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Research Article

Some Formulae of Products of the Apostol-Bernoulli and Apostol-Euler Polynomials

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Some formulae of products of the Apostol-Bernoulli and Apostol-Euler polynomials are established by applying the generating function methods and some summation transform techniques, and various known results are derived as special cases.

1. Introduction

The classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are usually defined by means of the following generating functions:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad(|t|<2\pi),\qquad\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad(|t|<\pi).\tag{1.1}$$

In particular, $B_n=B_n(0)$ and $E_n=2^nE_n(1/2)$ are called the classical Bernoulli numbers and Euler numbers, respectively. These numbers and polynomials play important roles in many branches of mathematics such as combinatorics, number theory, special functions, and analysis. Numerous interesting identities and congruences for them can be found in many papers; see, for example, [1–4].

Some analogues of the classical Bernoulli and Euler polynomials are the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ and Apostol-Euler polynomials $\mathcal{E}_n(x;\mu)$. They were respectively introduced by Apostol [5] (see also Srivastava [6] for a systematic study) and Luo [7, 8] as follows:

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\frac{t^n}{n!} \quad \left(|t| < 2\pi \text{ if } \lambda = 1; |t| < \left|\log \lambda\right| \text{ otherwise}\right), \tag{1.2}$$

$$\frac{2e^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x;\lambda)\frac{t^n}{n!} \quad \left(|t| < \pi \text{ if } \lambda = 1; |t| < \left|\log(-\lambda)\right| \text{ otherwise}\right). \tag{1.3}$$

Moreover, $\mathcal{B}_n(\lambda) = \mathcal{B}_n(0;\lambda)$ and $\mathcal{E}_n(\lambda) = 2^n\mathcal{E}_n(1/2;\lambda)$ are called the Apostol-Bernoulli numbers and Apostol-Euler numbers, respectively. Obviously $\mathcal{B}_n(x;\lambda)$ and $\mathcal{E}_n(x;\lambda)$ reduce to $B_n(x)$ and $E_n(x)$ when $\lambda = 1$. Some arithmetic properties for the Apostol-Bernoulli and Apostol-Euler polynomials and numbers have been well investigated by many authors. For example, in 1998, Srivastava and Todorov [9] gave the close formula for the Apostol-Bernoulli polynomials in terms of the Gaussian hypergeometric function and the Stirling numbers of the second kind. Following the work of Srivastava and Todorov, Luo [7] presented the close formula for the Apostol-Euler polynomials in a similar technique. After that, Luo [10] obtained some multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials. Further, Luo [11] showed the Fourier expansions for the Apostol-Bernoulli and Apostol-Euler polynomials by applying the Lipschitz summation formula and derived some explicit formulae at rational arguments for these polynomials in terms of the Hurwitz zeta function.

In the present paper, we will further investigate the arithmetic properties of the Apostol-Bernoulli and Apostol-Euler polynomials and establish some formulae of products of the Apostol-Bernoulli and Apostol-Euler polynomials by using the generating function methods and some summation transform techniques. It turns out that various known results are deduced as special cases.

2. The Restatement of the Results

For convenience, in this section we always denote by $\delta_{1,\lambda}$ the Kronecker symbol given by $\delta_{1,\lambda} = 0$ or 1 according to $\lambda \neq 1$ or $\lambda = 1$, and we also denote by $\max(a, b)$ the maximum number of the real numbers a, b and by $[x]$ the maximum integer less than or equal to the real number x. We now give the formula of products of the Apostol-Bernoulli polynomials in the following way.

