Research Article

Existence of Multiple Positive Periodic Solutions to Two Species Parasitical Model with Impulsive Effects and Harvesting Terms

Zhouhong Li

Department of Mathematics, Yuxi Normal University, Yuxi, Yunnan 653100, China

Correspondence should be addressed to Zhouhong Li; zhouhli@yeah.net

By applying Mawhin's continuation theorem of coincidence degree theory and some skills of inequalities, we establish the existence of four positive solutions for two species parasitical system with impulsive effects and harvesting terms. Finally, an example is given to illustrate the effectiveness of our results.

1. Introduction

In recent years, the existence of periodic solutions in biological models has been widely studied. Models with harvesting terms are often considered. Generally, the model with harvesting terms is described as follows:

$$\dot{x}_1 = x_1 f(x_1, x_2) - h, \qquad \dot{x}_2 = x_2 g(x_1, x_2) - k, \quad (1)$$

where x_1 and x_2 are functions of two species, respectively; h and k are harvesting terms standing for the harvests (see [1, 2]). Because of the effect of changing environment such as the weather, season, and food, the number of species population periodically varies with the time. The rate of change usually is not a constant. Motivated by this, we consider the periodic nonautonomous population models. For example, two species parasitical system with harvesting terms is as follows [3]:

$$\dot{x} = x(t)(a_1(t) - b_1(t)x(t)) - h_1(t),$$
$$\dot{y} = y(t)(a_2(t) - b_2(t)y(t) + c(t)x(t)) - h_2(t), \quad (2)$$

where $x(t)$ and $y(t)$ denote the densities of the host and the parasites, respectively; $a_i(t)$, $b_i(t)$, $c(t)$, and $h_i(t)$ $(i = 1, 2)$ are all positive continuous functions and denote the intrinsic growth rate, death rate, obtaining nutriment rate from the host, and harvesting rate, respectively. In the model (2), the parasitical influence on its host is negligible. As we know, in population dynamics, many evolutionary processes experience short-time rapid chance after undergoing relatively

long sooth variation. Examples include stocking of species and annual immigration. Incorporating these phenomena gives us impulsive differential equations. For the theory of impulsive differential equations, we refer the reader to [4, 5].

However, to the best of our knowledge, there are few results on the existence of multiple periodic solutions for the delay parasitical with impulsive effects in the literatures. This motivates us to consider the existence of multiple periodic solutions for following parasitical with impulsive effects and harvesting terms nonautonomous model:

$$\dot{x}_1(t) = x_1(t)(a_1(t) - b_1(t)x_1(t - \tau_{11}(t)))$$
$$- h_1(t), \quad t \neq t_k,$$
$$\dot{x}_2(t) = x_2(t)(a_2(t) - b_2(t)x_2(t - \tau_{22}(t))$$
$$+ c(t)x_1(t - \tau_{21}(t))) - h_2(t), \quad t \neq t_k, \quad (3)$$
$$\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = g_{ik}x_i(t_k), \quad t = t_k,$$

where $g_{ik} \in (-1, +\infty)$ $(i = 1, 2; k \in \mathbb{N} = \{1, 2\})$. $\{t_k\}_{k \in \mathbb{N}}$ is a strictly increasing sequence with $t_1 > 0$ and $\lim_{t \to \infty} t_k = \infty$. $x_i(t)$ $(i = 1, 2)$ is the ith species population density. $a_i(t) > 0$ $(i = 1, 2)$ denotes the intrinsic growth rate; $b_i(t) > 0$ and $h_i(t) > 0$ $(i = 1, 2)$ stand for death rate, obtaining nutriment rate from the host, and harvesting rate, respectively. $c(t) > 0$ represents obtaining nutriment rate from the host; $\tau_{21}(t) \geq 0$ stands for the time-lag in the process of transformation from the 1th species to the 2th species.

$\tau_{ii}(t) \geq 0$ $(i = 1, 2)$ represents the time-lag in the process of intraspecific competition.

In addition, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Therefore, the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (e.g, seasonal effects of weather, food supplies, mating habits, etc.), which leads us to assume that $a_i(t)$, $b_i(t)$, $c(t)$, $\tau_{ii}(t)$, $\tau_{21}(t)$, and $h_i(t)$ $(i = 1, 2)$ are all continuous ω-periodic functions. For impulsive effects, we further assume that there exists a $q \in \mathbb{N}$ such that $g_{i(k+q)} = g_{ik}$ $(i = 1, 2)$ and $t_{k+q} = t_k + \omega$.

Since a very basic and important problem in the study of a population growth model with a periodic environment is the global existence and stability of a positive periodic solution, which plays a similar role as a globally stable equilibrium does in an autonomous model, also, on the existence of positive periodic solutions to system (3), few results are found in the literatures. This motivates us to investigate the existence of a positive periodic or multiple positive periodic solutions for system (3). In fact, it is more likely for some biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena. Therefore, it is essential for us to investigate the existence of multiple positive periodic solutions for population models. Our main aim of this paper is by using Mawhin's continuation theorem of coincidence degree theory to establish the existence of four positive periodic solutions for system (3). For the work concerning the multiple existence of periodic solutions of periodic population models which was done using coincidence degree theory, we refer the reader to [6–8].

This paper is organized as follows. In Section 2, by using the continuation theorem of coincidence degree theory and the skills of inequalities, we establish the existence of at least four positive periodic solutions of system (3). An example is presented in the last section to illustrate the effectiveness of our results.

2. Existence of at Least Four Positive Periodic Solutions

We first summarize a few concepts from the book by Gaines and Mawhin [9].

Let X and Z be real normed vector spaces. Let L : Dom $L \subset X \to Z$ be a linear mapping and $N : X \times [0, 1] \to Z$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if dim Ker L = codim Im $L < \infty$, and Im L is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that Im P = Ker L and Ker Q = Im L = Im$(I - Q)$, and X = Ker $L \oplus$ Ker P, Z = Im $L \oplus$ Im Q. It follows that $L|_{\text{Dom } L \cap \text{Ker } P} : (I - P)X \to$ Im L is invertible, and its inverse is denoted by K_P. If Ω is a bounded open subset of X, the mapping N is called L-compact on $\overline{\Omega} \times [0, 1]$, and if $QN(\overline{\Omega} \times [0, 1])$ is bounded, $K_P(I - Q)N : \overline{\Omega} \times [0, 1] \to X$ is compact. Because Im Q is isomorphic to Ker L, there exists an isomorphism $J :$ Im $Q \to$ Ker L.

Lemma 1 (see [9]). *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\overline{\Omega} \times [0, 1]$. Assume*

  (a) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda N(x, \lambda)$ is such that $x \notin \partial\Omega \cap$ Dom L;*

  (b) *$QN(x, 0)x \neq 0$ for each $x \in \partial\Omega \cap$ Ker L;*

  (c) *$\deg(JQN(x, 0), \Omega \cap$ Ker $L, 0) \neq 0$.*

Then $Lx = N(x, 1)$ has at least one solution in $\overline{\Omega} \cap$ Dom L.

Let T be a given positive constant, and a finite number of points of the sequence $\{\tau_k\}$ lies in the interval $[0, T]$. Let $PC([0, T], \mathbb{R}^n)$ be the set of functions $x : [0, T] \to \mathbb{R}^n$ which are piecewise continuous in $[0, T]$ and have points of discontinuous $\tau_k \in [0, T]$, where they are continuous from the left. In the set $PC([0, T], \mathbb{R}^n)$ introduce the norm $\|x\|$ = $\sup |x(t)| : t \in [0, T]$ with which $PC([0, T], \mathbb{R}^n)$ becomes a Banach space with the uniform convergence topology.

