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Research Article

Complexity Analysis of a Cournot-Bertrand Duopoly Game Model with Limited Information

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A Cournot-Bertrand mixed duopoly game model with limited information about the market and opponent is considered, where the market has linear demand and two firms have the same fixed marginal cost. The principles of decision-making are bounded rational. One firm chooses output and the other chooses price as decision variable, with the assumption that there is a certain degree of differentiation between the products offered by firms to avoid the whole market being occupied by the one that applies a lower price. The existence of Nash equilibrium point and its local stability of the game are investigated. The complex dynamics, such as bifurcation scenarios and route to chaos, are displayed using parameter basin plots by numerical experiment. The influences of the parameters on the system performance are discussed from the perspective of economics.

1. Introduction

An *oligopoly* is a market structure between monopoly and perfect competition, in which the market is completely controlled by only a few number of firms producing the same or homogeneous productions [1, 2]. If there are two firms, it is called a *duopoly* while if there are three competitors, it is known as a *triopoly*.

Cournot oligopoly [3] and Bertrand oligopoly [4] are the two most notable models in oligopoly theory. In the Cournot model, firms control their production level, which influences the market price, while in the Bertrand model, firms choose the price of a unit of product to affect the market demand.

A large amount of the literature deals with Cournot or Bertrand competition in oligopolistic market [1, 2, 5–7], but there are only a considerably lower number of works devoted to Cournot-Bertrand competition, which are characterized by the fact that the market can be subdivided into two groups of firms, the first of which optimally adjusts prices and the other one optimally adjusts their output to ensure maximum profit [8].

Cournot-Bertrand model exists in realistic economy. For instance, in duopoly market, one firm competes in a dominant position, and it chooses output as decision variable while the other one is in disadvantage, and it chooses price as decision variable in order to gain more market share. As we have known so far, Bylka and Komar [9] and Singh and Vives [10] are the first authors to analyze duopolies, where one firm competes on quantities and the other on prices. Häckner [11], Zanchettin [12], and Arya et al. [13] pointed that in some cases Cournot-Bertrand competition may be optimal. Recently, C. H. Tremblay and V. J. Tremblay [14] analyzed the role of product differentiation for the static properties of the Nash equilibrium of a Cournot-Bertrand duopoly. Naimzada and Tramontana [8] considered a Cournot-Bertrand duopoly model, which is characterized by linear difference equations. They also analyzed the role of best response dynamics and of the adaptive adjustment mechanism for the stability of the equilibrium.

In this paper, we set up a Cournot-Bertrand duopoly model, assuming that two firms choose output and price as decision variable, respectively, and they all have bounded rational expectations. The gaming system can be described by nonlinear difference equations, which modifies and extends the results of Naimzada and Tramontana [8], which considered the firms with static expectations and described by linear difference equations. The research will lead to a good guidance for the enterprise decision-makers to do the best decision-making.

The paper is organized as follows the Cournot-Bertrand game model with bounded rational expectations is described in Section 2. In Section 3, the existence and stability of equilibrium points are studied. Dynamical behaviors under some change of control parameters of the game are investigated by numerical simulations in Section 4. Finally, a conclusion is drawn in Section 5.

2. The Cournot-Bertrand Game Model with Bounded Rational Expectations

We consider a market served by two firms and firm i produces good x_i, $i = 1, 2$. There is a certain degree of differentiation between the products x_1 and x_2. Firm 1 competes in output q_1 as in a Cournot duopoly, while firm 2 fixes its price p_2 like in the Bertrand case. Suppose that firms make their strategic choices simultaneously and each firm knows the production and the price of each other firm.

