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Research Article

Blow-Up and Global Existence for a Degenerate Parabolic System with Nonlocal Sources

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This paper investigates the blow-up and global existence of nonnegative solutions for a class of nonlocal degenerate parabolic system. By using the super- and subsolution techniques, the critical exponent of the system is determined. That is, if $P_c = p_1q_1 - (m - p_2)(n - q_2) < 0$, then every nonnegative solution is global, whereas if $P_c > 0$, there are solutions that blowup and others that are global according to the size of initial values $u_0(x)$ and $v_0(x)$. When $P_c = 0$, we show that if the domain is sufficiently small, every nonnegative solution is global while if the domain is large enough that is, if it contains a sufficiently large ball, there is no global solution.

1. Introduction and Description of Results

In this paper, we investigate the blowup and global existence of nonnegative solutions for the following degenerate parabolic system with nonlocal sources:

$$\begin{gathered} u_t = \Delta u^m + v^{p_1}\|u\|_\alpha^{p_2}, \quad (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v^n + u^{q_1}\|v\|_\beta^{q_2}, \quad (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$ and $u_0(x), v_0(x)$ are nonnegative bounded functions in Ω, constants $m, n > 1$, $\alpha, \beta \geq 1$, $p_1, q_1, p_2, q_2 > 0$, where $\|\cdot\|_\alpha^\alpha = \int_\Omega |\cdot|^\alpha \mathrm{d}x$.

Equation (1.1) constitutes a simple example of a reaction diffusion system exhibiting a nontrivial coupling on the unknowns $u(x,t), v(x,t)$, such as heat propagations in a two-component combustible mixture [1], chemical processes [2], and interaction of two biological groups without self-limiting [3]. And they are worth to study because of the applications to heat and mass transport processes. In addition, there exist interesting interactions among the multi-nonlinearities described by the eight exponents m, n, p_1, p_2, q_1, q_2 and α, β in the problem (1.1).

In the past two decades, many physical phenomena were formulated into nonlocal mathematical models (see [4–7] and references therein) and studied by many authors. Degenerate parabolic equations involving a nonlocal source, which arise in a population model that communicates through chemical means, were studied in [8, 9]. At the same time, there are many important results that have appeared on blowup problems for nonlinear parabolic system. We will recall some of those results concerning the first initial-boundary problem. For the other related works on the global existence and blowup of solutions of nonlinear parabolic system, we refer the readers to [10, 11] and references therein.

In [4], Escobedo and Herrero studied the system

$$u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q, \qquad x \in \Omega,\ t > 0 \tag{1.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with null Dirichlet boundary conditions. The authors show that if $pq \leq 1$, every solution of (1.2) is global, whereas if $pq > 1$, there are solutions that blowup and others that are global according to the size of initial values $u_0(x)$ and $v_0(x)$.

In [12], Galaktionov et al. considered the system

$$u_t = \Delta u^{\nu+1} + v^p, \quad v_t = \Delta v^{\mu+1} + u^q, \quad (x,t) \in \Omega \times (0,T) \tag{1.3}$$

with homogeneous Dirichlet boundary conditions, and they proved that $p_c = pq - (\nu+1)(\mu+1)$ is the critical exponent. Zheng [13] and Li et al. [14] studied the following systems:

$$\begin{gathered} u_t = \Delta u + u^{p_1} v^{q_1}, \quad v_t = \Delta v + u^{p_2} v^{q_2}, \quad (x,t) \in \Omega \times \mathbf{R}_+, \\ u_t = \Delta u + \int_\Omega u^m(x,t) v^n(x,t)\,\mathrm{d}x, \quad v_t = \Delta v + \int_\Omega u^p(x,t) v^q(x,t)\,\mathrm{d}x, \quad x \in \Omega,\ t > 0, \end{gathered} \tag{1.4}$$

respectively. They obtained some results on the global solutions, the blowup solutions and the blowup profiles. Lately, Deng et al. in [15] considered the following nonlocal degenerate parabolic system:

$$u_t = \Delta u^m + a\|v\|_\alpha^p, \quad v_t = \Delta v^n + b\|u\|_\beta^q, \quad (x,t) \in \Omega \times (0,T) \tag{1.5}$$

with homogeneous Dirichlet boundary conditions. Several interesting results are established as follows.

(i) If $pq < mn$, then every nonnegative solution of (1.5) is global.

(ii) If $pq = mn$, then if the domain is sufficiently small, the nonnegative solution of (1.5) is global, whereas if the domain contains a sufficiently large ball and $u_0(x), v_0(x) > 0$, the nonnegative solution blows up in finite time.

(iii) If $pq > mn$, then there are solutions of (1.5) that blowup and others that are global according to the size of initial data $u_0(x)$ and $v_0(x)$.

Our present work is motivated by [12–15] mentioned above. The main purpose of this paper is to extend and improve the results in [15]. At the same time, we will prove that $p_c = p_1q_1 - (m - p_2)(n - q_2)$ is also the critical exponent of system (1.1). Our main results are as follows, two theorems concern the global existence and blowup conditions of the solutions.

Theorem 1.1. *If one of the following conditions holds, then the nonnegative solution of system* (1.1) *exists globally.*

(1) $m > p_2,\ n > q_2$ *and* $p_1q_1 < (m - p_2)(n - q_2)$.