Definition 2. The set $\mathscr{F} \subset PC([0, T], \mathbb{R}^n)$ is said to be quasiequicontinuous in $[0, T]$, if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in \mathscr{F}$; $k \in \mathbb{Z}$; $t_1, t_2 \in (\tau_{k-1}, \tau_k] \cap [0, T]$ and $|t_1 - t_2| < \delta$, then $|x(t_1) - x(t_2)| < \epsilon$.

The following result called compactness criterion gives a necessary and sufficient condition for relative compactness in $PC([0, T], \mathbb{R}^n)$.

Lemma 3 (see [4]). *The set $\mathscr{F} \subset PC([0, T], \mathbb{R}^n)$ is relatively compact if and only if*

  (a) *$\mathscr{F}$ is bounded, namely, $\|x\| \leq c$, for each $x \in \mathscr{F}$ and some $c > 0$;*

  (b) *$\mathscr{F}$ is quasiequicontinuous in $[0, T]$.*

For the sake of convenience, we denote $\overline{f} = (1/\omega) \int_0^\omega f(t) dt$; here $f(t)$ is a continuous ω-periodic function. Let PC_ω denote the space of ω-periodic functions $\psi : \mathbb{R} \to \mathbb{R}$ which are continuous for $t \neq t_k$, are continuous from the left for $t \in \mathbb{R}$, and have possible discontinuities of the first kind at points $t = t_k$; that is, the limit from the right of t_k exists but may be different from the value at t_k. We also denote $PC_\omega^1 = \{\psi \in PC_\omega : \dot{\psi} \in PC_\omega\}$. Obviously, ψ is continuous if $\psi \in PC_\omega^1$.

For simplicity, we need to introduce some notations as follows:

$$g_i = \frac{1}{2} \sum_{k=1}^q \ln\left(1 + g_{ik}\right),$$

$$G_i = \frac{1}{2} \sum_{k=1}^q \left[\ln\left(1 + g_{ik}\right) + \left|\ln\left(1 + g_{ik}\right)\right|\right],$$

$$\rho_i = e^{\overline{a}_i \omega + G_i},$$

$$l_1^\pm = \frac{\left(\overline{a}_1 + 2g_1/\omega\right)\rho_1 \pm \sqrt{\left[\left(\overline{a}_1 + 2g_1/\omega\right)\rho_1\right]^2 - 4\overline{b}_1\overline{h}_1\rho_1}}{2\overline{b}_1},$$

$$l_2^\pm = \left(\left(\overline{c}l_1^+ + a_2 + \frac{2g_2}{\omega} \right) \rho_2 \right.$$

$$\left. \pm \sqrt{ \left[\left(\overline{c}l_1^+ + a_2 + \frac{2g_2}{\omega} \right) \rho_2 \right]^2 - 4\overline{b}_2\overline{h}_2\rho_2 } \right) (2\overline{b}_2)^{-1},$$

$$A_i^\pm = \frac{(\overline{a}_i + 2g_i/\omega) \pm \sqrt{(\overline{a}_i + 2g_i/\omega)^2 - 4\overline{b}_i\overline{h}_i\rho_i}}{2\overline{b}_i},$$

$$B_i^\pm = \frac{A_i^\pm}{\rho_i}, \qquad H_i = \frac{l_i^-}{\rho_i},$$

$$(4)$$

where $i = 1, 2$.

Throughout this paper, we need the following assumptions:

(T_1) $(\overline{a}_i + 2g_i/\omega)\rho_i^{-1/2} > 2\sqrt{\overline{b}_i\overline{h}_i}$ $(i = 1, 2)$;

(T_2) $\rho_i > 1$ $(i = 1, 2)$.

The following results will play an important role in the proof of our main result.

Lemma 4 (see [5]). *Suppose $\psi \in PC_\omega^1$, then*

$$0 \le \sup_{s \in [0,\omega]} \psi(s) - \inf_{s \in [0,\omega]} \psi(s)$$

$$\le \frac{1}{2} \left[\int_0^\omega |\psi'(s)| \, ds + \sum_{k=1}^q |\Delta \psi(t_k)| \right].$$

$$(5)$$

Lemma 5. *Let $x > 0$, $y > 0$, $z > 0$, and $x > 2\sqrt{yz}$; for the functions $f(x, y, z) = (x + \sqrt{x^2 - 4yz})/2z$ and $g(x, y, z) = (x - \sqrt{x^2 - 4yz})/2z$, the following assertions hold.*

(1) *$f(x, y, z)$ and $g(x, y, z)$ are monotonically increasing and monotonically decreasing on the variable $x \in (0, \infty)$, respectively.*

(2) *$f(x, y, z)$ and $g(x, y, z)$ are monotonically decreasing and monotonically increasing on the variable $y \in (0, \infty)$, respectively.*

(3) *$f(x, y, z)$ and $g(x, y, z)$ are monotonically decreasing and monotonically increasing on the variable $z \in (0, \infty)$, respectively.*

Proof. In fact, for all $x > 0$, $y > 0$, and $z > 0$, we have

$$\frac{\partial f}{\partial x} = \frac{x + \sqrt{x^2 - 4yz}}{2z\sqrt{x^2 - 4yz}} > 0,$$

$$\frac{\partial g}{\partial x} = \frac{\sqrt{x^2 - 4yz} - x}{2z\sqrt{x^2 - 4yz}} < 0,$$

$$\frac{\partial f}{\partial y} = \frac{-1}{\sqrt{x^2 - 4yz}} < 0,$$

$$\frac{\partial g}{\partial y} = \frac{1}{\sqrt{x^2 - 4yz}} > 0,$$

$$\frac{\partial f}{\partial z} = \frac{-\left(x + \sqrt{x^2 - 4yz}\right)^2}{4z^2\sqrt{x^2 - 4yz}} < 0,$$

$$\frac{\partial g}{\partial z} = \frac{\left(x - \sqrt{x^2 - 4yz}\right)^2}{4z^2\sqrt{x^2 - 4yz}} > 0.$$

$$(6)$$

By the relationship of the derivative and the monotonicity, the above assertions obviously hold. The proof of Lemma 5 is complete. □

Lemma 6. *Assume that (T_1) and (T_2) hold, then one has the following inequalities:*

(1) $\ln l_i^+ > \ln A_i^+ > \ln A_i^- > \ln l_i^- > \ln H_i$ $(i = 1, 2)$;

(2) $\ln l_i^+ > \ln B_i^+ > \ln B_i^- > \ln l_i^- > \ln H_i$ $(i = 1, 2)$;

(3) $\min\{\ln A_i^+, \ln B_i^+\} > \max\{\ln A_i^-, \ln B_i^-\}$ $(i = 1, 2)$.

