Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 937670, 12 pages
doi:10.1155/2012/937670

Research Article

Positive Solutions for a Class of Third-Order Three-Point Boundary Value Problem

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Received 25 November 2011; Accepted 8 February 2012

Academic Editor: Yong Zhou

We investigate the problem of existence of positive solutions for the nonlinear third-order three-point boundary value problem $u'''(t) + \lambda a(t) f(u(t)) = 0$, $0 < t < 1$, $u(0) = u'(0) = 0$, $u''(1) = \alpha u''(\eta)$, where λ is a positive parameter, $\alpha \in (0,1)$, $\eta \in (0,1)$, $f : (0,\infty) \to (0,\infty)$, $a : (0,1) \to (0,\infty)$ are continuous. Using a specially constructed cone, the fixed point index theorems and Leray-Schauder degree, this work shows the existence and multiplicities of positive solutions for the nonlinear third-order boundary value problem. Some examples are given to demonstrate the main results.

1. Introduction

This paper deals with the following third-order nonlinear boundary value problem:

$$\begin{gathered} u'''(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad u''(1) = \alpha u''(\eta). \end{gathered} \tag{1.1}$$

Third-order boundary value problems arise in a variety of different areas of applied mathematics and physics. In the few years, there has been increasing interest in studying certain third-order boundary value problems for nonlinear differential equation and have received much attention. To identify a few, we refer the reader to [1–6].

Recently, El-Shahed [1] discussed the following third-order two-point boundary value problem:

$$\begin{gathered} u'''(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad \alpha u'(1) + \beta u''(1) = 0. \end{gathered} \tag{1.2}$$

The methods employed in [1] are Kransnoselskii's fixed-point theorem of cone.

In later work, by placing restrictions on the nonlinear term f, Sun [2] studied the following boundary value problems and obtained the three solution via leggett-williams fixed point theorem:

$$\begin{gathered} u'''(t) = a(t)f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \\ u(0) = \delta u(\eta) = 0, \qquad u'(\eta) = 0, \qquad u''(1) = 0. \end{gathered} \tag{1.3}$$

The upper and lower solution is a powerful tool for proving existence for boundary value problems, Ma [7] studied the multiplicity of positive solutions of three-point boundary value problem of second-order ordinary differential equations. Du et al. [5] investigated a class of third-order nonlinear problem.

Motivated by the work of the above papers, the purpose of this article is to study the existence of solution for boundary value problem (1.1) using a new technique (different from the proof of [1, 2, 7]) and we get a new existence result. The tools are based on the fixed point index theorems and Leray-Schauder degree.

The paper is organized as follows: Section 2 states some definitions and some lemmas which are important to obtain our main result. Section 3 is devoted to the existence result of BVP (1.1). Section 4 gives some examples to illustrate our main results.

2. Preliminary

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called a cone of E if it satisfies the following two conditions:

(1) $x \in K$, $\lambda \geq 0$ implies $\lambda x \in K$;

(2) $x \in K$, $-x \in K$ implies $x = 0$.

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Lemma 2.3. *Let $y \in C[0,1]$, then the following boundary value problem:*

$$u'''(t) + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u'(0) = 0, \qquad u''(1) = \alpha u''(\eta), \tag{2.2}$$

has the unique solution

$$u(t) = \int_0^1 G(t,s) y(s) ds, \tag{2.3}$$

where

$$G(t,s) = \begin{cases} -\dfrac{1}{2}(t-s)^2 + \dfrac{t^2}{2}, & s \leq \eta,\ s \leq t, \\ \dfrac{t^2}{2}, & t \leq s \leq \eta, \\ -\dfrac{1}{2}(t-s)^2 + \dfrac{t^2}{2(1-\alpha)}, & \eta \leq s \leq t, \\ \dfrac{t^2}{2(1-\alpha)}, & \eta \leq s,\ t \leq s. \end{cases} \tag{2.4}$$

Proof. From (2.1), we have

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + At^2 + Bt + C. \tag{2.5}$$

In particular,

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + At^2 + Bt + C, \\ u'(t) &= -t\int_0^t y(s)ds + \int_0^t sy(s)ds + 2At + B, \\ u''(t) &= -\int_0^t y(s)ds + 2A. \end{aligned} \tag{2.6}$$

Combining this with boundary conditions (2.2), we conclude that

$$A = \frac{\int_0^1 y(s)ds}{2(1-\alpha)} - \frac{\alpha\int_0^\eta y(s)ds}{2(1-\alpha)}, \quad B = 0, \quad C = 0. \tag{2.7}$$

