Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2013, Article ID 137890, 8 pages
http://dx.doi.org/10.1155/2013/137890

Research Article

Eigenvalue of Fractional Differential Equations with p-Laplacian Operator

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Received 27 December 2012; Accepted 19 February 2013

Academic Editor: Fuyi Xu

We investigate the existence of positive solutions for the fractional order eigenvalue problem with p-Laplacian operator $-\mathscr{D}_t{}^{\beta}(\varphi_p(\mathscr{D}_t{}^{\alpha}x))(t) = \lambda f(t, x(t))$, $t \in (0, 1)$, $x(0) = 0$, $\mathscr{D}_t{}^{\alpha}x(0) = 0$, $\mathscr{D}_t{}^{\gamma}x(1) = \sum_{j=1}^{m-2} a_j \mathscr{D}_t{}^{\gamma}x(\xi_j)$, where $\mathscr{D}_t{}^{\beta}$, $\mathscr{D}_t{}^{\alpha}$, $\mathscr{D}_t{}^{\gamma}$ are the standard Riemann-Liouville derivatives and p-Laplacian operator is defined as $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. $f : (0, 1) \times (0, +\infty) \to [0, +\infty)$ is continuous and f can be singular at $t = 0, 1$ and $x = 0$. By constructing upper and lower solutions, the existence of positive solutions for the eigenvalue problem of fractional differential equation is established.

1. Introduction

Differential equations of fractional order have been recently proved to be valuable tools in the modeling of many phenomena arising from science and engineering, such as viscoelasticity, electrochemistry, control, porous media, and electromagnetism. For detail, see the monographs of Kilbas et al. [1], Miller and Ross [2], and Podlubny [3] and the papers [4–23] and the references therein.

In [16], the authors investigated the nonlinear nonlocal boundary value problem:

$$\begin{gathered} \mathscr{D}_t{}^{\beta}\left(\varphi_p\left(\mathscr{D}_t{}^{\alpha}x\right)\right)(t) + f(t, x(t)) = 0, \quad t \in (0, 1), \\ x(0) = 0, \qquad \mathscr{D}_t{}^{\alpha}x(0) = 0, \qquad x(1) = ax(\xi), \end{gathered} \tag{1}$$

where $0 < \alpha \le 2$, $0 < \beta \le 1$, $0 \le a \le 1$, $0 < \xi < 1$. By using Krasnoselskii's fixed point theorem and the Leggett-Williams theorem, some sufficient conditions for the existence of positive solutions to the above BVP are obtained. In [17], by using the upper and lower solutions method, under suitable monotone conditions, the authors investigated the existence of positive solutions to the following nonlocal problem:

$$\begin{gathered} \mathscr{D}_t{}^{\beta}\left(\varphi_p\left(\mathscr{D}_t{}^{\alpha}x\right)\right)(t) + f(t, x(t)) = 0, \quad t \in (0, 1), \\ x(0) = 0, \qquad \mathscr{D}_t{}^{\alpha}x(0) = 0, \qquad x(1) = ax(\xi), \\ \mathscr{D}_t{}^{\alpha}x(1) = b\mathscr{D}_t{}^{\alpha}x(\eta), \end{gathered} \tag{2}$$

where $0 < \alpha$, $\beta \le 2$, $0 \le a$, $b \le 1$, $0 < \xi$, $\eta < 1$. Recently, by means of the fixed point theorem on cones, Chai [18] investigated two-point boundary value problem of fractional differential equation with p-Laplacian operator:

$$\begin{gathered} \mathscr{D}_t{}^{\beta}\left(\varphi_p\left(\mathscr{D}_t{}^{\alpha}x\right)\right)(t) + f(t, x(t)) = 0, \quad t \in (0, 1), \\ x(0) = 0, \qquad \mathscr{D}_t{}^{\alpha}x(0) = 0, \qquad x(1) + a\mathscr{D}_t{}^{\gamma}x(1) = 0. \end{gathered} \tag{3}$$

Some existence and multiplicity results of positive solutions are obtained.