The inverse demand functions of products of variety 1 and 2 come from the maximization by the representative consumer of the following utility function:

$$U(q_1, q_2) = q_1 + q_2 - \frac{1}{2}\left(q_1^2 + 2dq_1q_2 + q_2^2\right) \tag{1}$$

subject to the budget constraint $p_1q_1 + p_2q_2 + y = M$ and are given by the following equations (the detailed proof see [15]):

$$\begin{aligned} p_1(t) &= 1 - q_1(t) - dq_2(t), \\ p_2(t) &= 1 - q_2(t) - dq_1(t), \end{aligned} \tag{2}$$

where the parameter $d \in (0, 1)$ denotes the index of product differentiation or product substitution. The degree of product differentiation will increase as $d \rightarrow 0$. Products x_1 and x_2 are homogeneous when $d = 1$, and each firm is a monopolist when $d = 0$, while a negative $d \in (-1, 0)$ implies that products are complements. Assume that the two firms have the same marginal cost $c > 0$, and the cost function has the linear form:

$$C_i(q_i(t)) = cq_i(t), \quad i = 1, 2. \tag{3}$$

We can write the demand system in the two strategic variables, $q_1(t)$ and $p_2(t)$:

$$\begin{aligned} p_1(t) &= 1 - d - \left(1 - d^2\right) q_1(t) + dp_2(t), \\ q_2(t) &= 1 - p_2(t) - dq_1(t). \end{aligned} \tag{4}$$

The profit functions of firm 1 and 2 are in the form:

$$\begin{aligned} \Pi_1(t) &= q_1(t)\left(1 - d + dp_2(t) - q_1(t) + d^2q_1(t)\right) \\ &\quad - cq_1(t), \\ \Pi_2(t) &= p_2(t)\left(1 - p_2(t) - dq_1(t)\right) \\ &\quad - c\left(1 - p_2(t) - dq_1(t)\right). \end{aligned} \tag{5}$$

We assume that the two firms do not have a complete knowledge of the market and the other player, and they build decisions on the basis of the expected marginal profit. If the marginal profit is positive (negative), they increases (decreases) their production or price in the next period; that is, they are bounded rational players [5, 15, 16]. Then the Cournot-Bertrand mixed dynamical system can be described by the nonlinear difference equations:

$$\begin{aligned} q_1(t+1) &= q_1(t) + \alpha q_1(t)\left(1 - c - d + dp_2(t)\right. \\ &\qquad \left. -2q_1(t) + 2d^2q_1(t)\right), \\ p_2(t+1) &= p_2(t)(t) + \beta p_2(t)\left(1 + c - 2p_2(t) - dq_1(t)\right), \end{aligned} \tag{6}$$

where $\alpha > 0$ and $\beta > 0$ represent the two players' adjustment speed in each relation, respectively.

3. Equilibrium Points and Local Stability

The system (6) has four equilibrium points:

$$\begin{aligned} &E_0 = (0, 0), \qquad E_1\left(0, \frac{1+c}{2}\right), \\ &E_2\left(\frac{1-c-d}{2\left(1-d^2\right)}, 0\right), \qquad E^*\left(q_1^*, p_2^*\right), \end{aligned} \tag{7}$$

where $q_1^* = (2 - 2c - d + cd)/(4 - 3d^2)$, $p_2^* = (2 + 2c - d + cd - d^2 - 2cd^2)/(4 - 3d^2)$. E_0, E_1, and E_2 are the boundary equilibrium points, and E^* is the unique Nash equilibrium point provided that $q_1^* > 0$ and $p_2^* > 0$, that requires $c < 1$. Otherwise, there will be one firm out of the market.

In order to investigate the local stability of the equilibrium points, let J be the Jacobian matrix of system (6) corresponding to the state variables (q_1, p_2), then

$$J(q_1, p_2) = \begin{pmatrix} J_{11} & \alpha dq_1 \\ -\beta dp_2 & J_{22} \end{pmatrix}, \tag{8}$$

where $J_{11} = 1 + \alpha(1 - c - d + dp_2 + 4(d^2 - 1)q_1)$, $J_{22} = 1 + \beta(1 + c - 4p_2 - dq_1)$. The stability of equilibrium points will be determined by the nature of the equilibrium eigenvalues of the Jacobian matrix evaluated at the corresponding equilibrium points.