(2) $m > p_2,\ n > q_2,\ p_1q_1 = (m - p_2)(n - q_2)$ *and the domain* $(|\Omega|)$ *is sufficiently small.*

(3) $m > p_2,\ n > q_2,\ p_1q_1 > (m - p_2)(n - q_2)$ *and the initial data* $u_0(x), v_0(x)$ *are sufficiently small.*

(4) $m \le p_2$ *or* $n \le q_2$ *and the initial data* $u_0(x), v_0(x)$ *are sufficiently small.*

Theorem 1.2. *If one of the following conditions holds, then the nonnegative solution of system* (1.1) *blows up in a finite time.*

(1) $m > p_2,\ n > q_2, p_1q_1 > (m - p_2)(n - q_2)$ *and the initial data* $u_0(x), v_0(x)$ *are sufficiently large.*

(2) $m > p_2,\ n > q_2, p_1q_1 = (m - p_2)(n - q_2)$ *and the domain contains a sufficiently large ball, moreover,* $u_0(x)$ *and* $v_0(x)$ *are large enough.*

(3) $m \le p_2$ *or* $n \le q_2$ *and initial data* $u_0(x), v_0(x)$ *are sufficiently large.*

This paper is organized as follows. In the next Section, we establish the local existence theorem and give some auxiliary lemmas. In Section 3, which concerns global existence, we prove Theorem 1.1. In Section 4, which deals with the blowup phenomenon, we prove Theorem 1.2.

2. Local Existence and Comparison Principle

Similar to the Propositions 2.1 and 2.2 of [15], we give the maximum principle and the comparison principle for the nonlocal parabolic system. For convenience, we denote $Q_T = \Omega \times (0,T),\ \overline{Q}_T = \overline{\Omega} \times [0,T],\ S_T = \partial\Omega \times (0,T)$, where $0 < T < +\infty$.

As it is now well known that degenerate equation needs not possess classical solution, we begin by giving a precise definition of a weak solution for system (1.1). To this end, define the class of test functions

$$\Psi \equiv \left\{ \varphi(x,t) \in C\left(\overline{Q}_T\right); \varphi_t, \Delta\varphi \in C(Q_T) \cap L^2(Q_T); \varphi \ge 0; \varphi(x,t)\Big|_{x\in\partial\Omega} = 0 \right\}. \tag{2.1}$$

Definition 2.1. A pair of vector function $(\overline{u}(x,t), \overline{v}(x,t))$ defined on $\overline{Q}_T$ is called a super-solution of (1.1), if all the following conditions hold:

(1) $\overline{u}(x,t), \overline{v}(x,t) \in L^\infty(Q_T)$;

(2) if $(x,t) \in S_T,\ \overline{u}(x,t), \overline{v}(x,t) \ge 0$, and for all $x \in \Omega$, $\overline{u}(x,0) \ge u_0(x),\ \overline{v}(x,0) \ge v_0(x)$;

(3) for every $t \in [0,T]$ and any $\psi_1, \psi_2 \in \Psi$,

$$\begin{aligned}
&\int_\Omega \left(\overline{u}(x,t)\psi_1(x,t) - u_0(x)\psi_1(x,0)\right)\mathrm{d}x \geq \int_0^t \int_\Omega \left(\overline{u}\psi_{1s} + \overline{u}^m \Delta\psi_1 + \overline{v}^{p_1} \|\overline{u}\|_\alpha^{p_2} \psi_1\right)\mathrm{d}x\,\mathrm{d}s, \\
&\int_\Omega \left(\overline{v}(x,t)\psi_2(x,t) - v_0(x)\psi_2(x,0)\right)\mathrm{d}x \geq \int_0^t \int_\Omega \left(\overline{v}\psi_{2s} + \overline{v}^n \Delta\psi_2 + \overline{u}^{q_1} \|\overline{v}\|_\beta^{q_2} \psi_2\right)\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{2.2}$$

A subsolution $(\tilde{u}(x,t), \tilde{v}(x,t))$ can be defined in a similar way.

Next, we state the maximum principle and comparison principle, and the proofs that are quite standard, we omit them here.

Lemma 2.2 (maximum principle). *Suppose that $\omega_1(x,t), \omega_2(x,t) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfy*

$$\begin{gathered}
M_1\omega = \omega_{1t} - d_1\Delta\omega_1 - \sum_{j=1}^N a_{1j}\omega_{1x_j} - c_{11}\omega_1 - c_{12}\omega_2 - c_{13}\int_\Omega c_{14}\omega_1 dx \geq 0, \quad (x,t) \in Q_T, \\
M_2\omega = \omega_{2t} - d_2\Delta\omega_2 - \sum_{j=1}^N a_{2j}\omega_{2x_j} - c_{21}\omega_2 - c_{22}\omega_1 - c_{23}\int_\Omega c_{24}\omega_2 dx \geq 0, \quad (x,t) \in Q_T, \\
\omega_1(x,t) \geq 0, \quad \omega_2(x,t) \geq 0, \quad (x,t) \in S_T, \\
\omega_1(x,0) \geq 0, \quad \omega_2(x,0) \geq 0, \quad x \in \Omega,
\end{gathered} \tag{2.3}$$

where $d_i(x,t)$, $c_{ij}(x,t)$ $(i = 1,2; j = 1,2,3,4)$ and $a_{ij}(x,t)$, $(i = 1,2; j = 1,2,\ldots,N)$ are the continuous and the bounded functions on $\overline{Q}_T$, respectively, and

$$d_i(x,t), c_{i2}(x,t), c_{i3}(x,t), c_{i4}(x,t) \geq 0, \quad i = 1,2, \quad (x,t) \in \Omega \times (0,T]. \tag{2.4}$$

Then $\omega_i(x,t) \geq 0$ on $\overline{Q}_T$.