As far as we know, no result has been obtained for the existence of positive solution for the fractional order eigenvalue problem with p-Laplacian operator:

$$-\mathscr{D}_t{}^{\beta}\left(\varphi_p\left(\mathscr{D}_t{}^{\alpha}x\right)\right)(t)=\lambda f(t,x(t)),\quad t\in(0,1),$$
$$x(0)=0,\qquad \mathscr{D}_t{}^{\alpha}x(0)=0,\tag{4}$$
$$\mathscr{D}_t{}^{\gamma}x(1)=\sum_{j=1}^{m-2}a_j\mathscr{D}_t{}^{\gamma}x\left(\xi_j\right),$$

where $\mathscr{D}_t{}^{\beta}$, $\mathscr{D}_t{}^{\alpha}$, $\mathscr{D}_t{}^{\gamma}$ are the standard Riemann-Liouville derivatives with $1<\alpha\leq 2$, $0<\beta\leq 1$, $0<\gamma\leq 1$, $0\leq\alpha-\gamma-1$, $0<\xi_1<\xi_2<\cdots<\xi_{p-2}<1$, $a_j\in[0,+\infty)$ with $c=\sum_{j=1}^{m-2}a_j\xi_j^{\alpha-\gamma-1}<1$, p-Laplacian operator is defined as $\varphi_p(s)=|s|^{p-2}s$, $p>1$, f can be singular at $t=0,1$, and $x=0$. In order to obtain the existence of positive solutions of the fractional order eigenvalue problem (4), we will apply the upper and lower solutions method associated with the Schauder's fixed point theorem. It is worth emphasizing that the problem (4) not only includes the well-known Sturm-Liouville boundary value problems and the nonlocal boundary value problems as special case, but also f can be singular at $t=0,1$ and $x=0$.

The organization of this paper is as follows. In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. In Section 3, we put forward and prove our main results. Finally, we will give an example to demonstrate our main results.

2. Preliminaries and Lemmas

In this section, we introduce some preliminary facts which are used throughout this paper.

Definition 1 (see [1–3]). The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $x:(0,+\infty)\to\mathbb{R}$ is given by

$$I^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)\,ds\tag{5}$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2 (see [1–3]). The Riemann-Liouville fractional derivative of order $\alpha>0$ of a function $x:(0,+\infty)\to\mathbb{R}$ is given by

$$\mathscr{D}_t{}^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-\alpha-1}x(s)\,ds,\tag{6}$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Proposition 3 (see [1–3]). *(1) If $x\in L^1(0,1)$, $\nu>\sigma>0$, then*

$$I^{\nu}I^{\sigma}x(t)=I^{\nu+\sigma}x(t),\qquad \mathscr{D}_t{}^{\sigma}I^{\nu}x(t)=I^{\nu-\sigma}x(t),$$
$$\mathscr{D}_t{}^{\sigma}I^{\sigma}x(t)=x(t).\tag{7}$$

(2) If $\nu>0$, $\sigma>0$, then

$$\mathscr{D}_t{}^{\nu}t^{\sigma-1}=\frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)}t^{\sigma-\nu-1}.\tag{8}$$

Proposition 4 (see [1–3]). *Let $\alpha>0$, and $f(x)$ is integrable, then*

$$I^{\alpha}\mathscr{D}_t{}^{\alpha}f(x)=f(x)+c_1x^{\alpha-1}+c_2x^{\alpha-2}+\cdots+c_nx^{\alpha-n},\tag{9}$$

where $c_i\in\mathbb{R}$ $(i=1,2,\dots,n)$ and n is the smallest integer greater than or equal to α.

Definition 5. A continuous function $\psi(t)$ is called a lower solution of the BVP (4), if it satisfies

$$-\mathscr{D}_t{}^{\beta}\left(\varphi_p\left(\mathscr{D}_t{}^{\alpha}\psi\right)\right)(t)\leq\lambda f(t,\psi(t)),\quad t\in(0,1),$$
$$\psi(0)\geq 0,\qquad \mathscr{D}_t{}^{\gamma}\psi(1)\geq\sum_{j=1}^{m-2}a_j\mathscr{D}_t{}^{\gamma}\psi\left(\xi_j\right),\tag{10}$$
$$\mathscr{D}_t{}^{\alpha}\psi(0)\geq 0.$$

Definition 6. A continuous function $\phi(t)$ is called an upper solution of the BVP (4), if it satisfies

$$-\mathscr{D}_t{}^{\beta}\left(\varphi_p\left(\mathscr{D}_t{}^{\alpha}\phi\right)\right)(t)\geq\lambda f(t,\phi(t)),\quad t\in(0,1),$$
$$\phi(0)\leq 0,\qquad \mathscr{D}_t{}^{\gamma}\phi(1)\leq\sum_{j=1}^{m-2}a_j\mathscr{D}_t{}^{\gamma}\phi\left(\xi_j\right),\tag{11}$$
$$\mathscr{D}_t{}^{\alpha}\phi(0)\leq 0.$$