Proposition 1. *The boundary equilibria E_0, E_1, and E_2 of system (6) are unstable equilibrium points when $c < 1$.*

Proof. For equilibrium E_0, the Jacobian matrix of system (6) is equal to

$$J(E_0) = \begin{pmatrix} 1+\alpha(1-c-d) & 0 \\ 0 & 1+\beta(1+c) \end{pmatrix}. \quad (9)$$

These eigenvalues that correspond to equilibrium E_0 are as follows:

$$\lambda_1 = 1+\alpha(1-c-d), \qquad \lambda_2 = 1+\beta(1+c). \quad (10)$$

Evidently $\lambda_2 > 1$, then the equilibrium point E_0 is unstable.

Also at E_1 the Jacobian matrix J becomes a triangular matrix

$$J(E_1) = \begin{pmatrix} 1+\alpha(1-c)\left(1-\dfrac{d}{2}\right) & 0 \\ -\dfrac{1}{2}\beta d(1+c) & 1-\beta(1+c) \end{pmatrix}. \quad (11)$$

These eigenvalues that correspond to equilibrium E_1 are as follows:

$$\lambda_1 = 1+\alpha(1-c)\left(1-\frac{d}{2}\right), \qquad \lambda_2 = 1-\beta(1+c). \quad (12)$$

When $c < 1$, evidently $\lambda_1 > 1$. So, the equilibrium point E_1 is unstable. Similarly we can prove that E_2 is also unstable. □

From an economic point of view we are more interested to the study of the local stability properties of the Nash equilibrium point E^*, whose properties have been deeply analyzed in [14].

The Jacobian matrix evaluated at the Nash equilibrium point E^* is as follows

$$J(E^*) = \begin{pmatrix} \dfrac{-4+3d^2+2\alpha(1-c)\left(2-d-2d^2+d^3\right)}{(3d^2-4)} & \dfrac{\alpha d(c-1)(2-d)}{4-3d^2} \\ \dfrac{\beta d\left(2-d-d^2+c\left(2+d-2d^2\right)\right)}{(3d^2-4)} & \dfrac{-4+3d^2-2\beta\left(-2+d+d^2+c\left(-2-d+2d^2\right)\right)}{(3d^2-4)} \end{pmatrix}. \quad (13)$$

The trace and determinant of $J(E^*)$ are denoted as $\mathrm{Tr}(J(E^*))$ and $\mathrm{Det}(J(E^*))$, respectively. With respect to the point E_0, E_1, and E_2, now it is more difficult to explicitly calculate the eigenvalues, but it is still possible to evaluate the stability of the Nash equilibrium point by using the following stability conditions, known as Jury's conditions [17]:

$$\begin{aligned} &\text{(i)}\ \ A := 1+\mathrm{Tr}(J(E^*))+\mathrm{Det}(J(E^*)) > 0, \\ &\text{(ii)}\ \ B := 1-\mathrm{Tr}(J(E^*))+\mathrm{Det}(J(E^*)) > 0, \\ &\text{(iii)}\ \ C := 1-\mathrm{Det}(J(E^*)) > 0. \end{aligned} \quad (14)$$

The above inequalities define a region in which the Nash equilibrium point E^* is local stable. Also, we can learn more about the stability region via numerical simulations. In order to study the complex dynamics of system (6), it is convenient to take the parameters values as follows:

$$c = 0.1, \qquad d = 0.2. \quad (15)$$

Figure 1 shows in the (α, β) parameters plane the stability and instability regions. From the figure, we can find that too high speed of adjustment will make the Nash equilibrium point E^* lose stability. We also find that the adjustment speed of price is more sensitive than the speed of output, and when about $\alpha > 2.5$, the Nash equilibrium point will lose stability, while about $\beta > 2.0$ the Nash equilibrium point will do that.

