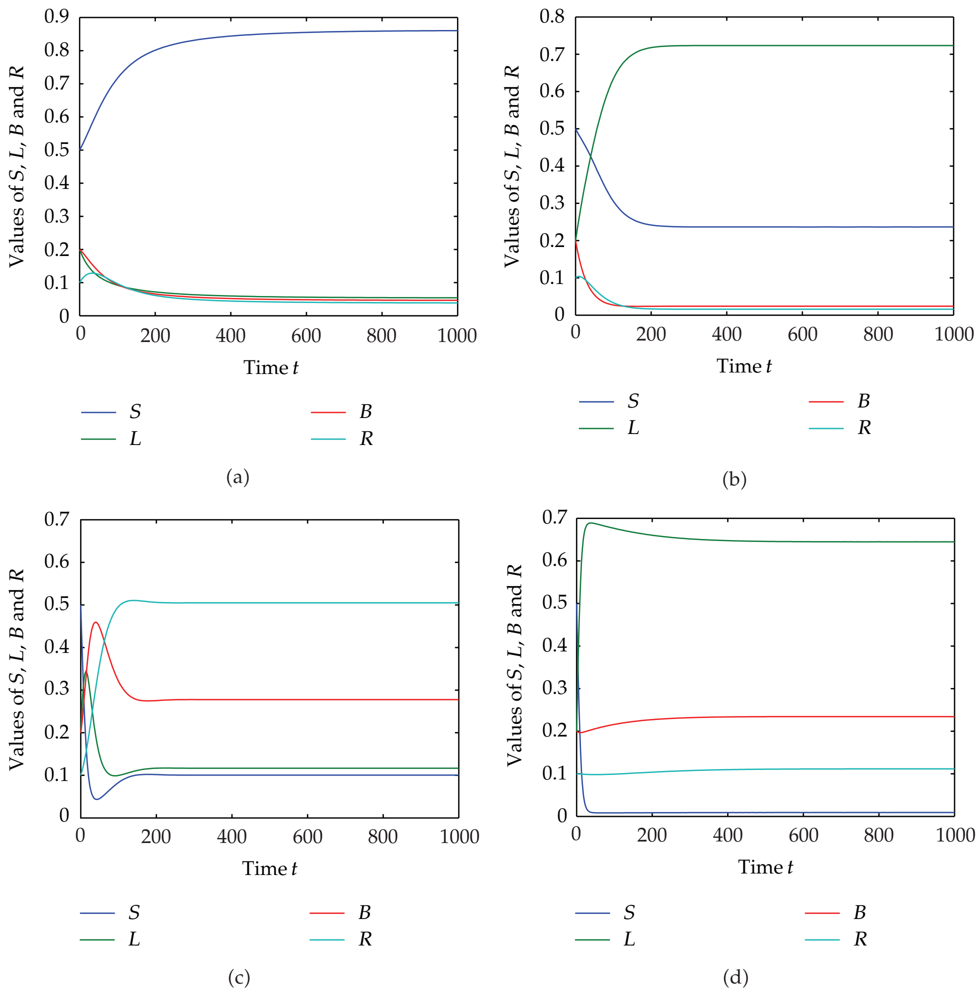


Figure 3: (a) Evolutions of S, L, B, and R in the case of $\mathcal{R}^0 = 1.1607$, (b) evolutions of S, L, B, and R in the case of $\mathcal{R}^0 = 4.2273$, (c) evolutions of S, L, B, and R in the case of $\mathcal{R}^0 = 9.9440$ and (d) evolutions of S, L, B and R in the case of $\mathcal{R}^0 = 109.0909$.