Therefore, BVP (2.1)-(2.2) has a unique solution:

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds - \frac{\alpha t^2\int_0^\eta y(s)ds}{2(1-\alpha)} + \frac{t^2\int_0^1 y(s)ds}{2(1-\alpha)} \\ &= \begin{cases} \displaystyle\int_0^t \left[-\frac{1}{2}(t-s)^2 + \frac{t^2}{2}\right]y(s)ds + \int_t^\eta \frac{t^2}{2}y(s)ds + \int_\eta^1 \frac{t^2}{2(1-\alpha)}y(s)ds, & t \le \eta, \\ \displaystyle\int_0^\eta \left[-\frac{1}{2}(t-s)^2 + \frac{t^2}{2}\right]y(s)ds + \int_\eta^t \left[-\frac{1}{2}(t-s)^2 + \frac{t^2}{2(1-\alpha)}\right]y(s)ds \\ \displaystyle\quad + \int_t^1 \frac{t^2}{2(1-\alpha)}y(s)ds, & t \ge \eta, \end{cases} \\ &= \int_0^1 G(t,s)y(s)ds. \end{aligned} \tag{2.8}$$

The proof is completed. □

Lemma 2.4. *For all* $(t,s) \in [0,1] \times [0,1]$, *one has* $G(t,s) \ge 0$.

Lemma 2.5. *for all* $(t,s) \in [\tau,1] \times [0,1]$, *one has*

$$\gamma G(1,s) \le G(t,s) \le G(1,s), \tag{2.9}$$

where $\gamma = \alpha\tau^2/2$, *and* τ *statisfies* $\int_\tau^1 G(t,s)a(s)ds > 0$.

Proof. For $s \le t,\ s \le \eta$,

$$\begin{gathered} G(t,s) = -\frac{1}{2}(t-s)^2 + \frac{t^2}{2} = \frac{s(2t-s)}{2} \le G(1,s), \\ \frac{G(t,s)}{G(1,s)} = \frac{2t-s}{2-s} = \frac{t+t-s}{2-s} \ge \frac{t}{2}. \end{gathered} \tag{2.10}$$

For $t \le s \le \eta$,

$$\begin{gathered} G(t,s) = \frac{t^2}{2} \le G(1,s), \\ \frac{G(t,s)}{G(1,s)} = \frac{t^2/2}{1/2} = t^2. \end{gathered} \tag{2.11}$$

For $\eta \le s \le t$,

$$\begin{gathered} G(t,s) = -\frac{1}{2}(t-s)^2 + \frac{t^2}{2(1-\alpha)} = \frac{\alpha t^2 + 2ts(1-\alpha) + s^2(1-\alpha)}{2(1-\alpha)} \le G(1,s), \\ \frac{G(t,s)}{G(1,s)} = \frac{\alpha t^2 + 2ts(1-\alpha) + s^2(1-\alpha)}{\alpha + 2s(1-\alpha) + s^2(1-\alpha)} \ge \alpha t^2. \end{gathered} \tag{2.12}$$

For $\eta \le s,\ t \le s$,

$$\begin{gathered} G(t,s) = \frac{t^2}{2(1-\alpha)} \le G(1,s), \\ \frac{G(t,s)}{G(1,s)} = t^2. \end{gathered} \tag{2.13}$$

Thus,

$$\frac{\alpha t^2}{2} G(1,s) \le G(t,s) \le G(1,s), \qquad \text{for } (t,s) \in [0,1] \times [0,1]. \tag{2.14}$$

Therefore,

$$\gamma G(1,s) \le G(t,s) \le G(1,s), \qquad \forall (t,s) \in [\tau,1] \times [0,1]. \tag{2.15}$$