Proof. Since

$$\left(\overline{a}_1 + \frac{2g_1}{\omega} \right) \rho_1 > \overline{a}_1 + \frac{2g_1}{\omega}$$

$$> \frac{\overline{a}_1 + 2g_1/\omega}{\sqrt{\rho_1}} > 2\sqrt{\overline{b}_1\overline{h}_1} > 0,$$

$$\left(\overline{c}l_2^+ + 2a_2 + \frac{2g_2}{\omega} \right) \rho_2 > \overline{a}_2 + \frac{2g_2}{\omega}$$

$$(7)$$

$$> \frac{2g_2/\omega + \overline{a}_2}{\sqrt{\rho_2}} > 2\sqrt{\overline{b}_2\overline{h}_2} > 0,$$

$$\overline{b}_i > 0, \quad \overline{h}_i > 0, \quad i = 1, 2.$$

Applying Lemma 5, we have

$$0 < H_1 = \frac{l_1^-}{\rho_1} < l_1^- = g\left(\left(\overline{a}_1 + \frac{2g_1}{\omega} \right) \rho_1, \overline{b}_1, \overline{h}_1\rho_1 \right)$$

$$< g\left(\overline{a}_1 + \frac{2g_1}{\omega}, \overline{b}_1, \overline{h}_1\rho_1 \right) = A_1^-$$

$$< A_1^+ = f\left(\overline{a}_1 + \frac{2g_1}{\omega}, \overline{b}_1, \overline{h}_1\rho_1 \right)$$

$$< f\left(\left(\overline{a}_1 + \frac{2g_1}{\omega} \right) \rho_1, \overline{b}_1, \overline{h}_1\rho_1 \right) = l_1^+,$$

$$0 < H_2 = \frac{l_2^-}{\sqrt{\rho_2}} < l_2^- = g\left(\left(\overline{c}l_2^+ + \overline{a}_2 + \frac{2g_2}{\omega} \right) \rho_2, \overline{b}_2, \overline{h}_2\rho_2 \right)$$

$$< g\left(\frac{2g_2}{\omega} + \overline{a}_2, \overline{b}_2, \overline{h}_2\rho_2\right) = A_2^-$$

$$< A_2^+ = f\left(\frac{2g_2}{\omega} + \overline{a}_2, \overline{b}_2, \overline{h}_2\rho_2\right)$$

$$< f\left(\left(\overline{c}l_1^+ + \overline{a}_2 + \frac{2g_2}{\omega}\right)\rho_2, \overline{b}_2, \overline{h}_2\rho_2\right) = l_2^+. \tag{8}$$

Thus, we have $\ln H_i < \ln l_i^- < \ln A_i^- < \ln A_i^+ < \ln l_i^+$, $i = 1, 2$. The proof of (1) of Lemma 6 is complete. For (2) of Lemma 6, we similarly have

$$0 < H_1 = \frac{l_1^-}{\rho_1} < l_1^- = g\left(\left(\overline{a}_1 + \frac{2g_1}{\omega}\right)\rho_1, \overline{b}_1, \overline{h}_1\rho_1\right)$$

$$< g\left(\overline{a}_1 + \frac{2g_1}{\omega}, \overline{b}_1, \overline{h}_1\rho_1\right) = B_1^-$$

$$< B_1^+ = f\left(\overline{a}_1 + \frac{2g_1}{\omega}, \overline{b}_1, \overline{h}_1\rho_1\right)$$

$$< f\left(\left(\overline{a}_1 + \frac{2g_1}{\omega}\right)\rho_1, \overline{b}_1, \overline{h}_1\rho_1\right) = l_1^+,$$

$$0 < H_2 = \frac{l_2^-}{\rho_2} < l_2^- = g\left(\left(\overline{a}_2 + \frac{2g_2}{\omega}\right)\rho_2, \overline{b}_2, \overline{h}_2\rho_2\right) \tag{9}$$

$$< g\left(\left(\frac{2g_2}{\omega} + \overline{a}_2\right)\rho_2, \overline{b}_2, \frac{\overline{h}_2}{\rho_2}\right) = B_2^- < B_2^+$$

$$= f\left(\left(\frac{2g_2}{\omega} + \overline{a}_2\right)\rho_2, \overline{b}_2, \frac{\overline{h}_2}{\rho_2}\right)$$

$$< f\left(\left(\overline{c}l_1^+ + \overline{a}_2 + \frac{2g_2}{\omega}\right)\rho_2, \overline{b}_2, \overline{h}_2\rho_2\right) = l_2^+,$$

which imply that $\ln H_i < \ln l_i^- < \ln B_i^- < \ln B_i^+ < \ln l_i^+$ ($i = 1, 2$); that is, the inequality (2) of Lemma 6 holds. Let us now prove the inequality (3) of Lemma 6. In fact, since

$$\max\{\ln A_i^-, \ln B_i^-\}$$

$$= \frac{\ln A_i^- + \ln B_i^- + |\ln A_i^- - \ln B_i^-|}{2} \quad (i = 1, 2),$$

$$\min\{\ln A_i^+, \ln B_i^+\} \tag{10}$$

$$= \frac{\ln A_i^+ + \ln B_i^+ + |\ln A_i^+ - \ln B_i^+|}{2} \quad (i = 1, 2),$$

we have

$$\min\{\ln A_i^+, \ln B_i^+\} - \max\{\ln A_i^-, \ln B_i^-\}$$

$$= \left(\ln A_i^+ + \ln B_i^+ - \ln A_i^- - \ln B_i^- \right. \tag{11}$$

$$\left. - |\ln A_i^- - \ln B_i^-| - |\ln A_i^+ - \ln B_i^+|\right)(2)^{-1}.$$

If $A_i^+ \geq B_i^+$, then $A_i^- \leq B_i^-$, and

$$\min\{\ln A_i^+, \ln B_i^+\} - \max\{\ln A_i^-, \ln B_i^-\}$$

$$= \ln B_i^+ - \ln B_i^- > 0. \tag{12}$$

If $A_i^+ < B_i^+$, then $A_i^- > B_i^-$, and

$$\min\{\ln A_i^+, \ln B_i^+\} - \max\{\ln A_i^-, \ln B_i^-\}$$

$$= \ln A_i^+ - \ln A_i^- > 0. \tag{13}$$

Hence, we drive $\min\{\ln A_i^+, \ln B_i^+\} > \max\{\ln A_i^-, \ln B_i^-\}$. From the above, we have completed the proof of Lemma 6. $\square$

Theorem 7. *Assume that (T_1) and (T_2) hold. Then system (3) has at least four positive ω-periodic solutions.*

Proof. By making the substitution

$$x_i(t) = e^{u_i(t)}, \quad i = 1, 2, \tag{14}$$

system (3) can be reformulated as

$$\dot{u}_1(t) = a_1(t) - b_1(t)e^{u_1(t-\tau_{11}(t))}$$

$$- h_1(t)e^{-u_1(t)}, \quad t \neq t_k,$$

$$\dot{u}_2(t) = a_2(t) - b_2(t)e^{u_2(t-\tau_{22}(t))}$$

$$+ c(t)e^{u_1(t-\tau_{21}(t))} - h_2(t)e^{-u_2(t)}, \quad t \neq t_k, \tag{15}$$

$$\Delta u_i(t_k) = u_i(t_k^+) - u_i(t_k^-)$$

$$= \ln(1 + g_{ik}), \quad i = 1, 2, \quad t = t_k.$$

Let

$$X = \left\{u = (u_1, u_2)^T : u_i \in PC_\omega^1, \quad i = 1, 2\right\}, \quad Z = X \times \mathbb{R}^{2q}, \tag{16}$$

and define

$$\|u\|_0 = \sum_{i=1}^{2} \sup_{t \in [0,\omega]} |u_i(t)|, \quad u \in X, \tag{17}$$

$$\|z\|_1 = \|u\|_0 + \sum_{j=1}^{q} |\xi_j|, \quad z = (u, \xi_1, \xi_2, \ldots, \xi_q) \in Z.$$