$$
\begin{aligned}
\frac{\partial \mathcal{R}^0/\mathcal{R}^0}{\partial \varepsilon/\varepsilon} &= \frac{\varepsilon}{\mathcal{R}^0}\frac{\partial \mathcal{R}^0}{\partial \varepsilon} = \frac{-\varepsilon\gamma}{(\mu+\gamma+\varepsilon)(\mu+\varepsilon)} < 0, \\
\frac{\partial \mathcal{R}^0/\mathcal{R}^0}{\partial \gamma/\gamma} &= \frac{\gamma}{\mathcal{R}^0}\frac{\partial \mathcal{R}^0}{\partial \gamma} = \frac{-\varepsilon\gamma}{(\mu+\gamma+\varepsilon)(\mu+\gamma)} < 0, \\
\frac{\partial \mathcal{R}^0/\mathcal{R}^0}{\partial \mu/\mu} &= \frac{\mu}{\mathcal{R}^0}\frac{\partial \mathcal{R}^0}{\partial \mu} = \frac{-\mu[\gamma(\mu+\gamma)+(\mu+\gamma+\varepsilon)(\mu+\gamma)]}{(\mu+\varepsilon)(\mu+\gamma(\mu+\gamma+\varepsilon))} < 0.
\end{aligned}
\tag{4.1}
$$

It can be seen that, among these four parameters, $\mathcal{R}^0$, in proportion with β, is the most sensitive to the change in β. As opposed to this, the other three parameters ε, γ, and μ have an inversely proportional relationship with $\mathcal{R}^0$, an increase in ε or γ, or μ will bring about a decrease in $\mathcal{R}^0$, with a proportionally smaller size of decrease.

Below, let us explain how these properties of model (2.2) can be utilized to control the spread of computer virus.

(1) Filtering and blocking suspicious messages with firewall located at the gateway of a domain, the parameter β can be kept low and, hence, the chance that a virus-free computer within the domain is infected by a viral computer outside the domain can be significantly decreased, yielding a lower threshold value $\mathcal{R}^0$.

(2) Timely updating and running antivirus software of the newest version on computers, the breaking out rate of latent computers, ε, and the cure rate of breaking out computers, γ, can be remarkably enhanced, leading to a low $\mathcal{R}^0$.

(3) Timely disconnecting computers from the Internet when the connections are unnecessary, the disconnecting rate of computers, μ, can be made high, bringing about an ideal $\mathcal{R}^0$.

In practice, all of these measures are strongly recommended to achieve a threshold value well below unity, so that viruses within the Internet approach extinction.

5. Conclusion

In nearly all previous computer virus propagation models with latent compartment, to our knowledge, latent computers are assumed not to infect other computers, which does not accord with the real situations. To overcome this defect, the SLBRS model proposed in this paper assumes that all latent computers have infectivity. The dynamics of this model has been fully studied. The results concerning this model include the following. (1) two equilibria, the virus-free equilibrium E_0 and the viral equilibrium E_*, as well as the basic reproduction ratio $\mathcal{R}^0$ are obtained. (2) The dynamical behavior is determined completely by the value of $\mathcal{R}^0$: $\mathcal{R}^0 \leq 1$ implies the global stability of E_0, whereas $\mathcal{R}^0 > 1$ implies the global stability of E_*. (3) By conducting a sensitive analysis of $\mathcal{R}^0$ with respect to various model parameters and on the condition that $\mathcal{R}^0 \ll 1$, a series of measures of strategies is proposed for controlling the spread of virus through the Internet effectively.

Our proof of the global stability of viral equilibrium when $\mathcal{R}^0 > 1$ utilizes a general approach established in [43], and relies on the construction of a new Lyapunov function for the second compound system.

Appendices

A. The General Mathematical Framework of Geometric Approach to Global Stability

The presentation here follows that in [22]. Let $x \mapsto f(x) \in \mathbb{R}^n$ be a C^1 function for x in an open set $D \subset \mathbb{R}^n$. Consider the differential equation:

$$x' = f(x). \tag{A.1}$$

Denote by $x(t, x_0)$ the solution to (A.1) such that $x(0, x_0) = x_0$. We make the following two assumptions.

(H1) There exists a compact absorbing set $K \subset D$.

(H2) Equation (A.1) has a unique equilibrium $\overline{x}$ in D.

The equilibrium $\overline{x}$ is said to be *globally stable* in D if it is locally stable and all trajectories in D *converge to* $\overline{x}$. The following global-stability problem is formulated in [22].