Theorem 2.1. *Let m and n be any positive integers. Then,*

$$\begin{aligned}\mathfrak{B}_m(x;\lambda)\mathcal{B}_n(y;\mu) = {} & n\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{B}_{m-k}\left(y-x;\frac{1}{\lambda}\right)\frac{\mathcal{B}_{n+k}(y;\lambda\mu)}{n+k} \\ & + m\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}(y-x;\mu)\frac{\mathcal{B}_{m+k}(x;\lambda\mu)}{m+k} \\ & + (-1)^{m+1}\delta_{1,\lambda\mu}\frac{m!n!}{(m+n)!}\mathcal{B}_{m+n}\left(y-x;\frac{1}{\lambda}\right).\end{aligned} \tag{2.1}$$

Proof. Multiplying both sides of the identity

$$\frac{1}{\lambda e^{u}-1}\cdot\frac{1}{\mu e^{v}-1}=\left(\frac{\lambda e^{u}}{\lambda e^{u}-1}+\frac{1}{\mu e^{v}-1}\right)\frac{1}{\lambda\mu e^{u+v}-1} \tag{2.2}$$

by uve^{xu+yv}, we obtain

$$\frac{ue^{xu}}{\lambda e^{u}-1}\cdot\frac{ve^{yv}}{\mu e^{v}-1}=\lambda v\frac{ue^{(1+x-y)u}}{\lambda e^{u}-1}\cdot\frac{e^{y(u+v)}}{\lambda\mu e^{u+v}-1}+u\frac{ve^{(y-x)v}}{\mu e^{v}-1}\cdot\frac{e^{x(u+v)}}{\lambda\mu e^{u+v}-1}. \tag{2.3}$$

It follows from (2.3) that

$$\begin{aligned}
&\delta_{1,\lambda\mu}\frac{uv}{u+v}\left(\lambda\frac{e^{(1+x-y)u}}{\lambda e^{u}-1}+\frac{e^{(y-x)v}}{\mu e^{v}-1}\right)\\
&\quad=\frac{ue^{xu}}{\lambda e^{u}-1}\cdot\frac{ve^{yv}}{\mu e^{v}-1}-\lambda v\frac{ue^{(1+x-y)u}}{\lambda e^{u}-1}\left(\frac{e^{y(u+v)}}{\lambda\mu e^{u+v}-1}-\frac{\delta_{1,\lambda\mu}}{u+v}\right)\\
&\qquad-u\frac{ve^{(y-x)v}}{\mu e^{v}-1}\left(\frac{e^{x(u+v)}}{\lambda\mu e^{u+v}-1}-\frac{\delta_{1,\lambda\mu}}{u+v}\right).
\end{aligned} \tag{2.4}$$

By the Taylor theorem we have

$$\frac{e^{x(u+v)}}{\lambda\mu e^{u+v}-1}-\frac{\delta_{1,\lambda\mu}}{u+v}=\sum_{n=0}^{\infty}\frac{\partial^{n}}{\partial u^{n}}\left(\frac{e^{xu}}{\lambda\mu e^{u}-1}-\frac{\delta_{1,\lambda\mu}}{u}\right)\frac{v^{n}}{n!}. \tag{2.5}$$

Since $\mathcal{B}_0(x;\lambda)=1$ when $\lambda=1$ and $\mathcal{B}_0(x;\lambda)=0$ when $\lambda\neq 1$ (see e.g., [8]), by (1.2) and (2.5) we get

$$\frac{e^{x(u+v)}}{\lambda\mu e^{u+v}-1}-\frac{\delta_{1,\lambda\mu}}{u+v}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{n+m+1}(x;\lambda\mu)}{n+m+1}\cdot\frac{u^{m}}{m!}\cdot\frac{v^{n}}{n!}. \tag{2.6}$$

Putting (1.2) and (2.6) in (2.4), with the help of the Cauchy product, we derive

$$\begin{aligned}
&\delta_{1,\lambda\mu}\frac{uv}{u+v}\left(\lambda\frac{e^{(1+x-y)u}}{\lambda e^{u}-1}+\frac{e^{(y-x)v}}{\mu e^{v}-1}\right)\\
&\quad=-\lambda\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{m}\binom{m}{k}\mathcal{B}_{m-k}(1+x-y;\lambda)\frac{\mathcal{B}_{n+k+1}(y;\lambda\mu)}{n+k+1}\right]\frac{u^{m}}{m!}\cdot\frac{v^{n+1}}{n!}\\
&\qquad-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}(y-x;\mu)\frac{\mathcal{B}_{m+k+1}(x;\lambda\mu)}{m+k+1}\right]\frac{u^{m+1}}{m!}\cdot\frac{v^{n}}{n!}\\
&\qquad+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{B}_{m}(x;\lambda)\mathcal{B}_{n}(y;\mu)\frac{u^{m}}{m!}\cdot\frac{v^{n}}{n!}.
\end{aligned} \tag{2.7}$$