Then both $(X, \|\cdot\|_0)$ and $(Z, \|\cdot\|_1)$ are Banach spaces. Let

$$\text{Dom } L = \{u \in X : \dot{x} \in X\}, L : \text{Dom } L \longrightarrow Z,$$

$$L\left(u = (u_1, u_2)^T\right)$$

$$= \left(\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix}, \left\{\begin{pmatrix} \Delta u_1(t_k) \\ \Delta u_2(t_k) \end{pmatrix}\right\}_{k=1}^{q}\right), \tag{18}$$

$$N : X \longrightarrow Z,$$

$$N(u, \lambda) = \left(\begin{pmatrix} F_1(t, \lambda) \\ F_2(t, \lambda) \end{pmatrix}, \left\{\begin{pmatrix} \ln(1 + g_{1k}) \\ \ln(1 + g_{2k}) \end{pmatrix}\right\}_{k=1}^{q}\right),$$

where

$$F_1(t, \lambda) = a_1(t) - b_1(t) e^{u_1(t-\tau_{11}(t))} - h_1(t) e^{-u_1(t)},$$

$$F_2(t, \lambda) = a_2(t) - b_2(t) e^{u_2(t-\tau_{22}(t))}$$
$$+ \lambda c(t) e^{u_1(t-\tau_{21}(t))} - h_2(t) e^{-u_2(t)},$$

$$P : X \to X, \qquad Q : Z \to Z,$$

$$P\left(u = (u_1, u_2)^T\right) = (\overline{u}_1, \overline{u}_2)^T, \quad u \in X, \tag{19}$$

$$Q\left(\binom{u_1}{u_2}, \left\{\binom{m_{1k}}{m_{2k}}\right\}_{k=1}^q\right)$$
$$= \left(\binom{\overline{u}_1 + \dfrac{1}{\omega}\sum_{k=1}^q m_{1k}}{\overline{u}_2 + \dfrac{1}{\omega}\sum_{k=1}^q m_{2k}}, \left\{\binom{0}{0}\right\}_{k=1}^q\right).$$

Obviously,

$$\text{Ker } L = \left\{u = (u_1, u_2)^T \in X : \right.$$
$$\left. (u_1(t), u_2(t)(t)) = (c_1, c_2) \in \mathbb{R}^2, \quad t \in \mathbb{R}\right\},$$

$$\text{Im } L$$
$$= \left\{\left(\binom{u_1}{u_2}, \left\{\binom{m_{1k}}{m_{2k}}\right\}_{k=1}^q\right) \in Z : \begin{array}{c} \omega\overline{u}_1 + \displaystyle\sum_{k=1}^q m_{1k} = 0 \\ \omega\overline{u}_2 + \displaystyle\sum_{k=1}^q m_{2k} = 0 \end{array}\right\}.$$

$$\tag{20}$$

Since $\text{Im } L$ is closed in Z, $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L$, and $\dim \text{Ker } L = 2 = \text{codim Im } L$; we know that L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : \text{Im } L \to \text{Ker } P \bigcap \text{Dom } L$ is given by

$$K_P\left(z = \left(\binom{u_1}{u_2}, \left\{\binom{m_{1k}}{m_{2k}}\right\}_{k=1}^q\right)\right)$$
$$= \left(\begin{array}{c} \displaystyle\int_0^t u_1(s)\,ds + \sum_{0<t_k<t} m_{1k} - \frac{1}{\omega}\sum_{k=1}^q m_{1k} - \frac{1}{\omega}\int_0^\omega\int_0^t u_1(s)\,ds\,dt \\ \displaystyle\int_0^t u_2(s)\,ds + \sum_{0<t_k<t} m_{2k} - \frac{1}{\omega}\sum_{k=1}^q m_{2k} - \frac{1}{\omega}\int_0^\omega\int_0^t u_2(s)\,ds\,dt \end{array}\right).$$
$$\tag{21}$$

Then

$$QN(u, \lambda) = \left(\left(\begin{array}{c} \dfrac{1}{\omega}\displaystyle\int_0^\omega F_1(s, \lambda)\,ds + \dfrac{2g_1}{\omega} \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega F_2(s, \lambda)\,ds + \dfrac{2g_2}{\omega} \end{array}\right), \left\{\binom{0}{0}\right\}_{k=1}^q\right),$$

$$K_P(I - Q)N(u, \lambda)$$
$$= \left(\begin{array}{c} \displaystyle\int_0^t F_1(s, \lambda)\,ds + \sum_{0<t_k<t} \ln(1 + g_{1k}) \\ \displaystyle\int_0^t F_1(s, \lambda)\,ds + \sum_{0<t_k<t} \ln(1 + g_{1k}) \end{array}\right)$$
$$+ \left(\frac{1}{2} - \frac{t}{\omega}\right)\left(\begin{array}{c} \displaystyle\int_0^\omega F_1(s, \lambda)\,ds + 2g_1 \\ \displaystyle\int_0^\omega F_2(s, \lambda)\,ds + 2g_2 \end{array}\right)$$
$$- \frac{1}{\omega}\left(\begin{array}{c} \displaystyle\int_0^\omega\int_0^t F_1(s, \lambda)\,ds\,dt + 2g_1 \\ \displaystyle\int_0^\omega\int_0^t F_2(s, \lambda)\,ds\,dt + 2g_2 \end{array}\right),$$
$$\tag{22}$$

where $F_i(s, \lambda)$ $(i = 1, 2)$ are very similar to that of $F_i(t, \lambda)$ $(i = 1, 2)$. By the Lebesgue convergence theorem, QN and $K_P(I - Q)N$ are continuous. Moreover, because of periodicity, it follows from Lemma 3 that $QN(\overline{\Omega})$, $K_P(I - Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Thus, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$.

In order to use Lemma 1, we have to find at least four appropriate open bounded subsets of X. Considering the operator equation $Lu = \lambda N(u, \lambda)$, $\lambda \in (0, 1)$, we have

$$\dot{u}_1(t) = \lambda\left(a_1(t) - b_1(t) e^{u_1(t-\tau_{11}(t))} - h_1(t) e^{-u_1(t)}\right),$$

$$\dot{u}_2(t) = \lambda\left(a_2(t) - b_2(t) e^{u_2(t-\tau_{22}(t))}\right.$$
$$\left. + \lambda c(t) e^{u_1(t-\tau_{21}(t))} - h_2(t) e^{-u_2(t)}\right), \tag{23}$$

$$\Delta u_j(t_k) = \lambda \ln(1 + g_{jk}), \quad j = 1, 2.$$

Assume that $u \in X$ is a ω-periodic solution of system (23) for some $\lambda \in (0, 1)$. Integrating (23) from 0 to ω, we have

$$-2g_1 = \int_0^\omega \left\{a_1(t) - b_1(t) e^{u_1(t-\tau_{11}(t))} - h_1(t) e^{-u_1(t)}\right\} dt,$$

$$-2g_2 = \int_0^\omega \left\{a_2(t) - b_2(t) e^{u_2(t-\tau_{22}(t))}\right.$$
$$\left. + \lambda c(t) e^{u_1(t-\tau_{21}(t))} - h_2(t) e^{-u_2(t)}\right\} dt. \tag{24}$$

Then

$$\int_0^\omega \left\{b_1(t) e^{u_1(t-\tau_{11}(t))} + h_1(t) e^{-u_1(t)}\right\} dt = \overline{a}_1\omega + 2g_1,$$

$$\int_0^\omega \left\{b_2(t) e^{u_2(t-\tau_{22}(t))} - \lambda c(t) e^{u_1(t-\tau_{21}(t))} + h_2(t) e^{-u_2(t)}\right\} dt$$
$$= \overline{a}_2\omega + 2g_2. \tag{25}$$

Furthermore, note that $u = (u_1, u_2)^T \in X$. There exists $\xi_i, \eta_i \in [0, \omega]$, $i = 1, 2$, such that $u_i(\xi_i) = \sup_{t\in[0,\omega]} u_i(t)$, $u_i(\eta_i) = \inf_{t\in[0,\omega]} u_i(t)$.