For forthcoming analysis, we first consider the following linear fractional differential equation:

$$\mathscr{D}_t{}^{\alpha}x(t)+h(t)=0,\quad t\in(0,1),$$
$$x(0)=0,\qquad \mathscr{D}_t{}^{\gamma}x(1)=\sum_{j=1}^{m-2}a_j\mathscr{D}_t{}^{\gamma}x\left(\xi_j\right).\tag{12}$$

Lemma 7 (see [15]). *If $1<\alpha\leq 2$ and $h\in L^1[0,1]$, then the boundary value problem (12) has the unique solution*

$$x(t)=\int_0^1G(t,s)h(s)\,ds,\tag{13}$$

where

$$G(t,s)=g_1(t,s)+\frac{t^{\alpha-1}}{1-\sum_{j=1}^{m-2}a_j\xi_j^{\alpha-\gamma-1}}\sum_{j=1}^{m-2}a_jg_2\left(\xi_j,s\right)\tag{14}$$

is the Green function of the boundary value problem (12) *and*

$$g_1(t,s)=\begin{cases}\dfrac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1,\\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0\le t\le s\le 1,\end{cases}$$

$$g_2(t,s)=\begin{cases}\dfrac{(t(1-s))^{\alpha-\gamma-1}-(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1,\\ \dfrac{(t(1-s))^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0\le t\le s\le 1.\end{cases} \quad (15)$$

Lemma 8. *The Green function $G(t,s)$ in Lemma 7 has the following properties:*

(i) *$G(t,s)$ is continuous on $[0,1]\times[0,1]$;*

(ii) *$G(t,s)>0$ for any $s,t\in(0,1)$;*

(iii) *$t^{\alpha-1}\sigma_1(s)\le G(t,s)\le t^{\alpha-1}\sigma_2(s)$, for $t,s\in[0,1]$, where*

$$\sigma_1(s)=\frac{\sum_{j=1}^{m-2}a_j g_2(\xi_j,s)}{1-\sum_{j=1}^{m-2}a_j\xi_j^{\alpha-\gamma-1}},$$
$$\sigma_2(s)=\frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}+\sigma_1(s). \quad (16)$$

Let $q>1$ satisfy the relation $1/q+1/p=1$, where p is given by (4). To study BVP (4), we first consider the associated linear BVP:

$$\mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}x\right)\right)(t)+h(t)=0,\quad t\in(0,1),$$
$$x(0)=0,\qquad \mathscr{D}_t^{\alpha}x(0)=0,$$
$$\mathscr{D}_t^{\gamma}x(1)=\sum_{j=1}^{m-2}a_j\mathscr{D}_t^{\gamma}x(\xi_j), \quad (17)$$

for $h\in L^1[0,1]$ and $h\ge 0$. For convenience, let

$$b=(\Gamma(\beta))^{1-q}, \quad (18)$$

then we have the following lemma.

Lemma 9. *The associated linear BVP* (17) *has the unique positive solution*

$$x(t)=b\int_0^1 G(t,s)\left(\int_0^s(s-\tau)^{\beta-1}h(\tau)\,d\tau\right)^{q-1}ds. \quad (19)$$

Proof. In fact, let $w=\mathscr{D}_t^{\alpha}x,\ v=\varphi_p(w)$. By Proposition 4, the solution of initial value problem

$$\mathscr{D}_t^{\beta}v(t)+h(t)=0,\quad t\in(0,1),$$
$$v(0)=0 \quad (20)$$

is given by $v(t)=C_1t^{\beta-1}-I^{\beta}h(t),\ t\in[0,1]$. From the relations $v(0)=0,\ 0<\beta\le 1$, it follows that $C_1=0$, and so

$$v(t)=-I^{\beta}h(t),\quad t\in[0,1]. \quad (21)$$

Noting that $\mathscr{D}_t^{\alpha}x=w,\ w=\varphi_p^{-1}(v)$, it follows from (21) that the solution of (17) satisfies

$$\mathscr{D}_t^{\alpha}x(t)=\varphi_p^{-1}\left(-I^{\beta}h(t)\right),\quad t\in(0,1),$$
$$x(0)=0,\qquad \mathscr{D}_t^{\gamma}x(1)=\sum_{j=1}^{m-2}a_j\mathscr{D}_t^{\gamma}x(\xi_j). \quad (22)$$