Lemma 2.3 (comparison principle). *Let $(\overline{u}, \overline{v})$ and $(\tilde{u}, \tilde{v})$ be a nonnegative supersolution and a nonnegative subsolution of system* (1.1)*, respectively. Then $(\tilde{u}, \tilde{v}) \leq (\overline{u}, \overline{v})$ on Q_T if $(\tilde{u}(x,0), \tilde{v}(x,0)) \leq (\overline{u}(x,0), \overline{v}(x,0))$ and either*

$$\overline{u}, \overline{v} \geq \rho > 0 \quad \text{or} \quad \tilde{u}, \tilde{v} \geq \rho > 0 \tag{2.5}$$

hold.

Theorem 2.4 (local existence and continuation). *Assume $u_0, v_0 \geq 0$, $u_0, v_0 \in L^\infty(\Omega)$, there is a $T^* = T^*(u_0, v_0) > 0$ such that there exists a nonnegative weak solution $(u(x,t), v(x,t))$ of* (1.1) *for each $T < T^*$. Furthermore, either $T^* = +\infty$ or*

$$\limsup_{t \to T^*} \|u(\cdot,t)\|_\infty = +\infty \quad \text{or} \quad \limsup_{t \to T^*} \|v(\cdot,t)\|_\infty = +\infty. \tag{2.6}$$

Proof. Owing to the degeneracy of equations of (1.1), in order to prove the existence of solution, for $k = 1, 2, \ldots$, we first consider the following corresponding regularized system

$$\begin{gathered}
u_{kt} = \Delta f_k(u_k) + \left(g_k'(v_k)\right)^{p_1} \left\|g_k(u_k)\right\|_{\alpha}^{p_2}, \quad (x,t) \in Q_T, \\
v_{kt} = \Delta f_k'(v_k) + \left(g_k(u_k)\right)^{q_1} \left\|g_k'(v_k)\right\|_{\beta}^{q_2}, \quad (x,t) \in Q_T, \\
u_k(x,t) = v_k(x,t) = \frac{1}{k}, \quad (x,t) \in S_T, \\
u_k(x,0) = u_{0i}(x) + \frac{1}{k}, \quad v_k(x,0) = v_{0i}(x) + \frac{1}{k}, \quad x \in \Omega,
\end{gathered} \tag{2.7}$$

where

$$\begin{gathered}
f_k(u_k) = \begin{cases} u_k^m, & u_k \geq \dfrac{1}{k}, \\ \left(\dfrac{1}{k}\right)^m, & u_k < \dfrac{1}{k}, \end{cases} \qquad f_k'(v_k) = \begin{cases} v_k^n, & v_k \geq \dfrac{1}{k}, \\ \left(\dfrac{1}{k}\right)^n, & v_k < \dfrac{1}{k}, \end{cases} \\
g_k(u_k) = \begin{cases} u_k, & u_k \geq \dfrac{1}{k}, \\ \dfrac{1}{k}, & u_k < \dfrac{1}{k}, \end{cases} \qquad g_k'(v_k) = \begin{cases} v_k, & v_k \geq \dfrac{1}{k}, \\ \dfrac{1}{k}, & v_k < \dfrac{1}{k}, \end{cases}
\end{gathered} \tag{2.8}$$

and $u_{0i}(x), v_{0i}(x)$ are smooth approximation of $u_0(x), v_0(x)$ with $\operatorname{supp} u_{0i} \subset \Omega$ and $\operatorname{supp} v_{0i} \subset \Omega$, respectively. It is known that the system (2.7) has a unique classical solution $(u_k^i, v_k^i) \in C(\overline{\Omega} \times [0, T_i(k))) \cap C^{2,1}(\Omega \times (0, T_i(k)))$ for $0 < T_i(k) \leq \infty$ by the classical theory for parabolic equations, where $T_i(k)$ is the maximal existence time. By a direct computation and the classical maximum principle, we have $u_k^i, v_k^i \geq 1/k$. Hence (u_k^i, v_k^i) satisfies

$$\left(u_k^i\right)_t = \Delta\left(u_k^i\right)^m + \left(v_k^i\right)^{p_1} \left\|u_k^i\right\|_{\alpha}^{p_2}, \quad \left(v_k^i\right)_t = \Delta\left(v_k^i\right)^n + \left(u_k^i\right)^{q_1} \left\|v_k^i\right\|_{\beta}^{q_2}, \quad (x,t) \in Q_{T_i(k)} \tag{2.9}$$

with the corresponding initial and boundary conditions. At the same time, if $k_1 > k_2$, according to Lemma 2.2, we get

$$\left(u_{k_1}^i(x,t), v_{k_1}^i(x,t)\right) \leq \left(u_{k_2}^i(x,t), v_{k_2}^i(x,t)\right), \quad (x,t) \in \overline{\Omega} \times [0, T_i(k_2)), \tag{2.10}$$

and $T_i(k_1) \geq T_i(k_2)$. On the other hand, passing to the limit $i \to \infty$, it follows that

$$u_k(x,t) = \lim_{i \to \infty} u_k^i(x,t), \qquad v_k(x,t) = \lim_{i \to \infty} v_k^i(x,t), \tag{2.11}$$