On the one hand, according to the first equation of (23) and (25), we have

$$\int_0^\omega |\dot{u}_1(t)| \, dt < 2\bar{a}_1 \omega + 2g_1. \tag{26}$$

In the light of the first equation of (25) and (26) and Lemma 4, we get

$$\begin{aligned}
\omega \bar{b}_1 & e^{u_1(\xi_1) - (\bar{a}_1 \omega + G_1)} + \omega \bar{h}_1 e^{-u_1(\xi_1)} \\
& \leq \omega \bar{b}_1 e^{u_1(\xi_1) - (1/2)[\int_0^\omega |\dot{u}_1(s)| ds + \sum_{k=1}^q |\Delta u_1(t_k)|]} + \omega \bar{h}_1 e^{-u_1(\xi_1)} \\
& < \omega \bar{b}_1 e^{u_1(\eta_1)} + \omega \bar{h}_1 e^{-u_1(\xi_1)} \\
& \leq \int_0^\omega \left\{ b_1(t) e^{u_1(t - \tau_{11}(t))} + h_1(t) e^{-u_1(t)} \right\} dt \\
& = \bar{a}_1 \omega + 2g_1,
\end{aligned} \tag{27}$$

namely,

$$\bar{b}_1 e^{2u_1(\xi_1)} - \left(\bar{a}_1 + \frac{2g_1}{\omega} \right) e^{\bar{a}_1 \omega + G_1} e^{u_1(\xi_1)} + \bar{h}_1 e^{\bar{a}_1 \omega + G_1} < 0, \tag{28}$$

which implies that

$$\ln l_1^- < u_1(\xi_1) < \ln l_1^+. \tag{29}$$

Applying Lemma (23), we drive

$$\ln H_1 = \ln l_1^- - (\bar{a}_1 \omega + G_1) < u_1(\eta) \leq u_1(\xi_1) < \ln l_1^+. \tag{30}$$

According to (23) and (25), we get

$$\int_0^\omega |\dot{u}_2(t)| \, dt < 2\bar{a}_2 \omega + 2g_2. \tag{31}$$

By (25) and (31), we have

$$\begin{aligned}
\omega \bar{b}_2 & e^{u_2(\xi_2) - (\bar{a}_2 \omega + G_2)} - \omega \bar{c} l_1^+ + \omega \bar{h}_2 e^{-u_2(\xi_2)} \\
& \leq \omega \bar{b}_2 e^{u_2(\xi_2) - (1/2)[\int_0^\omega |\dot{u}_2(s)| ds + \sum_{k=1}^q |\Delta u_2(t_k)|]} \\
& \quad - \omega \bar{c} l_1^+ + \omega \bar{h}_2 e^{-u_2(\xi_2)} \\
& < \omega \bar{b}_2 e^{u_2(\eta_2)} - \omega \bar{c} l_1^+ + \omega \bar{h}_2 e^{-u_2(\xi_2)} \\
& < \int_0^\omega \left\{ b_2(t) e^{u_2(t - \tau_{22}(t))} \right. \\
& \qquad \left. - \lambda c(t) e^{u_1(t - \tau_{21}(t))} + h_2(t) e^{-u_2(t)} \right\} dt \\
& = \bar{a}_2 \omega + 2g_2,
\end{aligned} \tag{32}$$

that is,

$$\bar{b}_2 e^{2u_2(\xi_2)} - \left(\bar{c} l_1^+ + a_2 + \frac{2g_2}{\omega} \right) e^{\bar{a}_2 \omega + G_2} e^{u_2(\xi_2)} + \bar{h}_2 e^{\bar{a}_2 \omega + G_2} < 0, \tag{33}$$

which implies that

$$\ln l_2^- < u_2(\xi_2) < \ln l_2^+. \tag{34}$$

Applying Lemma 4, we drive

$$\ln H_2 = \ln l_2^- - (\bar{a}_2 \omega + G_2) < u_2(\eta) \leq u_2(\xi_2) < \ln l_2^+. \tag{35}$$

On the other hand, in the light of the first equation of (25) and (26), we have

$$\begin{aligned}
\bar{a}_1 \omega & + 2g_1 \\
& = \int_0^\omega \left\{ b_1(t) e^{u_1(t - \tau_{11}(t))} + h_1(t) e^{-u_1(t)} \right\} dt \\
& \leq \bar{b}_1 \omega e^{u_1(\xi_1)} + \bar{h}_1 \omega e^{-u_1(\eta_1)} \\
& < \bar{b}_1 \omega e^{u_1(\xi_1)} + \bar{h}_1 \omega e^{-u_1(\xi_1) + (1/2)[\int_0^\omega |\dot{u}_1(s)| ds + \sum_{k=1}^q |\Delta u_1(t_k)|]} \\
& < \bar{b}_1 \omega e^{u_1(\xi_1)} + \bar{h}_1 \omega e^{-u_1(\xi_1) + a_1 \omega + G_1},
\end{aligned} \tag{36}$$

namely,

$$\bar{b}_1 e^{2u_1(\xi_1)} - \left(\bar{a}_1 + \frac{2g_1}{\omega} \right) e^{u_1(\xi_1)} + \bar{h}_1 e^{\bar{a}_1 \omega + G_1} > 0, \tag{37}$$

which implies that

$$\ln A_1^+ < u_1(\xi_1) \quad \text{or} \quad u_1(\xi_1) < \ln A_1^-,$$

$$\begin{aligned}
\bar{a}_1 \omega & + 2g_1 \\
& = \int_0^\omega \left\{ b_1(t) e^{u_1(t - \tau_{11}(t))} + h_1(t) e^{-u_1(t)} \right\} dt \\
& \leq \omega \left(\bar{b}_1 e^{u_1(\xi_1)} + \bar{h}_1 e^{-u_1(\eta_1)} \right) \\
& < \omega \left(\bar{b}_1 e^{u_1(\eta_1) + (1/2)[\int_0^\omega |\dot{u}_1(s)| ds + \sum_{k=1}^q |\Delta u_1(t_k)|]} + \bar{h}_1 e^{-u_1(\eta_1)} \right) \\
& < \bar{b}_1 \omega e^{u_1(\eta_1) + \bar{a}_1 \omega + G_1} + \bar{h}_1 \omega e^{-u_1(\eta_1)},
\end{aligned} \tag{38}$$

that is,

$$\bar{b}_1 e^{2u_1(\eta_1)} - \left(\bar{a}_1 + \frac{2g_1}{\omega} \right) e^{-(\bar{a}_1 \omega + G_1)} e^{u_1(\eta_1)} + \bar{h}_1 e^{-(\bar{a}_1 \omega + G_1)} > 0, \tag{39}$$

which implies that

$$\ln B_1^+ < u_1(\eta_1) \quad \text{or} \quad u_1(\eta_1) < \ln B_1^-. \tag{40}$$