Global-stability problem. Under the assumptions (H1) and (H2), find conditions on the vector field f such that the local stability of $\overline{x}$ implies its global stability in D.

The assumptions (H1) and (H2) are satisfied if $\overline{x}$ is globally stable in D. For $n \geq 2$, a *Bendixson criterion* is a condition satisfied by f which precludes the existence of nonconstant periodic solutions of (A.1). A Bendixson criterion is said to be robust under C^1 local perturbations of f at $x_1 \in D$ if, for sufficiently small $\epsilon > 0$ and neighborhood U of x_1, it is also satisfied by $g \in C^1(D \to \mathbb{R}^n)$ such that the support $(f-g) \subset U$ and $|f-g|_{C^1} < \epsilon$, where

$$|f - g|_{C^1} = \sup\left\{ |f(x) - g(x)| + \left|\frac{\partial f}{\partial x}(x) - \frac{\partial g}{\partial x}(x)\right| : x \in D \right\}. \tag{A.2}$$

Such g will be called local ε-perturbations of f at x_1. It is easy to see that the classical Bendixson's condition $f(x) < 0$ for $n = 2$ is robust under C^1 local perturbations of f at each $x_1 \in R^2$. Bendixson criteria for higher dimensional systems that are C^1 robust are discussed in [21, 22, 44].

A point $x_0 \in D$ is wandering for (A.1) if there exists a neighborhood U of x_0 and $T > 0$ such that $U \cap x(t, U)$ is empty for all $t > T$. Thus, for example, all equilibria and limit points are nonwandering. The following is a version of the local C^1 closing lemma of Pugh [45, 46] as stated in [43].

Lemma A.1. *Let $f \in C^1(D \to \mathbb{R}^n)$. Suppose that x_0 is a nonwandering point of* (A.1) *and that $f(x_0) \neq 0$. Also assume that the positive semiorbit of x_0 has compact closure. Then, for each neighborhood U of x_0 and $\varepsilon > 0$, there exists a C^1 local ε-perturbation g of f at x_0 such that*

(1) sup $p(f - g) \subset U$ *and*

(2) *the perturbed system $x' = g(x)$ has a nonconstant periodic solution whose trajectory passes through x_0.*

The following general global-stability principle is established in [43].

Theorem A.2. *Suppose that assumptions (H1) and (H2) hold. Assume that* (A.1) *satisfies a Bendixson criterion that is robust under C^1 local perturbations of f at all nonequilibrium nonwandering points for* (A.1)*. Then, $\overline{x}$ is globally stable in D provided it is stable.*

The main idea of the proof in [43] for Theorem A.2 is as follows. Suppose that system (A.1) satisfies a Bendixson criterion. Then it does not have any nonconstant periodic solutions. Moreover, the robustness assumption on the Bendixson criterion implies that all nearby differential equations have no nonconstant periodic solutions. Thus, by Lemma A.1, all nonwandering points of (A.1) in D must be equilibria. In particular, each omega limit point in D must be an equilibrium. Therefore, $\omega(x_0) = \{\overline{x}\}$ for all $x_0 \in D$ since $\overline{x}$ is the only equilibrium in D.

A method of deriving a Bendixson criterion in $\mathbb{R}^n$ is developed in [35]. The idea is to show that the second compound equation,

$$z'(t) = \frac{\partial f^{[2]}}{\partial x}(x(t,x_0))z(t), \tag{A.3}$$

with respect to a solution $x(t,x_0) \subset D$ to (A.1), is uniformly asymptotically stable, and the exponential decay rate of all solutions to (A.3) is uniform for x_0 in each compact subset of D. Here, $\partial f^{[2]}/\partial x$ is the second additive compound matrix of the Jacobian matrix $\partial f/\partial x$; see Appendix B. It is an $\binom{n}{2} \times \binom{n}{2}$ matrix, and thus (A.3) is a linear system of dimension $\binom{n}{2}$. If D is simply connected, the above-mentioned stability property of (A.3) implies the exponential decay of the surface area of any compact two-dimensional surface in D, which in turn precludes the existence of any invariant simple closed rectifiable curve in D, including periodic orbits. The required uniform asymptotic stability of the linear system (A.3) can be proved by constructing a suitable Lyapunov function.