and (u_k, v_k) is a weak solution of

$$(u_k)_t = \Delta(u_k)^m + (v_k)^{p_1} \|u_k\|_{\alpha}^{p_2}, \quad (v_k)_t = \Delta(v_k)^n + (u_k)^{q_1} \|v_k\|_{\beta}^{q_2}, \quad (x,t) \in Q_{T(k)} \tag{2.12}$$

with the corresponding initial and boundary conditions on $Q_{T(k)}$, where $T(k) = \lim_{i\to\infty} T_i(k)$ is the maximal existence time. Here a weak solution of (2.12) is defined in a manner similar to that for (1.1), only the equalities for u and v; (2.2) may be replaced with

$$\begin{aligned}
&\int_\Omega \left(u_k(x,t)\psi_1(x,t) - \left(u_0(x) + \frac{1}{k} \right)\psi_1(x,0) \right) \mathrm{d}x \\
&\quad = \int_0^t \int_\Omega \left(u_k \psi_{1s} + u_k^m \Delta\psi_1 + v_k^{p_1} \|u_k\|_\alpha^{p_2} \psi_1 \right) \mathrm{d}x\, \mathrm{d}s + \frac{1}{k}\int_0^t \int_{\partial\Omega} \left(\frac{\partial \psi_1}{\partial \nu} \right) \mathrm{d}\sigma\, \mathrm{d}s, \\
&\int_\Omega \left(v_k(x,t)\psi_2(x,t) - \left(v_0(x) + \frac{1}{k} \right)\psi_2(x,0) \right) \mathrm{d}x \\
&\quad = \int_0^t \int_\Omega \left(v_k \psi_{2s} + v_k^n \Delta\psi_2 + u_k^{q_1} \|v_k\|_\beta^{q_2} \psi_2 \right) \mathrm{d}x\, \mathrm{d}s + \frac{1}{k}\int_0^t \int_{\partial\Omega} \left(\frac{\partial \psi_2}{\partial \nu} \right) \mathrm{d}\sigma\, \mathrm{d}s,
\end{aligned} \tag{2.13}$$

respectively. Then, passing to the limit $i \to \infty$, it happens that $(u_{k_1}, v_{k_1}) \le (u_{k_2}, v_{k_2})$ and $T(k_1) \ge T(k_2)$ if $k_1 > k_2$.

Therefore, the limit $T^* = \lim_{k\to\infty} T(k)$ exists, and the pointwise limit

$$u(x,t) = \lim_{k\to\infty} u_k(x,t), \qquad v(x,t) = \lim_{k\to\infty} v_k(x,t) \tag{2.14}$$

exists for any $(x,t) \in \overline{\Omega} \times [0, T^*)$. Furthermore, as the convergence of the sequence is monotone, passing to the limit $k \to \infty$ in (2.13), we get that $(u(x,t), v(x,t))$ is a nonnegative weak solution of (1.1). Thus the proof is completed. □

Denote

$$A = \begin{pmatrix} m - p_2 & -p_1 \\ -q_1 & n - q_2 \end{pmatrix}, \qquad L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}. \tag{2.15}$$

We give Lemmas 2.5 and 2.6 that will be used in the following; please see [16] for their proofs.

Lemma 2.5. *If $m > p_2$, $n > q_2$ and $p_1 q_1 < (m - p_2)(n - q_2)$, then there exist two positive constants l_1, l_2 such that $AL > (0,0)^T$.*

Lemma 2.6. *If $m \le p_2$ or $n \le q_2$ or $p_1 q_1 > (m - p_2)(n - q_2)$, then there exist two positive constants l_1, l_2 such that $AL < (0,0)^T$.*

3. Proof of Theorem 1.1

According to Lemma 2.3, we only need to construct bounded super-solutions for any $T > 0$. Let $\varphi(x)$ be the unique position solution of the following linear elliptic problem:

$$-\Delta\varphi(x) = 1, \quad x \in \Omega; \qquad \varphi(x) = 0, \quad x \in \partial\Omega. \tag{3.1}$$

Denote $C = \max_{x\in\Omega} \varphi(x)$, then $0 \le \varphi(x) \le C$. We define the functions $\overline{u}, \overline{v}$ as follows:

$$\overline{u}(x,t) = \left(k\left(\varphi(x)+1\right)\right)^{l_1}, \quad \overline{v}(x,t) = \left(k\left(\varphi(x)+1\right)\right)^{l_2}, \tag{3.2}$$

where $l_1, l_2 < 1$ such that $ml_1, nl_2 < 1$, and $k > 0$ will be fixed later. Clearly, for any $T > 0$, $(\overline{u}, \overline{v})$ is a bounded function and $\overline{u} \ge k^{l_1} > 0$, $\overline{v} \ge k^{l_2} > 0$. Then, we have

$$\begin{aligned}
\overline{u}_t - \Delta\overline{u}^m &= -k^{ml_1} ml_1(ml_1-1)(\varphi+1)^{ml_1-2}\left|\nabla\varphi\right|^2 + k^{ml_1} ml_1(\varphi+1)^{ml_1-1} \\
&\ge k^{ml_1} ml_1(\varphi+1)^{ml_1-1} \ge k^{ml_1} ml_1(C+1)^{ml_1-1}, \\
\overline{v}^{p_1}\|\overline{u}\|_\alpha^{p_2} &= k^{p_1l_2}(\varphi+1)^{p_1l_2}\left\|\left(k(\varphi+1)\right)^{l_1}\right\|_\alpha^{p_2} \le k^{p_1l_2+p_2l_1}(C+1)^{p_1l_2+p_2l_1}|\Omega|^{p_2/\alpha}, \\
\overline{v}_t - \Delta\overline{v}^n &\ge k^{nl_2} nl_2(C+1)^{nl_2-1}, \qquad \overline{u}^{q_1}\|\overline{v}\|_\beta^{q_2} \le k^{q_1l_1+q_2l_2}(C+1)^{q_1l_1+q_2l_2}|\Omega|^{q_2/\beta}.
\end{aligned} \tag{3.3}$$