By (25) and (31), we have

$$
2g_2 < \bar{a}_2 \omega + 2g_2
$$

$$
= \int_0^\omega \left\{ b_2(t) e^{u_2(t-\tau_{22}(t))} \right.
$$

$$
\left. -\lambda c(t) e^{u_1(t-\tau_{21}(t))} + h_2(t) e^{-u_2(t)} \right\} dt
$$

$$
< \bar{b}_2 \omega e^{u_2(\xi_2)} + \bar{h}_2 \omega e^{-u_2(\eta_2)}
$$

$$
< \bar{b}_2 \omega e^{u_2(\xi_2)} + \bar{h}_2 \omega e^{-u_2(\xi_2)+(1/2)[\int_0^\omega |\dot{u}_2(s)|ds + \sum_{k=1}^q |\Delta u_2(t_k)|]}
$$

$$
< \bar{b}_2 \omega e^{u_2(\xi_2)} + \bar{h}_2 \omega e^{-u_2(\xi_2)+\bar{a}_2\omega+G_2}, \tag{41}
$$

namely,

$$
\bar{b}_2 e^{2u_2(\xi_2)} - \left(\bar{a}_2 + \frac{2g_2}{\omega} \right) e^{u_2(\xi_2)} + \bar{h}_2 e^{\bar{a}_2\omega+G_2} > 0, \tag{42}
$$

which implies that

$$
\ln A_2^+ < u_2(\xi_2) \quad \text{or} \quad u_2(\xi_2) < \ln A_2^-,
$$

$$
2g_2 < \bar{a}_2 \omega + 2g_2
$$

$$
= \int_0^\omega \left\{ b_2(t) e^{u_2(t-\tau_{22}(t))} \right.
$$

$$
\left. -\lambda c(t) e^{u_1(t-\tau_{21}(t))} + h_2(t) e^{-u_2(t)} \right\} dt
$$

$$
< \bar{b}_2 \omega e^{u_2(\xi_2)} + \bar{h}_2 \omega e^{-u_2(\eta_2)}
$$

$$
< \bar{b}_2 \omega e^{u_2(\eta_2)+(1/2)[\int_0^\omega |\dot{u}_2(s)|ds + \sum_{k=1}^q |\Delta u_2(t_k)|]} + \bar{h}_2 \omega e^{-u_2(\eta_2)}
$$

$$
< \bar{b}_2 \omega e^{u_2(\eta_2)+\bar{a}_2\omega+G_2} + \bar{h}_2 \omega e^{-u_2(\eta_2)}, \tag{43}
$$

that is,

$$
\bar{b}_2 e^{2u_2(\eta_2)} - \left(\frac{2g_2}{\omega} + \bar{a}_2 \right) e^{-(\bar{a}_2\omega+G_2)} e^{u_2(\eta_2)} + \bar{h}_2 e^{-(\bar{a}_2\omega+G_2)} > 0, \tag{44}
$$

which implies that

$$
\ln B_2^+ < u_2(\eta_2) \quad \text{or} \quad u_2(\eta_2) < \ln B_2^-. \tag{45}
$$

In view of (30)–(45) and Lemma 6, we have

$$
\min\{\ln A_1^+, \ln B_1^+\} < u_1(\eta_1) < u_1(\xi_1) < \ln l_1^+,
$$
$$
\min\{\ln A_2^+, \ln B_2^+\} < u_2(\eta_2) < u_2(\xi_2) < \ln l_2^+ \tag{46}
$$

or

$$
\ln H_1 < u_1(\eta_1) < u_1(\xi_1) < \max\{\ln A_1^-, \ln B_1^-\},
$$
$$
\ln H_2 < u_2(\eta_2) < u_2(\xi_2) < \max\{\ln A_2^-, \ln B_2^-\}, \tag{47}
$$

which imply that, for all $t \in R$,

$$
\min\{\ln A_1^+, \ln B_1^+\} < u_1(t) < \ln l_1^+ \text{ or}
$$
$$
\ln H_1^- < u_1(t) < \max\{\ln A_1^-, \ln B_1^-\},
$$
$$
\min\{\ln A_2^+, \ln B_2^+\} < u_2(t) < \ln l_2^+ \text{ or} \tag{48}
$$
$$
\ln H_2^- < u_2(t) < \max\{\ln A_2^-, \ln B_2^-\}.
$$

Clearly, $\ln l_1^\pm$, $\ln l_2^\pm$, $\ln A_1^\pm$, $\ln A_2^\pm$, $\ln B_1^\pm$, and $\ln B_2^\pm$ are independent of λ. Now let

$$
\Omega_1 = \left\{ u = (u_1, u_2)^T \in X : \right.
$$
$$
\min\{\ln A_1^+, \ln B_1^+\} < u_1(t) < \ln l_1^+,
$$
$$
\left. \min\{\ln A_2^+, \ln B_2^+\} < u_2(t) < \ln l_2^+ \right\},
$$

$$
\Omega_2 = \left\{ u = (u_1, u_2)^T \in X : \right.
$$
$$
\min\{\ln A_1^+, \ln B_1^+\} < u_1(t)
$$
$$
\left. \ln H_2 < u_2(t) < \max\{\ln A_2^-, \ln B_2^-\} \right\},
$$

$$
\Omega_3 = \left\{ u = (u_1, u_2)^T \in X : \right. \tag{49}
$$
$$
\ln H_1 < u_1(t) < \max\{\ln A_1^-, \ln B_1^-\},
$$
$$
\left. \min\{\ln A_2^+, \ln B_2^+\} < u_2(t) < \ln l_2^+ \right\},
$$

$$
\Omega_4 = \left\{ u = (u_1, u_2)^T \in X : \right.
$$
$$
\ln H_1 < u_1(t) < \max\{\ln A_1^-, \ln B_1^-\},
$$
$$
\left. \ln H_2 < u_2(t) < \max\{\ln A_2^-, \ln B_2^-\} \right\}.
$$

Obviously, the number of the above sets is four. We denote these sets as Ω_k, $k = 1, 2, 3$, and 4. Ω_k are bounded open subsets of X, $\Omega_i \cap \Omega_j = \phi$, $i \neq j$. Thus Ω_k ($k = 1, 2, 3$, and 4) satisfies the requirement (a) in Lemma 1.

Now we show that (b) of Lemma 1 holds; that is, we prove when $u \in \partial\Omega_k \cap \text{Ker} L = \partial\Omega_k \cap R^2$, $QN(u,0) \neq (0,0)^T$, $k = 1, 2, 3$, and 4. If it is not true, then when $u \in \partial\Omega_k \cap \text{Ker} L = \partial\Omega_k \cap R^2$, $i = 1, 2, 3$, and 4, constant vector $u = (u_1, u_2)^T$ with $u \in \partial\Omega_k$ ($k = 1, 2, 3$, and 4) satisfies

$$
-2g_1 = \int_0^\omega \left\{ a_1(t) - b_1(t) e^{u_1(t-\tau_{11}(t))} - h_1(t) e^{-u_1(t)} \right\} dt,
$$

$$
-2g_2 = \int_0^\omega \left\{ a_2(t) - b_2(t) e^{u_2(t-\tau_{22}(t))} \right.
$$
$$
\left. + \lambda c(t) e^{u_1(t-\tau_{21}(t))} - h_2(t) e^{-u_2(t)} \right\} dt, \tag{50}
$$

that is,

$$
-\frac{2g_1}{\omega} = \bar{a}_1 - \bar{b}_1 e^{u_1} - \bar{h}_1 e^{-u_1},
$$
$$
-\frac{2g_2}{\omega} = \bar{a}_2 - \bar{b}_2 e^{u_2} - \bar{h}_2 e^{-u_2}. \tag{51}
$$

Similar to the process of (25)–(48), we obtain

$$\ln H_i < u_i^- < \max\left\{\ln A_i^-, \ln B_i^-\right\}$$

$$< \min\left\{\ln A_i^+, \ln B_i^+\right\} \tag{52}$$

$$< u_i^+ < \ln l_i^+ \quad (1 \le i \le 2).$$

Then u belongs to one of $\Omega_k \cap R^2$, $k = 1, 2, 3$, and 4. This contradicts the fact that $u \in \partial\Omega_k \cap R^2$, $k = 1, 2, 3$, and 4. This proves that (b) in **Lemma 1** holds.