4. The Effects of Parameters on System Stability

The parameter basin plots (also called 2D bifurcation diagrams) are a more powerful tool in the numerical analysis of nonlinear dynamics than the 1D bifurcation diagrams [18], which assigns different colors in a 2D parameter space to stable cycles of different periods. In this section, the parameter basin plots will be used to analyze the effects of players' adjustment speed and index of product differentiation on system stability. We set $c = 0.1$ and the initial values are chosen as $(q_1(0), p_2(0)) = (0.2, 0.1)$.

4.1. The Effects of Players' Adjustment Speed on System Stability. Figure 2 presents the parameter basin with respect to the parameters (α, β) when $d = 0.1$ and assigns different colors to stable steady states (dark blue); stable cycles of periods 2 (light blue), 4 (purple), and 8 (green) (the first four cycles in a period-doubling bifurcation route to chaos) and periods 3 (red), 5 (orange), and 7 (pink) (low order stable cycles of odd period); chaos (yellow); divergence (white) (which means one of the players will be out of the market in economics).

We can find that when the parameters (α, β) pass through the borders as the black arrows A and C, system (6) loses its stability through flip bifurcation (called period-doubling bifurcation in continuous system), as shown in Figures 3 and 4. But when the parameters cross the borders as the arrow B, the system's dynamic behavior is more complicated, and it first enters into chaos through Neimark-Sacker bifurcation

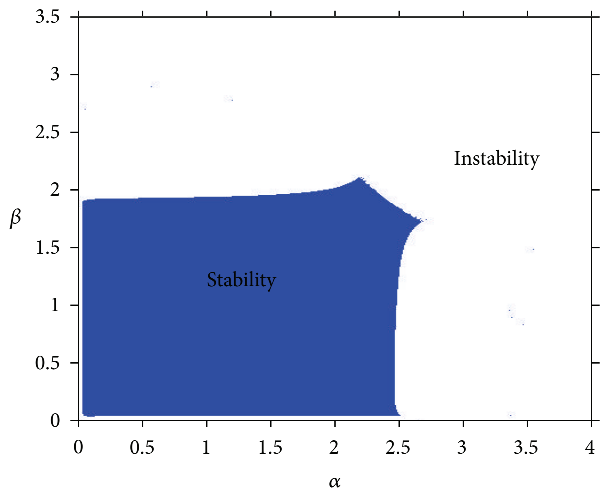

Figure 1: The stability and instability region.

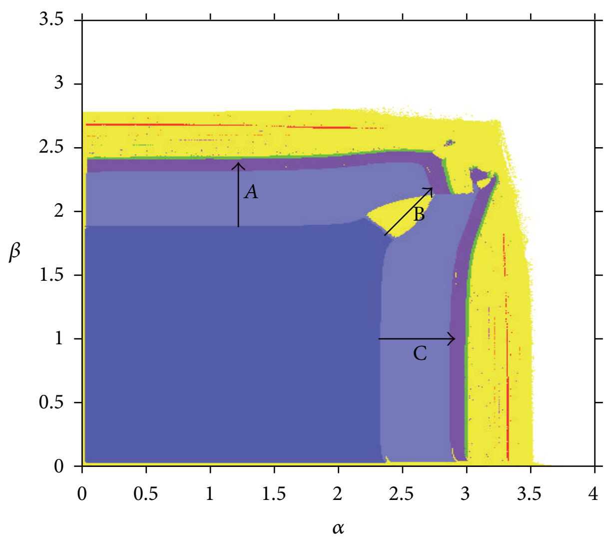

Figure 2: The parameter basin for $d = 0.1$.