If we denote the left-hand side of (2.7) by M_1 and

$$M_2 = \lambda\delta_{1,\lambda}\left(\frac{ve^{yv}}{\lambda\mu e^{v}-1}-\delta_{1,\lambda\mu}\right)+\delta_{1,\mu}\left(\frac{ue^{xu}}{\lambda\mu e^{u}-1}-\delta_{1,\lambda\mu}\right) - \delta_{1,\lambda}\frac{ve^{yv}}{\mu e^{v}-1}-\delta_{1,\mu}\left(\frac{ue^{xu}}{\lambda e^{u}-1}-\delta_{1,\lambda}\right), \tag{2.8}$$

then applying (1.2) to (2.8), in light of (2.7), we have

$$\begin{aligned} M_1+M_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Bigg[&\frac{-\lambda}{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k}\mathcal{B}_{m+1-k}(1+x-y;\lambda)\frac{\mathcal{B}_{n+k+1}(y;\lambda\mu)}{n+k+1} \\ &-\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\mathcal{B}_{n+1-k}(y-x;\mu)\frac{\mathcal{B}_{m+k+1}(x;\lambda\mu)}{m+k+1} \\ &+\frac{\mathcal{B}_{m+1}(x;\lambda)}{m+1}\cdot\frac{\mathcal{B}_{n+1}(y;\mu)}{n+1}\Bigg]\frac{u^{m+1}}{m!}\cdot\frac{v^{n+1}}{n!}. \end{aligned} \tag{2.9}$$

On the other hand, a simple calculation implies $M_1 = M_2 = 0$ when $\lambda\mu \neq 1$ and

$$M_1+M_2 = \delta_{1,\lambda\mu}\frac{uv}{u+v}\left(\lambda\frac{e^{(1+x-y)u}}{\lambda e^{u}-1}-\frac{\delta_{1,\lambda}}{u}+\frac{e^{(y-x)v}}{(1/\lambda)e^{v}-1}-\frac{\delta_{1,1/\lambda}}{v}\right) \tag{2.10}$$

when $\lambda\mu = 1$. Applying $u^n = \sum_{k=0}^{n}\binom{n}{k}(u+v)^k(-v)^{n-k}$ to (1.2), in view of changing the order of the summation, we obtain

$$\begin{aligned} \lambda\frac{e^{(1+x-y)u}}{\lambda e^{u}-1}-\frac{\delta_{1,\lambda}}{u} &= \lambda\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{\mathcal{B}_{n+1}(1+x-y;\lambda)}{(n+1)!}\binom{n}{k}(u+v)^k(-v)^{n-k} \\ &= \lambda\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\frac{\mathcal{B}_{n+1}(1+x-y;\lambda)}{(n+1)!}\binom{n}{k+1}(u+v)^{k+1}(-v)^{n-(k+1)} \\ &\quad+\lambda\sum_{n=0}^{\infty}\frac{\mathcal{B}_{n+1}(1+x-y;\lambda)}{n+1}\cdot\frac{(-v)^n}{n!}. \end{aligned} \tag{2.11}$$