The proof is completed. □

Lemma 2.6. *If $y \in C[0,1]$ and $y \ge 0$, then the unique solution $u(t)$ of the BVP* (2.1)-(2.2) *is nonnegative and satisfies*

$$\min_{t\in[\tau,1]} u(t) \ge \gamma \|u\|. \tag{2.16}$$

Proof. Let $y \in C^+[0,1]$, it is obvious that it is nonnegative. For any $t \in [0,1]$, by (2.3) and Lemma 2.5, it follows that

$$u(t) = \int_0^1 G(t,s) y(s) ds \le \int_0^1 G(1,s) y(s) ds, \tag{2.17}$$

and thus,

$$\|u\| \le \int_0^1 G(1,s)y(s)ds. \tag{2.18}$$

On the other hand, (2.3) and Lemma 2.5 imply, for any $t \in [\tau, 1]$,

$$u(t) = \int_0^1 G(t,s)y(s)ds \ge \gamma \int_0^1 G(1,s)y(s)ds. \tag{2.19}$$

Therefore,

$$\min_{t\in[\tau,1]} u(t) \ge \gamma \|u\|. \tag{2.20}$$

This completes the proof. □

Let $E = C[0,1]$ with the usual normal $\|u\| = \max_{t\in[0,1]}|u(t)|$.
Define the cone K by

$$K = \left\{ u \in C^+[0,1] : \min_{t\in[\tau,1]} u(t) \ge \gamma\|u\| \right\}. \tag{2.21}$$

Define an operator T by

$$Tu(t) = \lambda \int_0^1 G(t,s)a(s)f(u(s))ds. \tag{2.22}$$

By Lemma 2.3, BVP (1.1) has a positive solution $u = u(t)$ if and only if u is a fixed point of T.

Lemma 2.7. *Assume that* $0 < \lambda < \infty$. *Then,* $T : K \to K$ *is completely continuous.*

Proof. Firstly, it is easy to check that $T : K \to K$ is well defined. By Lemma 2.6, we know that $T(K) \subset K$.

Let Ω be any boundary subset of K, then there exists $r > 0$, $\|u\| \le r$, for all $u \in \Omega$. Therefore, we have

$$|Tu| = \lambda \left| \int_0^1 G(t,s)a(s)f(u(s))ds \right| \le \lambda \left| \int_0^1 G(1,s)a(s)f(u(s))ds \right|. \tag{2.23}$$

So $T\Omega$ is boundary. Moreover, for any $t_1, t_2 \in [0,1]$, $|t_1 - t_2| \le \delta$, $\delta > 0$, we have

$$|Tu(t_1) - Tu(t_2)| \le \lambda \int_0^1 |G(t_1,s) - G(t_2,s)| a(s) f(u(s))ds. \tag{2.24}$$

By the continuity of f and a, we have $a(t)$ and $f(u(t))$ are boundary on $u \in \Omega, t \in [0,1]$, which means that there exists a constant $M_a^f > 0$, depending only on Ω such that

$$\left|a(t)f(u(t))\right| < M_a^f, \tag{2.25}$$

and thus for any $\varepsilon > 0$,

$$\begin{gathered}|G(t_1,s) - G(t_2,s)| \leq \frac{\varepsilon}{\lambda M_a^f},\\ |Tu(t_1) - Tu(t_2)| < \varepsilon.\end{gathered} \tag{2.26}$$

Therefore, we can get $T\Omega$ is equicontinuity. Thirdly, we prove that T is continuous. Let $u_n \to u$ as $n \to \infty$, $u_n \in K$. Then, the continuity of f, we can get

$$\begin{aligned}|Tu_n(t) - Tu(t)| &= \left|\lambda\int_0^1 G(t,s)a(s)f(u_n(s))ds - \lambda\int_0^1 G(t,s)a(s)f(u(s))ds\right|\\ &= \left|\lambda\int_0^1 G(t,s)a(s)\big(f(u_n(s)) - f(u(s))\big)ds\right|\\ &\leq \left|\lambda\int_0^1 G(1,s)a(s)\big(f(u_n(s)) - f(u(s))\big)ds\right| \longrightarrow 0, \quad n \longrightarrow \infty.\end{aligned} \tag{2.27}$$

Then, $Tu_n(t) \to Tu(t)$. Therefore, T is continuous.The operator T is completely continuous by an application of the Ascoli-Arzela theorem. This completes the proof. □

Lemma 2.8 (see [7, 8]). *Let E be a real Banach space and let K be a cone in E. For $r \geq 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume $T : \overline{K}_r \to K$ is a completely continuous operator such that $Tx \neq x$ for $x \in \partial K_r = \{x \in K : \|x\| = r\}$.*

(1) *If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0. \tag{2.28}$$

(2) *If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1. \tag{2.29}$$

3. Main Results

Theorem 3.1. *Assume that*

(A1) *λ is a positive parameter, $\eta \in (0,1)$ and $\alpha \in (0,1)$;*

(A2) *$a : [0,1] \to (0,\infty)$ is continuous;*

(A3) *$f : [0,\infty) \to (0,\infty)$ is continuous;*

(A4) *$f_\infty := \lim_{u\to\infty}(f(u)/u) = \infty$.*

When λ is sufficiently small, (1.1) *has at least one positive solution, whereas for λ is sufficiently large,* (1.1) *has no positive solution.*