Denote

$$\begin{aligned}
C_1 &= \left(\frac{|\Omega|^{p_2/\alpha}}{ml_1}(C+1)^{p_1l_2+p_2l_1-ml_1+1}\right)^{1/(ml_1-p_1l_2-p_2l_1)}, \\
C_2 &= \left(\frac{|\Omega|^{q_2/\beta}}{nl_2}(C+1)^{q_1l_1+q_2l_2-nl_2+1}\right)^{1/(nl_2-q_1l_1-q_2l_2)}.
\end{aligned} \tag{3.4}$$

(1) If $m > p_2$, $n > q_2$ and $p_1q_1 < (m-p_2)(n-q_2)$, by Lemma 2.5, there exist two positive constants $l_1, l_2 < 1$ such that

$$p_1l_2 + p_2l_1 < ml_1, \qquad q_1l_1 + q_2l_2 < nl_2, \qquad ml_1, nl_2 < 1. \tag{3.5}$$

Therefore, we can choose k sufficiently large that $k > \max\{C_1, C_2\}$ and

$$\left(k(\varphi+1)\right)^{l_1} \ge u_0(x), \qquad \left(k(\varphi+1)\right)^{l_2} \ge v_0(x). \tag{3.6}$$

Now, it follows from (3.3)–(3.6) that $(\overline{u}, \overline{v})$ is a positive super-solution of (1.1).

(2) If $m > p_2$, $n > q_2$ and $p_1q_1 = (m-p_2)(n-q_2)$, then there exist two positive constants $l_1, l_2 < 1$ such that

$$p_1l_2 + p_2l_1 = ml_1, \qquad q_1l_1 + q_2l_2 = nl_2, \qquad ml_1, nl_2 < 1. \tag{3.7}$$

Without loss of generality, we may assume that $\Omega \subset\subset B$, where B is a sufficiently large ball. And we denote $\varphi_B(x)$ is the unique positive solution of the following linear elliptic problem:

$$-\Delta\varphi(x) = 1, \quad x \in B; \qquad \varphi(x) = 0, \quad x \in \partial B. \tag{3.8}$$

Let $C_0 = \max_{x\in B}\varphi_B(x)$, then $C \le C_0$. Therefore, as long as Ω is sufficiently small and such that

$$|\Omega| < \min\left\{\left(\frac{ml_1}{C_0+1}\right)^{\alpha/p_2}, \left(\frac{nl_2}{C_0+1}\right)^{\beta/q_2}\right\}. \tag{3.9}$$

Furthermore, choose k large enough to satisfy (3.6).Then, it follows from (3.3) and (3.6)–(3.9) that $(\overline{u}, \overline{v})$ is a positive super-solution of (1.1).

(3) If $m > p_2,\ n > q_2$ and $p_1q_1 > (m-p_2)(n-q_2)$, by Lemma 2.6, there exist two positive constants $l_1, l_2 < 1$ such that

$$p_1l_2 + p_2l_1 > ml_1, \qquad q_1l_1 + q_2l_2 > nl_2, \qquad ml_1, nl_2 < 1. \tag{3.10}$$

Hence, we can choose k sufficiently small that $k < \min\{C_1, C_2\}$, and provided $u_0(x), v_0(x)$ are also sufficiently small to satisfy (3.6). Then, from (3.3) and (3.6), (3.10), we know that $(\overline{u}, \overline{v})$ is a positive super-solution of (1.1).

(4) Finally, if $m \le p_2$ or $n \le q_2$, there exist also positive constants $l_1, l_2 < 1$ such that (3.10) and $ml_1, nl_2 < 1$. Similar to the proof of (3), we get that $(\overline{u}, \overline{v})$ is a positive super-solution of (1.1).

Thus the proof of Theorem 1.1 is completed.

4. Proof of Theorem 1.2

Due to the requirement of the comparison principle, we will construct blowup subsolutions in some subdomain of Ω in which $u, v > 0$. We use an idea from Souplet [17] and apply it to degenerate parabolic equation. By translation, one may assume without loss of generality that $0 \in \Omega$. Let $B = B(0,R) \subset \Omega$ be an open ball with radius R, and $\psi(x)$ is a nontrivial nonnegative continuous function, vanished on ∂B and $\psi(0) > 0$. Set

$$\tilde{u}(x,t) = \frac{1}{(T-t)^{l_1}} V\left[\frac{|x|}{(T-t)^{\sigma}}\right], \qquad \tilde{v}(x,t) = \frac{1}{(T-t)^{l_2}} V\left[\frac{|x|}{(T-t)^{\sigma}}\right], \tag{4.1}$$

with

$$V(r) = \frac{R^3}{6} - \frac{R}{2}r^2 + \frac{1}{3}r^3, \quad r = \frac{|x|}{(T-t)^{\sigma}}, \quad 0 \le r \le R, \tag{4.2}$$

where $l_1, l_2, \sigma > 0$ and $0 < T < 1$ are to be determined later. Clearly, $0 \le V(r) \le R^3/6$ and $V(r)$ is nonincreasing since $V'(r) = r(r-R) \le 0$. Note that, for T small enough,

$$\operatorname{Supp}\tilde{u}(\cdot,t) = \operatorname{Supp}\tilde{v}(\cdot,t) = \overline{B}\left(0, R(T-t)^{\sigma}\right) \subset \overline{B}(0, RT^{\sigma}) \subset \Omega, \quad 0 \le t < T. \tag{4.3}$$