By Lemma 7, the solution of (22) can be written as

$$x(t)=-\int_0^1 G(t,s)\varphi_p^{-1}\left(-I^{\beta}h(s)\right)ds,\quad t\in[0,1]. \quad (23)$$

Since $h(s)\ge 0,\ s\in[0,1]$, we have $\varphi_p^{-1}(-I^{\beta}h(s))=-(I^{\beta}h(s))^{q-1}$, $s\in[0,1]$, which implies that the solution of (22) is given by

$$x(t)=b\int_0^1 G(t,s)\left(\int_0^s(s-\tau)^{\beta-1}h(\tau)\,d\tau\right)^{q-1}ds,$$
$$t\in[0,1]. \quad (24)$$

□

The following lemma is a straightforward conclusion of Lemma 9.

Lemma 10. *If $x\in C([0,1],R)$ satisfies*

$$x(0)=0,\qquad \mathscr{D}_t^{\gamma}x(1)=\sum_{j=1}^{m-2}a_j\mathscr{D}_t^{\gamma}x(\xi_j), \quad (25)$$

and $-\mathscr{D}_t^{\alpha}x(t)\ge 0$ for any $t\in(0,1)$, then $x(t)\ge 0$, for $t\in[0,1]$.

3. Main Results

Set

$$e(t)=t^{\alpha-1}. \quad (26)$$

We present the following two assumptions.

(H1) $f:((0,1)\times(0,\infty)\to[0,+\infty))$ is continuous and decreasing in x.

(H2) For any $\mu>0,\ f(t,\mu)\not\equiv 0$, and

$$0<\int_0^1\sigma_2(s)\left(\int_0^s(s-\tau)^{\beta-1}f(\tau,\mu e(\tau))\,d\tau\right)^{q-1}ds<+\infty. \quad (27)$$

Let $E=C[0,1]$, and

$$P=\{y\in E:\ \text{there exist positive numbers}$$
$$0<l_x<1,\ L_x>1 \text{ such that } l_xe(t)\le x(t)\le L_xe(t),$$
$$t\in[0,1]\}. \quad (28)$$

Clearly, $e(t) \in P$, so P is nonempty. For any $x \in P$, define an operator T by

$$(T_\lambda x)(t) = \lambda b \int_0^1 G(t,s) \times \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau))\, d\tau \right)^{q-1} ds, \quad t \in [0,1]. \tag{29}$$

Theorem 11. *Suppose conditions* $(H1)$ *and* $(H2)$ *hold. In addition, if the following condition* $(S1)$ *holds:*

$(S1)$

$$\lim_{\kappa \to +\infty} \kappa^{1/(q-1)} f(t, \kappa x) = +\infty, \tag{30}$$

for $(t,x) \in (0,1) \times (0,\infty)$ *uniformly holds. Then there exists a constant* $\lambda^* > 0$ *such that the BVP* (4) *has at least one positive solution* w *for any* $\lambda \in (\lambda^*, +\infty)$*, and there exists one positive constant* $n > 1$ *such that*

$$e(t) \le w(t) \le n e(t), \quad t \in [0,1]. \tag{31}$$

Proof. The proof is divided into four steps.

Step 1. We show that T_λ is well defined on P and $T_\lambda(P) \subset P$, and T_λ is decreasing in x.

In fact, for any $x \in P$, by the definition of P, there exists two positive numbers $0 < l_x < 1$, $L_x > 1$ such that $l_x e(t) \le x(t) \le L_x e(t)$ for any $t \in [0,1]$. It follows from Lemma 8 and (H1)-(H2) that

$$\begin{aligned}(T_\lambda x)(t) &= \lambda b \int_0^1 G(t,s) \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau))\, d\tau \right)^{q-1} ds \\ &\le \lambda b e(t) \int_0^1 \sigma_2(s) \times \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, l_x e(\tau))\, d\tau \right)^{q-1} ds \\ &< +\infty.\end{aligned} \tag{32}$$

Now take $c = \max_{t\in[0,1]} x(t)$, by (H2), for any $s \in (0,1)$, $f(s,c) \not\equiv 0$. Thus by the continuity of $f(t,x)$ and Lemma 8 and (32), we have

$$\begin{aligned}(T_\lambda x)(t) &\ge \lambda b e(t) \int_0^1 \sigma_1(s) \times \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau))\, d\tau \right)^{q-1} ds \\ &\ge \lambda b e(t) \int_0^1 \sigma_1(s) \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, c)\, d\tau \right)^{q-1} ds \\ &> 0, \quad t \in (0,1).\end{aligned} \tag{33}$$