It follows from (1.2), (2.10), (2.11), and the symmetric relation for the Apostol-Bernoulli polynomials $\lambda\mathcal{B}_n(1-x;\lambda) = (-1)^n\mathcal{B}_n(x;1/\lambda)$ for any nonnegative integer n (see e.g., [8]) that

$$\begin{aligned}
M_1 + M_2 &= uv\delta_{1,\lambda\mu}\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}(-1)^k\frac{\mathcal{B}_{n+1}(y-x;1/\lambda)}{(n+1)!}\binom{n}{k+1}(u+v)^k v^{n-(k+1)} \\
&= \delta_{1,\lambda\mu}\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}(-1)^k\frac{\mathcal{B}_{n+1}(y-x;1/\lambda)}{(n+1)!}\binom{n}{k+1}\sum_{m=0}^{k}\binom{k}{m}u^{m+1}v^{n-m} \\
&= \delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\sum_{n=k+1}^{\infty}(-1)^k\frac{\mathcal{B}_{n+1}(y-x;1/\lambda)}{(n+1)!}\binom{n}{k+1}\binom{k}{m}u^{m+1}v^{n-m} \\
&= \delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty}(-1)^m\frac{\mathcal{B}_{n+1}(y-x;1/\lambda)}{(n+1)!}u^{m+1}v^{n-m} \\
&= \delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m\frac{m!n!\mathcal{B}_{n+m+2}(y-x;1/\lambda)}{(n+m+2)!}\cdot\frac{u^{m+1}}{m!}\cdot\frac{v^{n+1}}{n!}.
\end{aligned} \tag{2.12}$$

Thus, by equating (2.9) and (2.12) and then comparing the coefficients of $u^{m+1}v^{n+1}$, we complete the proof of Theorem 2.1 after applying the symmetric relation for the Apostol-Bernoulli polynomials. □

It follows that we show some special cases of Theorem 2.1. By setting $x = y$ in Theorem 2.1, we have the following.

Corollary 2.2. *Let m and n be any positive integers. Then,*

$$\begin{aligned}
\mathcal{B}_m(x;\lambda)\mathcal{B}_n(x;\mu) = {}& n\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{B}_{m-k}\left(\frac{1}{\lambda}\right)\frac{\mathcal{B}_{n+k}(x;\lambda\mu)}{n+k} \\
&+ m\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}(\mu)\frac{\mathcal{B}_{m+k}(x;\lambda\mu)}{m+k} \\
&+ (-1)^{m+1}\delta_{1,\lambda\mu}\frac{m!n!}{(m+n)!}\mathcal{B}_{m+n}\left(\frac{1}{\lambda}\right).
\end{aligned} \tag{2.13}$$

It is well known that the classical Bernoulli numbers with odd subscripts obey $\mathcal{B}_1 = -1/2$ and $\mathcal{B}_{2n+1} = 0$ for any positive integer n (see, e.g., [12]). Setting $\lambda = \mu = 1$ in Corollary 2.2, we immediately obtain the familiar formula of products of the classical Bernoulli polynomials due to Carlitz [13] and Nielsen [14] as follows.

Corollary 2.3. *Let m and n be any positive integers. Then,*

$$
\begin{aligned}
B_m(x)B_n(x) = \sum_{k=0}^{\max([m/2],[n/2])} & \left\{ n\binom{m}{2k} + m\binom{n}{2k} \right\} B_{2k} \frac{B_{m+n-2k}(x)}{m+n-2k} \\
& + (-1)^{m+1} \frac{m!n!}{(m+n)!} B_{m+n}.
\end{aligned}
\tag{2.14}
$$

Since the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ satisfy the difference equation $(\partial/\partial x)\mathcal{B}_n(x;\lambda) = n\mathcal{B}_{n-1}(x;\lambda)$ for any positive integer n (see, e.g., [8]), by substituting $x+y$ for x in Theorem 2.1 and then taking differences with respect to y, we get the following result after replacing x by $x-y$.