Obviously, $(\widetilde{u},\widetilde{v})$ becomes unbounded as $t \to T^-$ at the point $x=0$. Calculating directly, we obtain

$$\begin{aligned}
\widetilde{u}_t - \Delta\widetilde{u}^m &= \frac{l_1 V(r) + \sigma r V'(r)}{(T-t)^{l_1+1}} - \frac{m(m-1)V^{m-2}(r)(V'(r))^2}{(T-t)^{ml_1+2\sigma}} - \frac{mV^{m-1}(r)}{(T-t)^{ml_1+2\sigma}}(-NR+(N+1)r) \\
&\le \frac{l_1 R^3/6}{(T-t)^{l_1+1}} + \frac{mV^{m-1}(r)}{(T-t)^{ml_1+2\sigma}}(NR-(N+1)r), \quad (x,t)\in B\times(0,T), \\
\widetilde{v}_t - \Delta\widetilde{v}^n &\le \frac{l_2 R^3/6}{(T-t)^{l_2+1}} + \frac{nV^{n-1}(r)}{(T-t)^{nl_2+2\sigma}}(NR-(N+1)r), \quad (x,t)\in B\times(0,T).
\end{aligned} \tag{4.4}$$

notice $T<1$ is sufficiently small. Therefore, If $0\le r\le r_0 = NR/(N+1)$, we have $V(r) \ge R^3(3N+1)/(6(N+1)^3)$; then

$$\begin{aligned}
\widetilde{v}^{p_1}\|\widetilde{u}\|_\alpha^{p_2} &= \frac{V^{p_1}(r)}{(T-t)^{p_1 l_2}}\left(\int_B \frac{V^\alpha(r)}{(T-t)^{\alpha l_1}}\right)^{p_2/\alpha} \ge \frac{|B|^{p_2/\alpha}}{(T-t)^{p_1 l_2 + p_2 l_1}}\left(\frac{R^3(3N+1)}{6(N+1)^3}\right)^{p_1+p_2}, \\
\widetilde{u}^{q_1}\|\widetilde{v}\|_\beta^{q_2} &= \frac{V^{q_1}(r)}{(T-t)^{q_1 l_1}}\left(\int_B \frac{V^\beta(r)}{(T-t)^{\beta l_2}}\right)^{q_2/\beta} \ge \frac{|B|^{q_2/\beta}}{(T-t)^{q_1 l_1 + q_2 l_2}}\left(\frac{R^3(3N+1)}{6(N+1)^3}\right)^{q_1+q_2}.
\end{aligned} \tag{4.5}$$

Hence,

$$\begin{aligned}
&\widetilde{u}_t - \Delta\widetilde{u}^m - \widetilde{v}^{p_1}\|\widetilde{u}\|_\alpha^{p_2} \\
&\quad \le \frac{l_1 R^3/6}{(T-t)^{l_1+1}} + \frac{mV^{m-1}(r)}{(T-t)^{ml_1+2\sigma}}(NR-(N+1)r) - \frac{|B|^{p_2/\alpha}}{(T-t)^{p_1 l_2 + p_2 l_1}}\left(\frac{R^3(3N+1)}{6(N+1)^3}\right)^{p_1+p_2}, \\
&\widetilde{v}_t - \Delta\widetilde{v}^n - \widetilde{u}^{q_1}\|\widetilde{v}\|_\beta^{q_2} \\
&\quad \le \frac{l_2 R^3/6}{(T-t)^{l_2+1}} + \frac{nV^{n-1}(r)}{(T-t)^{nl_2+2\sigma}}(NR-(N+1)r) - \frac{|B|^{q_2/\beta}}{(T-t)^{q_1 l_1 + q_2 l_2}}\left(\frac{R^3(3N+1)}{6(N+1)^3}\right)^{q_1+q_2}.
\end{aligned} \tag{4.6}$$

Similarly, if $NR/(N+1) < r \le R$, then

$$\widetilde{u}_t - \Delta\widetilde{u}^m - \widetilde{v}^{p_1}\|\widetilde{u}\|_\alpha^{p_2} \le \frac{l_1 R^3/6}{(T-t)^{l_1+1}} + \frac{mV^{m-1}(r)}{(T-t)^{ml_1+2\sigma}}(NR-(N+1)r), \tag{4.7}$$

$$\widetilde{v}_t - \Delta\widetilde{v}^n - \widetilde{u}^{q_1}\|\widetilde{v}\|_\beta^{q_2} \le \frac{l_2 R^3/6}{(T-t)^{l_2+1}} + \frac{nV^{n-1}(r)}{(T-t)^{nl_2+2\sigma}}(NR-(N+1)r). \tag{4.8}$$