Take

$$\begin{aligned} l'_x &= \min\left\{1, \lambda b \int_0^1 \sigma_1(s) \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, c)\, d\tau \right)^{q-1} ds \right\}, \\ L'_x &= \max\left\{1, \lambda b \times \int_0^1 \sigma_2(s) \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, l_x e(\tau))\, d\tau \right)^{q-1} ds \right\}, \end{aligned} \tag{34}$$

then by (32) and (33),

$$l'_x e(t) \le (T_\lambda x)(t) \le L'_x e(t), \tag{35}$$

which implies that T_λ is well defined and $T_\lambda(P) \subset P$. And the operator T_λ is decreasing in x from (H1). Moreover, by direct computations, we also have

$$\begin{aligned} &-\mathscr{D}_t^{\beta} \left(\varphi_p \left(\mathscr{D}_t^{\alpha} (T_\lambda x) \right) \right)(t) = \lambda f(t, x(t)), \quad t \in (0,1), \\ &(T_\lambda x)(0) = 0, \qquad \mathscr{D}_t^{\alpha} (T_\lambda x)(0) = 0, \\ &\mathscr{D}_t^{\gamma} (T_\lambda x)(1) = \sum_{j=1}^{m-2} a_j \mathscr{D}_t^{\gamma} (T_\lambda x)(\xi_j). \end{aligned} \tag{36}$$

Step 2. In this step, we will focus on lower and upper solutions of the fractional boundary value problem (4).

By Lemma 8, we have

$$\begin{aligned} b \int_0^1 G(t,s) &\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, e(\tau))\, d\tau \right)^{q-1} ds \\ &\ge e(t)\, b \int_0^1 \sigma_1(s) \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, e(\tau)) d\tau \right)^{q-1} ds, \quad \forall t \in [0,1]. \end{aligned} \tag{37}$$

Let

$$\mu = \left(b \int_0^1 \sigma_1(s) \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, e(\tau)) d\tau \right)^{q-1} ds \right)^{-1}; \tag{38}$$

it follows from (37) that

$$\mu b \int_0^1 G(t,s) \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, e(\tau))\, d\tau \right)^{q-1} ds \ge e(t), \quad \forall t \in [0,1]. \tag{39}$$

On the other hand, take

$$\nu(t) = b \int_0^1 G(t,s) \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, e(\tau))\, d\tau \right)^{q-1} ds; \tag{40}$$

then by monotonicity of f in x and (37)–(40), for any $\lambda > \mu$, we have

$$\begin{aligned}\int_0^1 G(t,s)&\left(\int_0^s (s-\tau)^{\beta-1} f(\tau,\lambda\nu(\tau))\,d\tau\right)^{q-1} ds \\ &\le \int_0^1 G(t,s)\left(\int_0^s (s-\tau)^{\beta-1} f(\tau,\mu\nu(\tau))\,d\tau\right)^{q-1} ds \\ &\le \int_0^1 \sigma_2(s)\left(\int_0^s (s-\tau)^{\beta-1} f(\tau,e(\tau))\,d\tau\right)^{q-1} ds \\ &< +\infty.\end{aligned} \tag{41}$$

From (S1), we have

$$\lim_{\kappa\to+\infty} \kappa^{1/(q-1)} f(t,\kappa x) = +\infty, \tag{42}$$

uniformly on $(t,x) \in (0,1)\times(0,\infty)$. Thus there exists large enough $\lambda^* > \mu > 0$, such that, for any $t \in (0,1)$,

$$\lambda^{*\,1/(q-1)} f(s,\lambda^* e(s)) \ge \frac{\beta}{b}\left(\int_0^1 \sigma_1(s)\, s^{\beta(q-1)} ds\right)^{-1/(q-1)}, \tag{43}$$

which yields

$$\begin{aligned}\lambda^* b\int_0^1 G(t,s)&\left(\int_0^s (s-\tau)^{\beta-1} f(\tau,\lambda^* e(\tau))\,d\tau\right)^{q-1} ds \\ &\ge \beta\left(\int_0^1 \sigma_1(s)\, s^{\beta(q-1)} ds\right)^{-1/(q-1)} \\ &\quad\times \int_0^1 G(t,s)\left(\int_0^s (s-\tau)^{\beta-1} d\tau\right)^{q-1} ds \\ &\ge \beta\left(\int_0^1 \sigma_1(s)\, s^{\beta(q-1)} ds\right)^{-1/(q-1)} \\ &\quad\times \int_0^1 \sigma_1(s)\left(\int_0^s (s-\tau)^{\beta-1} d\tau\right)^{q-1} ds e(t) = e(t), \\ &\qquad \forall t \in [0,1].\end{aligned} \tag{44}$$