Corollary 2.4. *Let m and n be any positive integers. Then,*

$$
\begin{aligned}
& \frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{B}_{m-k}\left(y-x;\frac{1}{\lambda}\right)\mathcal{B}_{n-1+k}(y;\lambda\mu) - \frac{1}{m}\mathcal{B}_m(x;\lambda)\mathcal{B}_{n-1}(y;\mu) \\
& \quad = -\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}(y-x;\mu)\mathcal{B}_{m-1+k}(x;\lambda\mu) + \frac{1}{n}\mathcal{B}_n(y;\mu)\mathcal{B}_{m-1}(x;\lambda).
\end{aligned}
\tag{2.15}
$$

Setting $x=t$ and $y=1-t$ in Corollary 2.4, by $\lambda\mathcal{B}_n(1-x;\lambda) = (-1)^n\mathcal{B}_n(x;1/\lambda)$ for any nonnegative integer n, we get the following.

Corollary 2.5. *Let m and n be any positive integers. Then,*

$$
\begin{aligned}
& \frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{k}\mathcal{B}_{m-k}(2t;\lambda)\mathcal{B}_{n-1+k}\left(t;\frac{1}{\lambda\mu}\right) - \frac{1}{m}\mathcal{B}_m(t;\lambda)\mathcal{B}_{n-1}\left(t;\frac{1}{\mu}\right) \\
& \quad = \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\mathcal{B}_{n-k}\left(2t;\frac{1}{\mu}\right)\mathcal{B}_{m-1+k}(t;\lambda\mu) - \frac{1}{n}\mathcal{B}\left(t;\frac{1}{\mu}\right)\mathcal{B}_{m-1}(t;\lambda).
\end{aligned}
\tag{2.16}
$$

In particular, the case $\lambda = \mu = 1$ in Corollary 2.5 gives the following generalization for Woodcock's identity on the classical Bernoulli numbers, see [15, 16],

Corollary 2.6. *Let m and n be any positive integers. Then,*

$$
\begin{aligned}
& \frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{k}B_{m-k}(2t)B_{n-1+k}(t) - \frac{1}{m}B_m(t)B_{n-1}(t) \\
& \quad = \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}B_{n-k}(2t)B_{m-1+k}(t) - \frac{1}{n}B_n(t)B_{m-1}(t).
\end{aligned}
\tag{2.17}
$$

We next present some mixed formulae of products of the Apostol-Bernoulli and Apostol-Euler polynomials and numbers.

Theorem 2.7. *Let m and n be non-negative integers. Then,*

$$\begin{aligned}\mathcal{E}_m(x;\lambda)\mathcal{E}_n(y;\mu) &= 2\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{E}_{m-k}\left(y-x;\frac{1}{\lambda}\right)\frac{\mathcal{B}_{n+k+1}(y;\lambda\mu)}{n+k+1}\\ &\quad-2\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{n-k}(y-x;\mu)\frac{\mathcal{B}_{m+k+1}(x;\lambda\mu)}{m+k+1}\\ &\quad+(-1)^{m+1}2\delta_{1,\lambda\mu}\frac{m!n!}{(m+n+1)!}\mathcal{E}_{m+n+1}\left(y-x;\frac{1}{\lambda}\right).\end{aligned} \tag{2.18}$$

Proof. Multiplying both sides of the identity

$$\frac{1}{\lambda e^u+1}\cdot\frac{1}{\mu e^v+1}=\left(\frac{\lambda e^u}{\lambda e^u+1}-\frac{1}{\mu e^v+1}\right)\frac{1}{\lambda\mu e^{u+v}-1} \tag{2.19}$$

by $2e^{xu+yv}$, we obtain

$$\frac{1}{2}\cdot\frac{2e^{xu}}{\lambda e^u+1}\cdot\frac{2e^{yv}}{\mu e^v+1}=\lambda\frac{2e^{(1+x-y)u}}{\lambda e^u+1}\cdot\frac{e^{y(u+v)}}{\lambda\mu e^{u+v}-1}-\frac{2e^{(y-x)v}}{\mu e^v+1}\cdot\frac{e^{x(u+v)}}{\lambda\mu e^{u+v}-1}. \tag{2.20}$$