Finally, in order to show that (c) in **Lemma 1** holds, we only prove that for $u \in \partial\Omega_k \cap \mathrm{Ker}\,L = \partial\Omega_k \cap R^2$, $k = 1, 2, 3$, and 4, then it holds that $\deg\{JQN(u, 0), \Omega_k \cap \mathrm{Ker}\,L, (0, 0)^T\} \neq 0$. To this end, we define the mapping $\phi : \mathrm{Dom}\,L \times [0, 1] \rightarrow X$ by

$$\phi(u, \mu) = \mu QN(u, 0) + (1 - \mu)G(u), \tag{53}$$

here $\mu \in [0, 1]$ is a parameter, and $G(u)$ is defined by

$$G(u)$$

$$= \left(\left(\begin{array}{c} \dfrac{1}{\omega}\displaystyle\int_0^\omega \left(a_1(s) - b_1(s)e^{u_1(s)} - h_1(s)e^{-u_1(s)}\right)ds + \dfrac{2g_1}{\omega} \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega \left(a_2(s) - b_2(s)e^{u_2(s)} - h_2(s)e^{-u_2(s)}\right)ds + \dfrac{2g_2}{\omega} \end{array}\right), \left\{\left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right\}_{k=1}^q\right). \tag{54}$$

We show that for $u \in \partial\Omega_k \cap \mathrm{Ker}\,L = \partial\Omega_k \cap R^2$, $k = 1, 2, 3$, and 4, $\mu \in [0, 1]$, then it holds that $\phi(u, \mu) \neq (0, 0,)^T$. Otherwise, parameter μ and constant vector $u = (u_1, u_2)^T \in R^2$ satisfy $\phi(u, \mu) = (0, 0)^T$, that is,

$$0 = \mu \int_0^\omega \left(a_1(s) - b_1(s)e^{u_1} - h_1(s)e^{-u_1} + \frac{2g_1}{\omega}\right)ds$$

$$+ (1 - \mu)\int_0^\omega \left(a_1(s) - b_1(s)e^{u_1} - h_1(s)e^{-u_1} + \frac{2g_1}{\omega}\right)ds,$$

$$0 = \mu \int_0^\omega \left(a_2(s) - b_2(s)e^{u_2} - h_2(s)e^{-u_2} + \frac{2g_2}{\omega}\right)ds$$

$$+ (1 - \mu)\int_0^\omega \left(a_2(s) - b_2(s)e^{u_2} - h_2(s)e^{-u_2} + \frac{2g_2}{\omega}\right)ds, \tag{55}$$

that is,

$$\bar{a}_1 - \bar{b}_1 e^{u_1} - \bar{h}_1 e^{-u_1} = -\frac{2g_1}{\omega},$$

$$\bar{a}_2 - \bar{b}_2 e^{u_2} - \bar{h}_2 e^{-u_2} = -\frac{2g_2}{\omega}. \tag{56}$$

Following the argument of (25)–(48), we obtain

$$\ln H_i < u_i^- < \max\left\{\ln A_i^-, \ln B_i^-\right\}$$

$$< \min\left\{\ln A_i^+, \ln B_i^+\right\} \tag{57}$$

$$< u_i^+ < \ln l_i^+ \quad (1 = 1, 2).$$

Equation (57) gives that u belongs to one of $\Omega_k \cap R^2$, $k = 1, 2, 3$, and 4. This contradicts the fact that $u \in \partial\Omega_k \cap R^2$, $k = 1, 2, 3$, and 4. This proves $\phi(u, \mu) \neq (0, 0)^T$ holds. Note that the system of algebraic equations,

$$\bar{a}_1 - \bar{b}_1 e^x - \bar{h}_1 e^{-x} = -\frac{2g_1}{\omega},$$

$$\bar{a}_2 - \bar{b}_2 e^y - \bar{h}_2 e^{-y} = -\frac{2g_2}{\omega}, \tag{58}$$

has four distinct solutions since (T_1) and (T_2) hold:

$$(x_1^*, y_1^*) = (\ln x^+, \ln y^+),$$

$$(x_2^*, y_2^*) = (\ln x^+, \ln y^-),$$

$$(x_3^*, y_3^*) = (\ln x^-, \ln y^+), \tag{59}$$

$$(x_4^*, y_4^*) = (\ln x^-, \ln y^-),$$

where

$$x^\pm = \frac{\bar{a}_1 + 2g_1/\omega \pm \sqrt{(\bar{a}_1 + 2g_1/\omega)^2 - 4\bar{b}_1\bar{h}_1}}{2\bar{b}_1},$$

$$y^\pm = \frac{\bar{a}_2 + 2g_2/\omega \pm \sqrt{(\bar{a}_2 + 2g_2/\omega)^2 - 4\bar{b}_2\bar{h}_2}}{2\bar{b}_2}. \tag{60}$$

Similar to the proof of **Lemma 6**, it is easy to verify that

$$\ln H_1^- < \ln x^- < \max\left\{\ln B_1^-, \ln A_1^-\right\}$$

$$< \min\left\{\ln A_1^+, \ln B_1^+\right\} < \ln x^+ < \ln l_1^+,$$

$$\ln H_2^- < \ln y^- < \max\left\{\ln B_2^-, \ln A_2^-\right\} \tag{61}$$

$$< \min\left\{\ln A_2^+, \ln B_2^+\right\} < \ln y^+ < \ln l_2^+.$$

Therefore

$$(x_1^*, y_1^*) \in \Omega_1, \qquad (x_2^*, y_2^*) \in \Omega_2,$$

$$(x_3^*, y_3^*) \in \Omega_3, \qquad (x_4^*, y_4^*) \in \Omega_4, \tag{62}$$

uniquely belongs to the corresponding Ω_k. Since $\mathrm{Ker}\,L = \mathrm{Im}\,Q$, we can take $J = I$. A direct computation gives, for $k = 1, 2, 3$, and 4,

$$\deg\left\{JQN(u, 0), \Omega_k \cap \mathrm{Ker}\,L, (0, 0)^T\right\}$$

$$= \mathrm{sign}\begin{vmatrix} -\bar{b}_1 x^* + \dfrac{\bar{h}_1}{x^*} & 0 \\ 0 & -\bar{b}_2 y^* + \dfrac{\bar{h}_2}{y^*} \end{vmatrix} \tag{63}$$

$$= \mathrm{sign}\left[\left(-\bar{b}_1 x^{*2} + \bar{h}_1\right)\left(-\bar{b}_2 y^{*2} + \bar{h}_2\right)\right].$$

Since $(\bar{a}_1 + 2g_1/\omega) - \bar{b}_1 x^* - \bar{h}_1/x^* = 0$, $(\bar{a}_2 + 2g_2/\omega) - \bar{b}_2 y^* - \bar{h}_1/y^* = 0$.