Letting

$$\begin{aligned}\phi(t) &= \lambda^* b\int_0^1 G(t,s)\left(\int_0^s (s-\tau)^{\beta-1} f(\tau,e(\tau))\,d\tau\right)^{q-1} ds \\ &= \lambda^*\nu(t) = T_{\lambda^*}e, \\ \psi(t) &= \lambda^* b\int_0^1 G(t,s)\left(\int_0^s (s-\tau)^{\beta-1} f(\tau,\lambda^*\nu(\tau))\,d\tau\right)^{q-1} ds \\ &= T_{\lambda^*}\phi,\end{aligned} \tag{45}$$

and by Lemma 9, (39), (44), and (45), one has

$$\begin{gathered}\phi(t) = \lambda^* b\int_0^1 G(t,s)\left(\int_0^s (s-\tau)^{\beta-1} f(\tau,e(\tau))d\tau\right)^{q-1} ds \\ \ge e(t), \\ \phi(0) = 0, \qquad \mathscr{D}_t^{\phi}\phi(0) = 0, \\ \mathscr{D}_t^{\gamma}(\phi x)(1) = \sum_{j=1}^{m-2} a_j \mathscr{D}_t^{\gamma}\phi(\xi_j),\end{gathered} \tag{46}$$

$$\begin{gathered}\psi(t) = \lambda^* b\int_0^1 G(t,s)\left(\int_0^s (s-\tau)^{\beta-1} f(\tau,\lambda^*\nu(\tau))\,d\tau\right)^{q-1} ds \\ \ge e(t), \\ \psi(0) = 0, \qquad \mathscr{D}_t^{\phi}\psi(0) = 0, \\ \mathscr{D}_t^{\gamma}(\psi x)(1) = \sum_{j=1}^{m-2} a_j \mathscr{D}_t^{\gamma}\psi(\xi_j).\end{gathered} \tag{47}$$

By Step 1 and (46), (47), we know $\phi(t), \psi(t) \in P$. And it follows from (45)–(47) that

$$e(t) \le \psi(t) \le \phi(t), \quad \forall t \in [0,1]. \tag{48}$$

Consequently, it follows from (44)–(48) that

$$\begin{aligned}\mathscr{D}_t^{\beta}&\left(\varphi_p\left(\mathscr{D}_t^{\alpha}\psi\right)\right)(t) + \lambda^* f(t,\psi(t)) \\ &= \mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}\left(T_{\lambda^*}\phi\right)\right)\right)(t) + \lambda^* f(t,\psi(t)) \\ &= -\lambda^* f(t,\phi(t)) + \lambda^* f(t,\psi(t)) \ge 0, \\ \mathscr{D}_t^{\beta}&\left(\varphi_p\left(\mathscr{D}_t^{\alpha}\phi\right)\right)(t) + \lambda^* f(t,\phi(t)) \\ &= \mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}\left(T_{\lambda^*}e\right)\right)\right)(t) + \lambda^* f(t,\phi(t)) \\ &= -\lambda^* f(t,e(t)) + \lambda^* f(t,\phi(t)) \le 0;\end{aligned} \tag{49}$$

that is, $\phi(t)$ and $\psi(t)$ are a couple of lower and upper solutions of fractional boundary value problem (4) by (46)–(49), respectively.

Step 3. Let

$$F(t,x) = \begin{cases} f(t,\psi(t)), & x < \psi(t), \\ f(t,x(t)), & \psi(t) \le x \le \phi(t), \\ f(t,\phi(t)), & x > \phi(t). \end{cases} \tag{50}$$

It follows from (H1) and (46) that $F : (0,1)\times[0,+\infty) \to [0,+\infty)$ is continuous.

We will show that the fractional boundary value problem

$$-\mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}x\right)\right)(t)=\lambda^* F(t,x(t)),\quad t\in(0,1),$$
$$x(0)=0,\qquad \mathscr{D}_t^{\alpha}x(0)=0,$$
$$\mathscr{D}_t^{\gamma}x(1)=\sum_{j=1}^{m-2}a_j\mathscr{D}_t^{\gamma}x\left(\xi_j\right) \tag{51}$$

has a positive solution.