Let $x \mapsto p(x)$ be an $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function that is C^1 for $x \in D$. Assume that $P^{-1}(x)$ exists and is continuous for $x \in K$, the compact absorbing set.

Set

$$B = P_f P^{-1} + P\frac{\partial f^{[2]}}{\partial x}P^{-1}, \tag{A.4}$$

where the matrix P_f is obtained by replacing each entry p_{ij} of P by its derivative in the direction of f, p_{ij_f}. Let Z be a suitable vector norm for $z \in \mathbb{R}^n$, $N = \binom{n}{2}$, and let $\mu(B)$ be the *Lozinskii* measure of B with respect to the induced matrix norm $|\cdot|$ in $\mathbb{R}^n$, defined by

$$\mu(B) = \lim_{h\to 0^+}\frac{|I + hB| - 1}{h}. \tag{A.5}$$

Define a quantity $\overline{q}_2$ as

$$\overline{q}_2 = \limsup_{t\to\infty}\sup_{x_0\in\Omega}\frac{1}{t}\int_0^t \mu(B(x(s,x_0)))ds. \tag{A.6}$$

It is shown in [22] that if D is simply connected and $\overline{q}_2$, the function $|P(x)z|$ is a Lyapunov function for (A.3), and hence (A.1) has no orbit that gives rise to an invariant simple closed rectifiable curve, such as periodic orbits, homoclinic orbits, and heteroclinic cycles. Hence, $\overline{q}_2 < 0$ is a Bendixson criterion for (A.1) in D. Moreover, it is robust under C^1 local perturbations of f near any nonequilibrium point that is nonwandering. In particular, the following global-stability result is proved in Theorem 3.5 of [22].

Theorem A.3. *Assume that D is simply connected and that the assumptions (H1), (H2) hold. Then, the unique equilibrium $\overline{x}$ of* (A.1) *is globally stable in D if $\overline{q}_2 < 0$.*

We remark that, under the assumptions of Theorem A.3, the condition $\overline{q}_2 < 0$ also implies the local stability of $\overline{x}$, since, assuming the contrary, $\overline{x}$ is both the alpha and the omega limit point of a homoclinic orbit that is ruled out by the condition $\overline{q}_2 < 0$.

B. The Second Additive Compound Matrix

Let A be a linear operator on $\mathbb{R}^n$ and also denote its matrix representation with respect to the standard basis of R_N. Let $\wedge^2\mathbb{R}^n$ denote the exterior product of $\mathbb{R}^n$. A induces canonically a linear operator $A^{[2]}$ on $\wedge^2\mathbb{R}^n$: for $\mu_1, \mu_2 \in \mathbb{R}^n$, define

$$A^{[2]}(\mu_1 \wedge \mu_2) := A(\mu_1) \wedge \mu_2 + \mu_1 \wedge A(\mu_2), \tag{B.1}$$

and extend the definition over $\wedge^2\mathbb{R}^n$ by linearity. The matrix representation of $A^{[2]}$ with respect to the canonical basis in $\wedge^2\mathbb{R}^n$ is called *the second additive compound* matrix of A. This is an $\binom{n}{2} \times \binom{n}{2}$ matrix and satisfies the property $(A+B)^{[2]} = A^{[2]} + B^{[2]}$. In the special case, when $n = 2$, we have $A^{[2]}_{2\times 2} = \operatorname{tr} A$. In general, each entry of $A^{[2]}$ is a linear expression of those of A. For instance, when $n = 3$, the second additive compound matrix of $A = (a_{ij})$ is

$$A^{[2]} = \begin{bmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{bmatrix}. \tag{B.2}$$

For detailed discussions of compound matrices and their properties, we refer the reader to [39, 40]. A comprehensive survey on compound matrices and their relations to differential equations is given in [40].

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