(1) If $m > p_2,\ n > q_2$ and $p_1q_1 > (m-p_2)(n-q_2)$, by Lemma 2.6, there exist two positive constants l_1, l_2 large enough that

$$p_1l_2 + p_2l_1 > ml_1, \quad q_1l_1 + q_2l_2 > nl_2, \quad (m-1)l_1 > 1, \quad (n-1)l_2 > 1. \tag{4.9}$$

Then, we can choose $\sigma > 0$ sufficiently small such that

$$p_1l_2 + p_2l_1 > ml_1 + 2\sigma > ml_1 > l_1 + 1, \qquad q_1l_1 + q_2l_2 > nl_2 + 2\sigma > nl_2 > l_2 + 1. \tag{4.10}$$

Hence, for sufficiently small $T > 0$, (4.6)–(4.8) imply that

$$\tilde{u}_t - \Delta\tilde{u}^m - \tilde{v}^{p_1}\|\tilde{u}\|_\alpha^{p_2} \le 0, \quad \tilde{v}_t - \Delta\tilde{v}^n - \tilde{u}^{q_1}\|\tilde{v}\|_\beta^{q_2} \le 0, \quad (x,t) \in B \times (0,T). \tag{4.11}$$

Since $\psi(0) > 0$ and $\psi(x)$ is continuous, there exist two positive constants ρ and ε such that $\psi(x) \ge \varepsilon$ for all $x \in B(0,\rho) \subset B(0,R)$. Choose T small enough to insure $B(0, RT^\sigma) \subset B(0,\rho) \subset \Omega$, hence $\tilde{u} \le 0, \tilde{v} \le 0$ on $\partial\Omega \times (0,T)$, and from (4.3) it follows that $\tilde{u}(x,0) \le K\psi(x)$, $\tilde{v}(x,0) \le K\psi(x)$ for sufficiently large K. By comparison principle, we have $(\tilde{u},\tilde{v}) \le (u,v)$ provided that $u_0(x) \ge K\psi(x)$ and $v_0(x) \ge K\psi(x)$. It follows that (u,v) blows up in finite time.

(2) Next, we consider the case $m > p_2,\ n > q_2$ and $p_1q_1 = (m-p_2)(n-q_2)$. Clearly, there exist two positive constants l_1, l_2 such that

$$ml_1 = p_1l_2 + p_2l_1, \quad nl_2 = q_1l_1 + q_2l_2, \quad (m-1)l_1 > 1, \quad (n-1)l_2 > 1. \tag{4.12}$$

Denote by $\lambda_{B_R} > 0$ and $\phi_R(r)$ the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem:

$$-\phi''(r) - \frac{N-1}{r}\phi'(r) = \lambda\phi(r), \quad r \in (0,R);\ \phi'(0) = 0,\ \phi(R) = 0. \tag{4.13}$$

It is well known that $\phi_R(r)$ can be normalized as $\phi_R(r) > 0$ in B and $\phi_R(0) = \max_B\phi_R(r) = 1$. By the property (let $\tau = r/R$) of eigenvalues and eigenfunctions we see that $\lambda_{B_R} = R^{-2}\lambda_{B_1}$ and $\phi_R(r) = \phi_1(r/R) = \phi_1(\tau)$, where λ_{B_1} and $\phi_1(\tau)$ are the first eigenvalue and the corresponding normalized eigenfunction of the eigenvalue problem in the unit ball $B_1(0)$. Moreover,

$$\max_{B_1}\phi_1(\tau) = \phi_1(0) = \phi_R(0) = \max_B\phi_R(r) = 1. \tag{4.14}$$

Similar to (4.1), we define the functions $\tilde{u}(x,t), \tilde{v}(x,t)$ in the form

$$\tilde{u}(x,t) = \frac{1}{(T-t)^{l_1}}\phi_R^{l_1}(|x|), \qquad \tilde{v}(x,t) = \frac{1}{(T-t)^{l_2}}\phi_R^{l_2}(|x|). \tag{4.15}$$

In the following, we will prove that $(\tilde{u},\tilde{v})$ blows up in finite time in the ball $B = B(0,R)$. Because of so, $(\tilde{u},\tilde{v})$ does blow up in the larger domain Ω. Calculating directly, we have

$$\begin{aligned}
\tilde{u}_t - \Delta\tilde{u}^m - \tilde{v}^{p_1}\|\tilde{u}\|_\alpha^{p_2} &\le \frac{\phi_R^{l_1}}{(T-t)^{l_1+1}}\left(l_1 - \frac{1}{(T-t)^{ml_1-l_1-1}}(c_1 - \lambda_{B_R} m l_1)\right),\\
\tilde{v}_t - \Delta\tilde{v}^n - \tilde{u}^{q_1}\|\tilde{v}\|_\beta^{q_2} &\le \frac{\phi_R^{l_2}}{(T-t)^{l_2+1}}\left(l_2 - \frac{1}{(T-t)^{nl_2-l_2-1}}(c_2 - \lambda_{B_R} n l_2)\right),
\end{aligned} \tag{4.16}$$

where

$$c_1 = \phi_R^{p_1 l_2}\left\|\phi_R^{l_1}\right\|_\alpha^{p_2} \le K_1 R^{Np_2/\alpha}, \qquad c_2 = \phi_R^{q_1 l_1}\left\|\phi_R^{l_2}\right\|_\beta^{q_2} \le K_2 R^{Nq_2/\beta} \tag{4.17}$$

and K_1, K_2 are constants independent of R. Then, in view of $\lambda_{B_R} = R^{-2}\lambda_{B_1}$, we may assume that R, that is, the ball B, is sufficiently large that

$$\lambda_{B_R} < \min\left\{\frac{c_1}{ml_1}, \frac{c_2}{nl_2}\right\}. \tag{4.18}$$