It follows from (2.20) that

$$\begin{aligned}&\frac{\delta_{1,\lambda\mu}}{u+v}\left(\lambda\frac{2e^{(1+x-y)u}}{\lambda e^u+1}-\frac{2e^{(y-x)v}}{\mu e^v+1}\right)\\ &\quad=\frac{1}{2}\cdot\frac{2e^{xu}}{\lambda e^u+1}\cdot\frac{2e^{yv}}{\mu e^v+1}-\lambda\frac{2e^{(1+x-y)u}}{\lambda e^u+1}\left(\frac{e^{y(u+v)}}{\lambda\mu e^{u+v}-1}-\frac{\delta_{1,\lambda\mu}}{u+v}\right)\\ &\quad\quad+\frac{2e^{(y-x)v}}{\mu e^v+1}\left(\frac{e^{x(u+v)}}{\lambda\mu e^{u+v}-1}-\frac{\delta_{1,\lambda\mu}}{u+v}\right).\end{aligned} \tag{2.21}$$

Applying (1.3) and (2.6) to (2.21), in view of the Cauchy product, we get

$$\begin{aligned}&\frac{\delta_{1,\lambda\mu}}{u+v}\left(\lambda\frac{2e^{(1+x-y)u}}{\lambda e^u+1}-\frac{2e^{(y-x)v}}{\mu e^v+1}\right)\\ &\quad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{1}{2}\mathcal{E}_m(x;\lambda)\mathcal{E}_n(y;\mu)-\lambda\sum_{k=0}^{m}\binom{m}{k}\mathcal{E}_{m-k}(1+x-y;\lambda)\frac{\mathcal{B}_{n+k+1}(y;\lambda\mu)}{n+k+1}\right.\\ &\quad\quad\left.+\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{n-k}(y-x;\mu)\frac{\mathcal{B}_{m+k+1}(x;\lambda\mu)}{m+k+1}\right]\frac{u^m}{m!}\cdot\frac{v^n}{n!}.\end{aligned} \tag{2.22}$$

On the other hand, since the left-hand side of (2.22) vanishes when $\lambda\mu \neq 1$, it suffices to consider the case $\lambda\mu = 1$. Applying $u^n = \sum_{k=0}^{n}\binom{n}{k}(u+v)^k(-v)^{n-k}$ to (1.3), in view of changing the order of the summation, we have

$$\begin{aligned}
\lambda\frac{2e^{(1+x-y)u}}{\lambda e^u+1} &= \lambda\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{\mathcal{E}_n(1+x-y;\lambda)}{n!}\binom{n}{k}(u+v)^k(-v)^{n-k} \\
&= \lambda\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\frac{\mathcal{E}_n(1+x-y;\lambda)}{n!}\binom{n}{k+1}(u+v)^{k+1}(-v)^{n-(k+1)} \\
&\quad + \lambda\sum_{n=0}^{\infty}\mathcal{E}_n(1+x-y;\lambda)\frac{(-v)^n}{n!}.
\end{aligned} \tag{2.23}$$

It follows from (1.3), (2.23), and the symmetric relation for the Apostol-Euler polynomials $\lambda\mathcal{E}_n(1-x;\lambda) = (-1)^n\mathcal{E}_n(x;1/\lambda)$ for any non-negative integer n (see, e.g., [7]) that