Proof. If $q > 0$, then

$$\beta(q) = \max_{u\in K,\|u\|=q}\left[\int_0^1 G(t,s)a(s)f(u(s))ds\right] > 0. \tag{3.1}$$

For any number $0 < r_1$, let $\delta_1 = r_1/\beta(r_1)$, and set

$$K_{r_1} = \{u \in K : \|u\| < r_1\}. \tag{3.2}$$

Then, for $\lambda \in (0,\delta_1)$ any $u \in \partial K_{r_1}$, we have

$$Tu(t) < \delta_1\left[\int_0^1 G(t,s)f(u(s))ds\right] \le \delta_1\beta(r_1) = r_1. \tag{3.3}$$

Thus, Lemma 2.8 implies

$$i(T,K_{r_1},K) = 1. \tag{3.4}$$

Since $f_\infty = \infty$, there is $M > 0$, such that $f(u) \ge \mu u$, for $u > M$, where μ is chosen so that

$$\lambda\mu\gamma\int_\tau^1 G(1,s)a(s)ds > 1. \tag{3.5}$$

Let $r_2 > M/\gamma$, and set

$$K_{r_2} = \{u \in K : \|u\| < r_2\}. \tag{3.6}$$

If $u \in \partial K_{r_2}$, then

$$\min_{t\in[\tau,1]} u(t) \ge \gamma\|u\| \ge M. \tag{3.7}$$

Therefore,

$$\begin{aligned} Tu(1) &= \lambda\int_0^1 G(1,s)a(s)f(u(s))ds \\ &\ge \lambda\int_\tau^1 G(1,s)a(s)f(u(s))ds \\ &\ge \lambda\int_\tau^1 G(1,s)a(s)\mu u(s)ds \end{aligned}$$

$$\begin{aligned}&\geq \lambda\mu \int_{\tau}^{1} G(1,s)a(s)ds\gamma\|u\| \\ &\geq \lambda\mu\gamma \int_{\tau}^{1} G(1,s)a(s)ds\|u\| \\ &> \|u\|,\end{aligned} \tag{3.8}$$

which implies that

$$\|Tu\| \geq \|u\|, \tag{3.9}$$

for $u \in \partial K_{r_2}$. An application of Lemma 2.8 again shows that

$$i(T, K_{r_2}, K) = 0. \tag{3.10}$$

Since we can adjust r_1, r_2 so that $r_1 < r_2$, it follows the additivity of the fixed-point index that

$$i\left(T, K_{r_2} \setminus \overline{K}_{r_1}, K\right) = -1. \tag{3.11}$$

Thus, T has a fixed point in $K_{r_2} \setminus \overline{K}_{r_1}$ which is the desired positive solution of (1.1).

We verify that BVP of (1.1) has no positive solution for λ large enough.

Otherwise, there exist $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, with $\lim_{n\to\infty}\lambda_n = +\infty$, such that for any positive integer n, the BVP,

$$\begin{aligned}&u'''(t) + \lambda_n a(t) f(u(t)) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = 0, \qquad u''(1) = \alpha u''(\eta),\end{aligned} \tag{3.12}$$

has a positive solution $u_n(t)$. By (2.22), we have

$$u_n = \lambda_n \int_0^1 G(t,s)a(s)f(u_n(s)) \longrightarrow +\infty, \quad (n \longrightarrow \infty). \tag{3.13}$$

Thus,

$$u_n \longrightarrow \infty, \quad (n \longrightarrow \infty). \tag{3.14}$$

Since f_∞, for $c_0 > 0$, there exists $r_3 > 0$, such that $f(u)/u > c_0$, for $u \in [r_3, \infty)$, which implies that

$$f(u) > c_0 u, \quad \text{for } u \in [r_3, \infty). \tag{3.15}$$

Let n be large enough that $\|u_n\| \geq r_3$, then

$$\begin{aligned}\|u_n\| &\geq u_n(1)\\ &= \lambda_n \int_0^1 G(1,s)a(s)f(u_n(s))ds\\ &\geq \lambda_n \gamma \int_0^1 G(1,s)a(s)ds c_0 \|u_n\|\\ &> \|\mathrm{u}_n\|.\end{aligned} \tag{3.16}$$