To see this, we consider the operator $A_{\lambda^*}: C[0,1]\rightarrow C[0,1]$ defined as follows:

$$\left(A_{\lambda^*}x\right)(t)=\lambda^* b\int_0^1 G(t,s)\times\left(\int_0^s (s-\tau)^{\beta-1}F(\tau,x(\tau))\,d\tau\right)^{q-1}ds,\quad t\in[0,1]. \tag{52}$$

Obviously, a fixed point of the operator A_{λ^*} is a solution of the BVP (51). Noting that $\phi\in P$, then there exists a constant $0<l_\phi<1$ such that $\phi(t)\ge l_\phi e(t)$, $t\in[0,1]$. Thus for all $x\in E$, it follows from Lemma 8, (50), and (H2) that

$$\begin{aligned}\left(A_{\lambda^*}x\right)(t)&\le\lambda^* b\int_0^1\sigma_2(s)\left(\int_0^s(s-\tau)^{\beta-1}F(\tau,x(\tau))\,d\tau\right)^{q-1}ds\\&\le\lambda^* b\int_0^1\sigma_2(s)\left(\int_0^s(s-\tau)^{\beta-1}f(\tau,\phi(\tau))\,d\tau\right)^{q-1}ds\\&\le\lambda^* b\int_0^1\sigma_2(s)\left(\int_0^s(s-\tau)^{\beta-1}f\left(\tau,l_\phi e(\tau)\right)d\tau\right)^{q-1}ds\\&<+\infty,\end{aligned} \tag{53}$$

which implies that the operator A_{λ^*} is uniformly bounded.

From the uniform continuity of $G(t,s)$ and the Lebesgue dominated convergence theorem, we easily obtain that A is equicontinuous. Thus by the means of the Arzela-Ascoli theorem, we have that $A_{\lambda^*}: E\rightarrow E$ is completely continuous. The Schauder fixed point theorem implies that A_{λ^*} has at least a fixed point w such that $w=A_{\lambda^*}w$.

Step 4. We will prove that the boundary value problem (4) has at least one positive solution.

In fact, we only need to prove that

$$\psi(t)\le w(t)\le\phi(t),\qquad t\in[0,1]. \tag{54}$$

By (46), (47) and noticing that w is fixed point of A_{λ^*}, we know that

$$\begin{aligned}&\phi(0)=0,\qquad \mathscr{D}_t^{\gamma}\phi(1)=\sum_{j=1}^{m-2}a_j\mathscr{D}_t^{\gamma}\phi\left(\xi_j\right),\\&\mathscr{D}_t^{\alpha}\phi(0)=0,\\&w(0)=0,\qquad \mathscr{D}_t^{\gamma}w(1)=\sum_{j=1}^{m-2}a_j\mathscr{D}_t^{\gamma}w\left(\xi_j\right),\\&\mathscr{D}_t^{\alpha}w(0)=0.\end{aligned} \tag{55}$$

Notice that the definition of F and the function $f(t,x)$ is nonincreasing in x, we obtain

$$f(t,\phi(t))\le F(t,x(t))\le f(t,\psi(t)),\quad \forall x\in E. \tag{56}$$

So by (48) and (56),

$$f(t,\phi(t))\le F(t,x(t))\le f(t,e(t)),\quad \forall x\in E. \tag{57}$$

Thus one has by (57)

$$\begin{aligned}&\mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}\phi\right)\right)(t)-\mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}w\right)\right)(t)\\&=\mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}\phi\right)-\varphi_p\left(\mathscr{D}_t^{\alpha}w\right)\right)(t)\\&=-\lambda^* f(t,e(t))+\lambda^* F(t,w(t))\le 0,\\&\forall t\in[0,1].\end{aligned} \tag{58}$$

Let $z(t)=\varphi_p(\mathscr{D}_t^{\alpha}\phi(t))-\varphi_p(\mathscr{D}_t^{\alpha}w(t))$; then

$$\mathscr{D}_t^{\beta}z(t)\le 0,\quad t\in[0,1], \tag{59}$$

and (55) implies that $z(0)=0$. It follows from (21) that

$$z(t)\le 0, \tag{60}$$

and then

$$\varphi_p\left(\mathscr{D}_t^{\alpha}\phi(t)\right)-\varphi_p\left(\mathscr{D}_t^{\alpha}w(t)\right)\le 0. \tag{61}$$