$$\begin{aligned}
&\frac{\delta_{1,\lambda\mu}}{u+v}\left(\lambda\frac{2e^{(1+x-y)u}}{\lambda e^u+1} - \frac{2e^{(y-x)v}}{\mu e^v+1}\right) \\
&\quad = \delta_{1,\lambda\mu}\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}(-1)^{k+1}\frac{\mathcal{E}_n(y-x;1/\lambda)}{n!}\binom{n}{k+1}(u+v)^k v^{n-(k+1)} \\
&\quad = \delta_{1,\lambda\mu}\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}(-1)^{k+1}\frac{\mathcal{E}_n(y-x;1/\lambda)}{n!}\binom{n}{k+1}\sum_{m=0}^{k}\binom{k}{m}u^m v^{n-(m+1)} \\
&\quad = \delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\sum_{n=k+1}^{\infty}(-1)^{k+1}\frac{\mathcal{E}_n(y-x;1/\lambda)}{n!}\binom{n}{k+1}\binom{k}{m}u^m v^{n-(m+1)} \\
&\quad = \delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty}(-1)^{m+1}\frac{\mathcal{E}_n(y-x;1/\lambda)}{n!}u^m v^{n-(m+1)} \\
&\quad = \delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+1}\frac{m!n!\mathcal{E}_{m+n+1}(y-x;1/\lambda)}{(m+n+1)!}\cdot\frac{u^m}{m!}\cdot\frac{v^n}{n!}.
\end{aligned} \tag{2.24}$$

Thus, by equating (2.22) and (2.24) and then comparing the coefficients of $u^m v^n$, we complete the proof of Theorem 2.7 after applying the symmetric relation for the Apostol-Euler polynomials. □

Next, we give some special cases of Theorem 2.7. By setting $x = y$ in Theorem 2.7, we have the following.

Corollary 2.8. *Let m and n be non-negative integers. Then,*

$$\begin{aligned}\mathcal{E}_m(x;\lambda)\mathcal{E}_n(x;\mu) &= 2\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{E}_{m-k}\left(0;\frac{1}{\lambda}\right)\frac{\mathcal{B}_{n+k+1}(x;\lambda\mu)}{n+k+1}\\ &\quad -2\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{n-k}(0;\mu)\frac{\mathcal{B}_{m+k+1}(x;\lambda\mu)}{m+k+1}\\ &\quad +(-1)^{m+1}2\delta_{1,\lambda\mu}\frac{m!n!}{(m+n+1)!}\mathcal{E}_{m+n+1}\left(0;\frac{1}{\lambda}\right).\end{aligned} \tag{2.25}$$

Since the classical Euler polynomials $E_n(x)$ at zero arguments satisfy $E_0(0) = 1$, $E_{2n}(0) = 0$, and $E_{2n-1}(0) = (1-2^{2n})B_{2n}/n$ for any positive integer n (see, e.g., [12]), by setting $\lambda = \mu = 1$ in Corollary 2.8, we obtain the following.

Corollary 2.9. *Let m and n be non-negative integers. Then,*

$$\begin{aligned}E_m(x)E_n(x) = &-2\sum_{k=1}^{\max([(m+1)/2],[(n+1)/2])}\left\{\binom{m}{2k-1}+\binom{n}{2k-1}\right\}\frac{(1-2^{2k})B_{2k}}{k}\\ &\times\frac{B_{m+n+2-2k}(x)}{m+n+2-2k}+(-1)^{m+1}\frac{2m!n!}{(m+n+1)!}K_{m,n},\end{aligned} \tag{2.26}$$

where $K_{m,n} = 2(1-2^{m+n+2})B_{m+n+2}/(m+n+2)$ *when* $m+n \equiv 0 \pmod 2$ *and* $K_{m,n} = 0$ *otherwise.*

Theorem 2.10. *Let m be non-negative integer and n positive integer. Then,*

$$\begin{aligned}\mathcal{E}_m(x;\lambda)\mathcal{B}_n(y;\mu) &= \frac{n}{2}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{E}_{m-k}\left(y-x;\frac{1}{\lambda}\right)\mathcal{E}_{n+k-1}(y;\lambda\mu)\\ &\quad +\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}(y-x;\mu)\mathcal{E}_{m+k}(x;\lambda\mu).\end{aligned} \tag{2.27}$$