Choose n so that $c_0 \lambda_n \gamma \int_0^1 G(1,s)a(s)ds > 1$ which is a contradiction. The proof is completed. □

Theorem 3.2. *Assume that*

(B1) λ *is a positive parameter;* $\eta \in (0,1)$ *and* $\alpha \in (0,1)$*;*

(B2) $a : [0,1] \to (0,\infty)$ *is continuous and there exists* $m > 0$ *such that* $a(t) \geq m$*;*

(B3) $f : [0,\infty) \to (0,\infty)$ *is continuous;*

(B4) $f_\infty = \lim_{u\to\infty}(f(u)/u) = 0$, $f_0 = \lim_{u\to 0}(f(u)/u) = 0$*;*

(B5) *there exists* $\sigma > 0$*, for* $u \geq \sigma$*, such that* $f(u) \geq \beta$*, where* $\beta > 0$*, then there exists* $\delta_2 > 0$*, such that, for* $\lambda > \delta_2$*, BVP* (1.1) *has at least two positive solutions* u_λ^1, u_λ^2 *and* $\max u_\lambda^1 > \sigma$*.*

Proof. Let $\delta_2 = (M\gamma m\beta)^{-1}\sigma$, then for $\lambda > \delta_2$, Lemma 2.7 implies that $T : K \to K$ is completely continuous. Considering (B4), there exists $0 < r < \sigma$ such that $f(u) \leq u/2\Lambda\lambda$, for $0 \leq u \leq r$, where $\Lambda = \int_0^1 G(1,s)a(s)ds$.

So, for $u \in \partial K_r$, we have from (2.4)

$$\begin{aligned}(Tu)(t) &= \lambda\left[\int_0^1 G(t,s)a(s)f(s)ds\right]\\ &\leq \lambda \int_0^1 G(1,s)a(s)f(u(s))ds\\ &\leq \lambda\left[\int_0^1 G(1,s)a(\mathrm{s})ds\right]\frac{\|u\|}{2\Lambda\lambda}\\ &= \frac{\|u\|}{2} < \|u\| = r.\end{aligned} \tag{3.17}$$

Consequently, for $u \in \partial K_r$, we have $\|Tu\| < \|u\|$, by Lemma 2.8,

$$i(T, K_r, K) = 1. \tag{3.18}$$

Now considering (B4), there exists $h > 0$, for $u > h$, such that $f(u) \leq u/2\Lambda\lambda$. Letting $\rho = \max_{0\leq u\leq h} f(u)$, then

$$0 \leq f(u) \leq \frac{u}{2\Lambda\lambda} + \rho. \tag{3.19}$$

Choose

$$R > \max\{r, 2\Lambda\rho\lambda\}. \tag{3.20}$$

So for $u \in \partial K_R$, from (3.18) and (3.19), we have

$$\begin{aligned}(Tu)(t) &= \lambda\left[\int_0^1 G(t,s)a(s)f(u)ds\right]\\ &\le \lambda\left[\int_0^1 G(1,s)a(s)f(u)ds\right]\\ &\le \lambda\left[\int_0^1 G(1,s)a(s)ds\right]\left(\frac{1}{2\Lambda\lambda}\|u\| + \rho\right)\\ &< \frac{\|u\|}{2} + \frac{R}{2} = \|u\|,\end{aligned} \tag{3.21}$$

That is, by Lemma 2.8,

$$i(T, K_R, K) = 1. \tag{3.22}$$

On the other hand, for $u \in \overline{K}_\sigma^R = \{u \in K : \|u\| \le R,\ \min_{t\in J_\theta} u(t) \ge \sigma, \theta \in (0,1/2), J_\theta = [\theta, 1-\theta]\}$, (2.3) and (2.4) yield that

$$\|Tu\| \le \lambda\left[\int_0^1 G(t,s)a(s)ds\right]\left(\frac{1}{2\Lambda\lambda}\|u\| + \rho\right) < R. \tag{3.23}$$