Hence,

$$\deg \left\{ JQN\left(u,0\right), \Omega_1 \cap \mathrm{Ker}\, L, (0,0)^T \right\}$$

$$= \mathrm{sign}\left\{ \left(-\bar{b}_1(x_1^*)^2 + \bar{h}_1\right)\left(-\bar{b}_2(y_1^*)^2 + \bar{h}_2\right) \right\} = 1,$$

$$\deg \left\{ JQN\left(u,0\right), \Omega_2 \cap \mathrm{Ker}\, L, (0,0)^T \right\}$$

$$= \mathrm{sign}\left\{ \left(-\bar{b}_1(x_2^*)^2 + \bar{h}_1\right)\left(-\bar{b}_2(y_2^*)^2 + \bar{h}_2\right) \right\} = -1,$$

$$\deg \left\{ JQN\left(u,0\right), \Omega_3 \cap \mathrm{Ker}\, L, (0,0)^T \right\}$$

$$= \mathrm{sign}\left\{ \left(-\bar{b}_1(x_3^*)^2 + \bar{h}_1\right)\left(-\bar{b}_2(y_3^*)^2 + \bar{h}_2\right) \right\} = 1,$$

$$\deg \left\{ JQN\left(u,0\right), \Omega_4 \cap \mathrm{Ker}\, L, (0,0)^T \right\}$$

$$= \mathrm{sign}\left\{ \left(-\bar{b}_1(x_4^*)^2 + \bar{h}_1\right)\left(-\bar{b}_2(y_4^*)^2 + \bar{h}_2\right) \right\} = -1. \tag{64}$$

So far, we have proved that Ω_k ($k = 1, 2, 3,$ and 4) satisfies all the assumptions in Lemma 1. Hence, system (15) has at least four different ω-periodic solutions. Thus by Lemma 1, system (3) has at least four different positive ω-periodic solutions. This completes the proof of Theorem 7. $\square$

3. An Example

Example 8. Consider the following time-delay parasitical system with impulsive effectiveness and harvesting terms:

$$\dot{x}_1(t)$$
$$= x_1(t)\left(\frac{3 + \sin t}{2} - \frac{2 + \sin t}{10e^{2\pi}} x_1\left(t - |\sin t|\right) \right) - \frac{4 + \cos t}{20e^{2\pi}},$$

$$\dot{x}_2(t) = x_2(t)$$
$$\times \left(\frac{2 + \cos t}{25} - \frac{5 + \cos t}{10e^{\pi}} x_2\left(t - |\sin 2t|\right) \right.$$
$$\left. + \frac{2 + \sin 2t}{8} x_1\left(t - |\cos 2t|\right) \right) - \frac{2 + \cos t}{25e^{\pi}},$$

$$\Delta x_1\left(\frac{\pi}{2}\right) = e^{-\pi} - 1, \qquad \Delta x_1\left(\frac{3\pi}{2}\right) = e^{\pi} - 1,$$

$$\Delta x_2\left(\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}} - 1, \qquad \Delta x_2\left(\frac{3\pi}{2}\right) = e^{3\pi/2} - 1. \tag{65}$$

In this case, $a_1(t) = (3 + \sin t)/2$, $b_1(t) = (2 + \sin t)/10e^2\pi$, $h_1(t) = (4+\cos t)/20e^2\pi$, $\tau_{11}(t) = |\sin t|$, $a_2(t) = (2+\cos t)/25$, $b_2(t) = (5 + \cos t)/10$, $h_2(t) = (2 + \cos t)/25e^{\pi}$, $c(t) = (2+\sin 2t)/8$, $\tau_{22}(t) = |\cos 2t|$, and $\tau_{21}(t) = |\sin 2t|$. It is not difficult to calculate $\rho_1 > e^{4\pi} > 1$, $\rho_2 = e^{2\pi} > 1$. Since

$$l_1^{\pm} = \frac{\left(\bar{a}_1 + 2g_1/\omega\right)\rho_1 \pm \sqrt{\left[\left(\bar{a}_1 + 2g_1/\omega\right)\rho_1\right]^2 - 4\bar{b}_1\bar{h}_1\rho_1}}{2\bar{b}_1}$$

$$= \frac{\left(15e^{4\pi} \pm 3\sqrt{225e^{8\pi} - 16}\right)}{4e^{-2\pi}},$$

$$l_2^{\pm} = \left(\left(\bar{c}l_1^+ + \bar{a}_2 + 2g_2/\omega\right)\rho_2 \right.$$

$$\left. \pm \sqrt{\left[\left(\bar{c}l_1^+ + \bar{a}_2 + 2g_2/\omega\right)\rho_2\right]^2 - 4\bar{b}_2\bar{h}_2\rho_2} \right)\left(2\bar{b}_2\right)^{-1}$$

$$= \frac{15e^{2\pi} \pm \sqrt{225e^{4\pi} - 64}}{20e^{-\pi}}. \tag{66}$$

Then

$$\frac{\bar{a}_1 + 2g_1/\omega}{\sqrt{\rho_1}} = \frac{1}{2}e^{-2\pi} > \frac{2}{5}e^{-2\pi} = 2\sqrt{\bar{b}_1\bar{h}_1},$$

$$\frac{2g_2/\omega + \bar{a}_2}{\sqrt{\rho_2}} = \frac{21}{25}e^{-\pi} > \frac{2}{5}e^{-\pi} = 2\sqrt{\bar{b}_2\bar{h}_2}. \tag{67}$$

Therefore, all conditions of Theorem 7 are satisfied. By Theorem 7, system (65) has at least four positive 2π-periodic solutions.

Aknowledgment

References

[1] Z. Ma, *Mathematical Modelling and Studing on Species Ecology*, Education Press, Hefei, China, 1996.

[2] H. R. Thieme, *Mathematics in Population Biology*, Princeton Series in Theoretical and Computational Biology, Princeton University Press, Princeton, NJ, USA, 2003.

[3] K. Zhao and Y. Ye, "Four positive periodic solutions to a periodic Lotka-Volterra predatory-prey system with harvesting terms," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 4, pp. 2448–2455, 2010.

[4] D. Baĭnov and P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, vol. 66 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman Scientific & Technical, Harlow, UK, 1993.

[5] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific Publishing, Teaneck, NJ, USA, 1989.

[6] Y. Chen, "Multiple periodic solutions of delayed predator-prey systems with type IV functional responses," *Nonlinear Analysis. Real World Applications*, vol. 5, no. 1, pp. 45–53, 2004.

[7] Q. Wang, B. Dai, and Y. Chen, "Multiple periodic solutions of an impulsive predator-prey model with Holling-type IV functional response," *Mathematical and Computer Modelling*, vol. 49, no. 9-10, pp. 1829–1836, 2009.

[8] D. Hu and Z. Zhang, "Four positive periodic solutions to a Lotka-Volterra cooperative system with harvesting terms," *Nonlinear Analysis. Real World Applications*, vol. 11, no. 2, pp. 1115–1121, 2010.

[9] R. E. Gaines and J. L. Mawhin, *Coincidence Degree, and Nonlinear Differential Equations*, vol. 568 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1977.

Advances in
Operations Research

Advances in
Decision Sciences

Mathematical Problems
in Engineering

Algebra

Journal of
Probability and Statistics

The Scientific
World Journal

International Journal of
Differential Equations

International Journal of
Combinatorics

Hindawi

Submit your manuscripts at
http://www.hindawi.com

Advances in
Mathematical Physics

Journal of
Complex Analysis

Journal of
Mathematics

International Journal of
Stochastic Analysis

Abstract and
Applied Analysis

Discrete Dynamics in
Nature and Society

International
Journal of
Mathematics and
Mathematical
Sciences

Journal of
Discrete Mathematics

Journal of
Applied Mathematics

Journal of
Function Spaces

Journal of
Optimization