Proof. Multiplying both sides of the identity

$$\frac{1}{\lambda e^u+1}\cdot\frac{1}{\mu e^v-1} = \left(\frac{\lambda e^u}{\lambda e^u+1}+\frac{1}{\mu e^v-1}\right)\frac{1}{\lambda\mu e^{u+v}+1} \tag{2.28}$$

by $2ve^{xu+yv}$, we obtain

$$\frac{2e^{xu}}{\lambda e^u+1}\cdot\frac{ve^{yv}}{\mu e^v-1} = \frac{\lambda v}{2}\cdot\frac{2e^{(1+x-y)u}}{\lambda e^u+1}\cdot\frac{2e^{y(u+v)}}{\lambda\mu e^{u+v}+1}+\frac{ve^{(y-x)v}}{\mu e^v-1}\cdot\frac{2e^{x(u+v)}}{\lambda\mu e^{u+v}+1}. \tag{2.29}$$

By (1.3) and the Taylor theorem, we have

$$\frac{2e^{x(u+v)}}{\lambda\mu e^{u+v}+1}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{E}_{n+m}(x;\lambda\mu)\frac{u^m}{m!}\cdot\frac{v^n}{n!}. \tag{2.30}$$

Applying (1.2), (1.3), and (2.30) to (2.29), we get

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{E}_m(x;\lambda)\mathcal{B}_n(y;\mu)\frac{u^m}{m!}\cdot\frac{v^n}{n!}\\
&\quad=\frac{\lambda}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{m}\binom{m}{k}\mathcal{E}_{m-k}(1+x-y;\lambda)\mathcal{E}_{n+k}(y;\lambda\mu)\right]\frac{u^m}{m!}\cdot\frac{v^{n+1}}{n!}\\
&\quad\quad+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}(y-x;\mu)\mathcal{E}_{m+k}(x;\lambda\mu)\right]\frac{u^m}{m!}\cdot\frac{v^n}{n!},
\end{aligned} \tag{2.31}$$

which means

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{E}_m(x;\lambda)\mathcal{B}_{n+1}(y;\mu)}{n+1}\cdot\frac{u^m}{m!}\cdot\frac{v^{n+1}}{n!}\\
&\quad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{\lambda}{2}\sum_{k=0}^{m}\binom{m}{k}\mathcal{E}_{m-k}(1+x-y;\lambda)\mathcal{E}_{n+k}(y;\lambda\mu)\right.\\
&\quad\quad\left.+\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\mathcal{B}_{n+1-k}(y-x;\mu)\mathcal{E}_{m+k}(x;\lambda\mu)\right]\frac{u^m}{m!}\cdot\frac{v^{n+1}}{n!}.
\end{aligned} \tag{2.32}$$

Thus, by comparing the coefficients of $u^m v^{n+1}$ in (2.32), we conclude the proof of Theorem 2.10 after applying the symmetric relation for the Apostol-Euler polynomials. □

Obviously, by setting $x=y$ in Theorem 2.10, we have the following.

Corollary 2.11. *Let m be non-negative integer and n positive integer. Then,*

$$\begin{aligned}
\mathcal{E}_m(x;\lambda)\mathcal{B}_n(x;\mu)&=\frac{n}{2}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{E}_{m-k}\left(0;\frac{1}{\lambda}\right)\mathcal{E}_{n+k-1}(x;\lambda\mu)\\
&\quad+\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}(\mu)\mathcal{E}_{m+k}(x;\lambda\mu).
\end{aligned} \tag{2.33}$$

Since $B_1=-1/2$, $E_0(0)=1$, $E_{2n}(0)=0$, and $E_{2n-1}(0)=(1-2^{2n})B_{2n}/n$ for any positive integer n, by setting $\lambda=\mu=1$ in Corollary 2.11, we obtain the following.

Corollary 2.12. *Let m be non-negative integer and n positive integer. Then,*

$$E_m(x)B_n(x) = \sum_{k=1}^{\max([(m+1)/2],[n/2])} \left\{ n\frac{(2^{2k}-1)}{2k}\binom{m}{2k-1} + \binom{n}{2k} \right\} B_{2k}E_{m+n-2k}(x) + E_{m+n}(x). \tag{2.34}$$

Remark 2.13. For the equivalent forms of Corollaries 2.9 and 2.12, the interested readers may consult [14].

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