Furthermore, for $u \in \overline{K}_\sigma^R$, from (2.3) and (2.4), we obtain

$$\begin{aligned}\min_{t\in J_\theta}(Tu)(t) &= \min_{t\in J_\theta}\lambda\left[\int_0^1 G(1,s)a(s)f(u(s))ds\right]\\ &\ge \min_{t\in J_\theta}\lambda\int_\theta^{1-\theta} G(t,s)a(s)f(u(s))ds\\ &\ge \lambda\gamma\int_\theta^{1-\theta} G(1,s)a(s)f(u(s))ds\\ &\ge \lambda M\gamma m\beta > \delta_2 \mathrm{M}\gamma m\beta = \sigma,\end{aligned} \tag{3.24}$$

where $M = \int_\theta^{1-\theta} G(1,s)ds$. Let $u_0 \equiv (\sigma + R)/2$ and $H(t,u) = (1-t)Tu + tu_0$, then $H : [0,1] \times \overline{K}_\sigma^R \to K$ is continuous, and from the analysis above, we obtain for $(t,u) \in [0,1] \times \overline{K}_\sigma^R$:

$$H(t,u) \in K_\sigma^R. \tag{3.25}$$

Therefore, for $u \in \partial K_\sigma^R$, we have $H(t,u) \neq u$. Hence, by the normality property and the homotopy invariance property of the fixed point index, we obtain

$$i\left(T, K_\sigma^R, K\right) = i\left(u_0, K_\sigma^R, K\right) = 1. \tag{3.26}$$

Consequently, by the solution property of the fixed point index, T has a fixed point u_λ^1 and $u_\lambda^1 \in K_\sigma^R$. By Lemma 2.4, it follows that u_1^λ is a solution to BVP (1.1), and

$$\max_{t\in[0,1]} u_\lambda^1 \geq \min_{t\in J\theta} u_\lambda^1 > \gamma. \tag{3.27}$$

On the other hand, from (3.18) and (3.19) together with the additivity of the fixed point index, we get

$$i\left(T, K_R \setminus \left(\overline{K}_r \cup \overline{K}_\sigma^R\right)\right) = i(T, K_R, K) - i\left(T, K_\sigma^R, K\right) - i(T, K_r, K) = 1 - 1 - 1 = -1. \tag{3.28}$$

Hence, by the solution property of the fixed point index, T has a fixed point u_λ^2 and $u_\lambda^2 \in K_R \setminus (\overline{K}_r \cup \overline{K}_\sigma^R)$. By Lemma 2.3, it follows that u_λ^2 is also a solution to BVP (1.1), and $u_\lambda^1 \neq u_\lambda^2$. The proof is completed. □

4. Examples

Example 4.1. We consider the following third-order boundary value problems:

$$\begin{gathered} u'''(t) + \lambda(2t+1)e^u = 0, \\ u(0) = u'(0) = 0, \qquad u''(1) = \frac{3}{4}u''\left(\frac{1}{4}\right), \end{gathered} \tag{4.1}$$

here $\eta = 1/4, \alpha = 3/4, f(u(t)) = e^u, a(t) = 2t+1, f_\infty = \lim_{u\to\infty}(f(u)/u) = \infty, f$ is continuous, $a(t)$ is continuous. By direct calculations, we obtain that $\lambda < r_1(1-\alpha)$, for $r_1 > 0$. Therefore, by Theorem 3.1, there exists at least one solution $u(t)$ for BVP (4.1), whereas for λ large enough, (4.1) has no solution.

Example 4.2. Consider the following third-order ordinary differential equation:

$$\begin{gathered} u''' + \lambda(2t+1)f(u(t)) = 0, \\ u(0) = u'(0) = 0, \qquad u''(1) = \frac{1}{4}u''\left(\frac{1}{2}\right), \end{gathered} \tag{4.2}$$

where

$$f(u(t)) = \begin{cases} u^2 e^{-u}, & \text{if } u \le a, \\ a^{3/2}\sqrt{u}e^{-a}, & \text{if } u > a, \end{cases} \tag{4.3}$$

f is continuous, $a(t)$ is continuous. Here, $m = 1$, $\alpha = 1/4$, $\beta = a^2 e^{-a}$, $\sigma = a$, $a > 0$. Choose $\delta_2 = 6a/(2\theta^3 - 3\theta^2 + 3\theta - 1), \theta \in (0, 1/2)$, $\tau \in (0,1)$, when $\lambda > \delta_2$, by Theorem 3.2, there exist at least two solutions $u_\lambda^1(t), u_\lambda^2(t)$ for BVP (4.1).

Acknowledgment

This project is sponsored by the Natural Science Foundation of China (11101349, 11071205), the NFS of Jiangsu Province (BK2011042), the NSF of Education Department of Jiangsu Province (11KJB11003), and Jiangsu Government Scholarship Program.

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