Notice that φ_p is monotone increasing; we have

$$\mathscr{D}_t^{\alpha}\phi(t)\le\mathscr{D}_t^{\alpha}w(t),\quad \text{that is, }\mathscr{D}_t^{\alpha}(\phi-w)(t)\le 0. \tag{62}$$

It follows from Lemma 10 and (55) that

$$\phi(t)-w(t)\ge 0. \tag{63}$$

Thus we have $w(t)\le\phi(t)$ on $[0,1]$. By the same way, we also have $w(t)\ge\psi(t)$ on $[0,1]$. So

$$\psi(t)\le w(t)\le\phi(t),\quad t\in[0,1]. \tag{64}$$

Consequently, $F(t,w(t))=f(t,w(t))$, $t\in[0,1]$. Then $w(t)$ is a positive solution of the problem (4).

Finally, by (48) and (64) and $\phi\in P$, we have

$$e(t)\le\psi(t)\le w(t)\le\phi(t)\le l_\phi e(t)=ne(t), \tag{65}$$

where

$$n=l_\phi>1. \tag{66}$$

□

In the end of this work we also remark the above results to the problem (4) with which $f(t,x)$ is nonsingular at $x=0$ and $t=0,1$; that is, we have the following result.

Theorem 12. *If $f(t,x):[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous, decreasing in x and $f(t,\mu)\not\equiv 0$, for any $\mu>0$, then the boundary value problem* (4) *has at least one positive solution $w(t)$ for any $\lambda>0$, and there exists a constant $n>1$ such that*

$$e(t)\le w(t)\le ne(t). \tag{67}$$

Proof. The proof is similar to Theorem 11; we omit it here. □

Example 13. Consider the following boundary value problem:

$$\begin{gathered}-\mathscr{D}_t^{4/3}\left(\varphi_2\left(\mathscr{D}_t^{3/2}x\right)\right)(t)=\frac{1}{\sqrt{t^{1/2}x(t)}},\quad t\in(0,1),\\ x(0)=0,\qquad \mathscr{D}_t^{1/6}x(1)=\frac{1}{8}\mathscr{D}_t^{1/6}x\left(\frac{1}{4}\right)+\frac{1}{3}\mathscr{D}_t^{1/6}x\left(\frac{3}{4}\right),\\ \mathscr{D}_t^{3/2}x(0)=0.\end{gathered}\tag{68}$$

Let $\alpha=3/2$, $\beta=4/3$, $\gamma=1/6$, $p=2$, and

$$f(t,x)=\frac{1}{\sqrt{xt^{1/2}}}. \tag{69}$$

Firstly,

$$c=\sum_{j=1}^{p-2}a_j\xi_j^{\alpha-\gamma-1}=\frac{1}{8}\left(\frac{1}{4}\right)^{1/3}+\frac{1}{3}\left(\frac{3}{4}\right)^{1/3}=0.3816<1. \tag{70}$$

And, it is easy to check that (H1) holds. For any $\mu>0$, $f(t,\mu)\not\equiv 0$ and

$$\begin{aligned}0&<\int_0^1\sigma_2(s)\left(\int_0^s(s-\tau)^{\beta-1}f(\tau,\mu e(\tau))\,d\tau\right)^{q-1}ds\\ &=\int_0^1\sigma_2(s)\int_0^s(s-\tau)^{1/3}\tau^{-1/2}\mu^{-1/2}d\tau\,ds\\ &=\int_0^1 s^{5/6}\sigma_2(s)\int_0^1(1-\tau)^{1/3}\tau^{-1/2}\mu^{-1/2}d\tau\,ds\\ &=\mu^{-1/2}\int_0^1 s^{5/6}\sigma_2(s)\,ds\,B\left(\frac{4}{3},\frac{1}{2}\right)<+\infty,\end{aligned}\tag{71}$$

which implies that (H2) holds.

On the other hand,

$$\lim_{\kappa\to+\infty}\kappa^{1/(q-1)}f(t,\kappa x)=\lim_{\kappa\to+\infty}\kappa\frac{1}{\sqrt{\kappa x t^{1/2}}}=+\infty. \tag{72}$$

Thus (S1) also holds.

By Theorem 11, the boundary value problem (68) has at least one positive solution.

Acknowledgments

This work was supported by the Natural Sciences of Education and the Science Office Bureau of Sichuan Province of China, under Grants nos. 10ZC060, 2010JY0J41.

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