Hence, for sufficiently small $T > 0$, (4.16) implies that

$$\tilde{u}_t - \Delta\tilde{u}^m - \tilde{v}^{p_1}\|\tilde{u}\|_\alpha^{p_2} \le 0, \qquad \tilde{v}_t - \Delta\tilde{v}^n - \tilde{u}^{q_1}\|\tilde{v}\|_\beta^{q_2} \le 0. \tag{4.19}$$

Therefore, $(\tilde{u},\tilde{v})$ is a positive subsolution of (1.1) in the ball B, which blows up in finite time provided the initial data is sufficiently large that

$$\tilde{u}(x,0) = T^{-l_1}\phi_R^{l_1}(|x|) \le u_0(x), \qquad \tilde{v}(x,0) = T^{-l_2}\phi_R^{l_2}(|x|) \le v_0(x) \tag{4.20}$$

in the ball B.

(3) Finally, if $m \le p_2$ or $n \le q_2$, there also exist two positive constants l_1, l_2 to satisfy (4.9). Similar to the proof of case (1), we can get that $(\tilde{u},\tilde{v})$ is a subsolution of (1.1), which blows up in finite time.

Thus the proof of Theorem 1.2 is completed.

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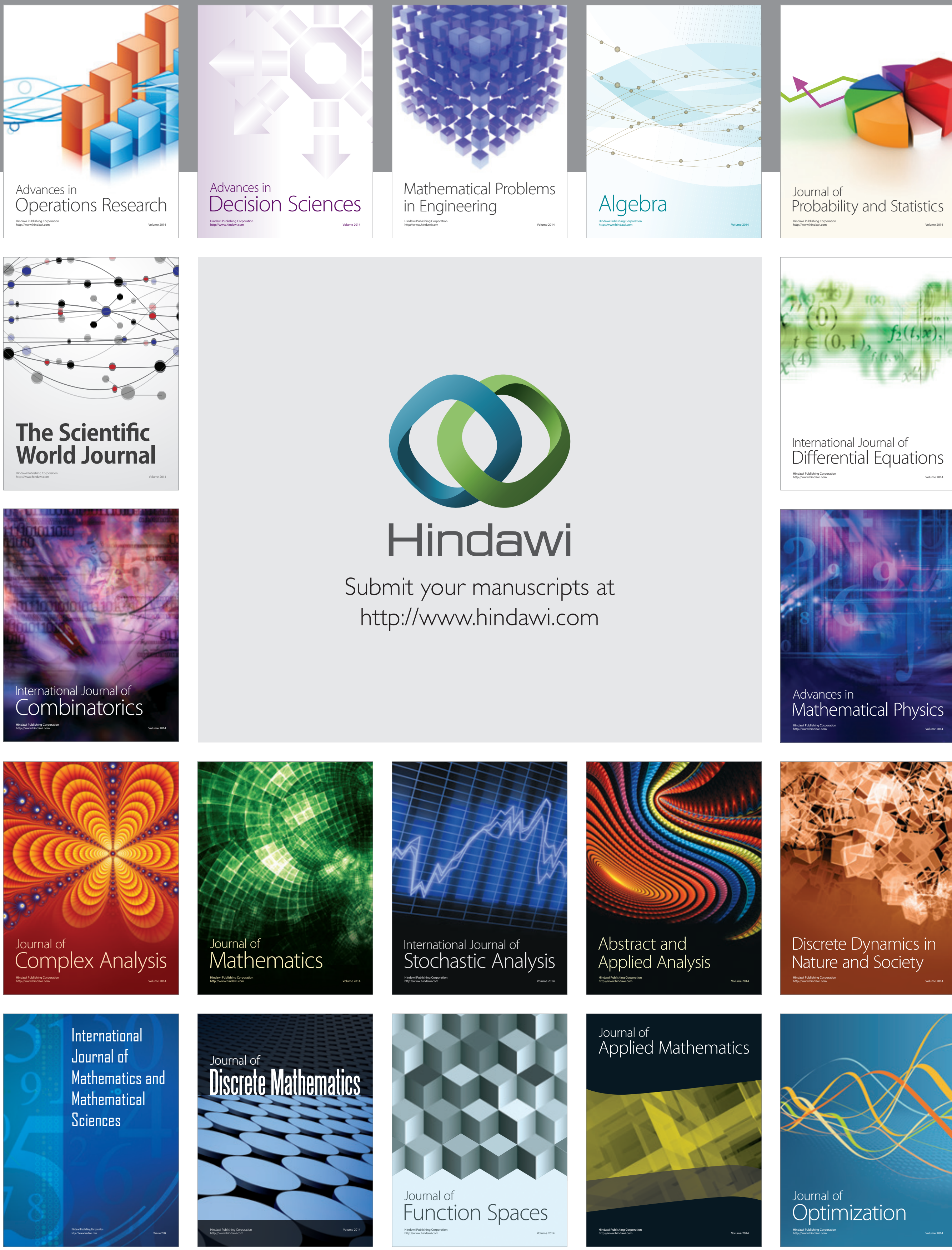

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