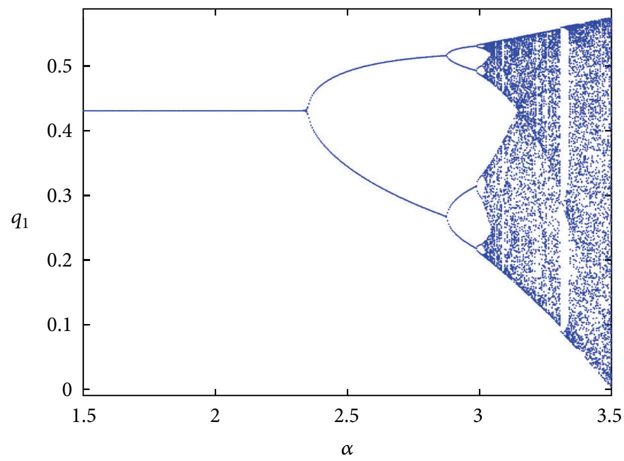

Figure 3: Bifurcation diagram for $\beta = 1$ and α varies from 1.5 to 3.5.

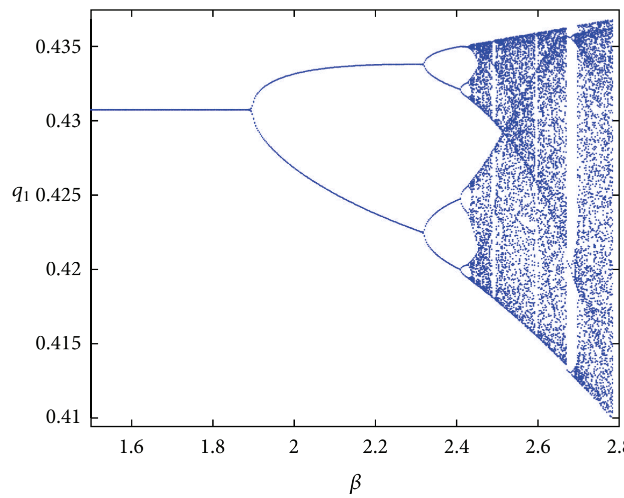

Figure 4: Bifurcation diagram for $\alpha = 1$ and β varies from 1.5 to 2.8.

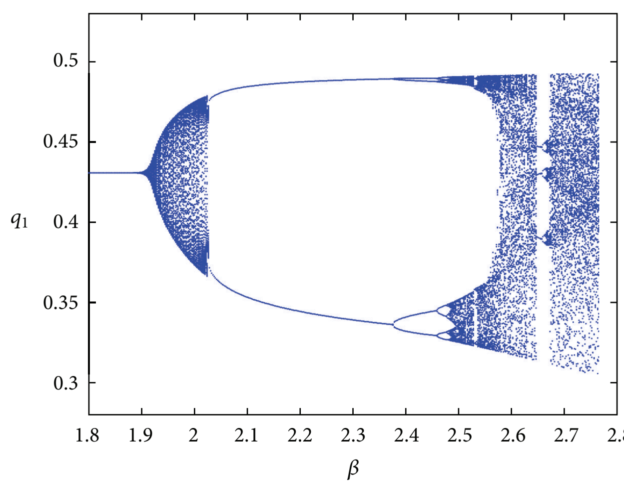

Figure 5: Bifurcation diagram for $\alpha = 2.3$ and β varies from 1.8 to 2.8.

(called Hopf bifurcation in continuous system) [19–21], second enters period 2, and then evolves into chaos through flip bifurcation separately, as shown in Figure 5. We also notice that in the yellow region (chaos) there is red line and orange points (odd cycle); that is, there is intermittent odd cycle in the chaos as shown in Figure 3 to Figure 5. It is well known that, for 1D continuous maps, a cycle with odd period implies chaotic dynamical behavior (the so-called topological chaos) according to the famous "period 3 implies chaos" result of Li and Yorke [22].

From the perspective of economics, the firms' adjustment speed α and β should be in a certain range; otherwise, the system will come forth the cycle fluctuation, and then into chaos, which means irregular, sensitive to initial values, unpredictable and bad for the economy. We also find that the adjustable range of α is larger than that of β, which means the

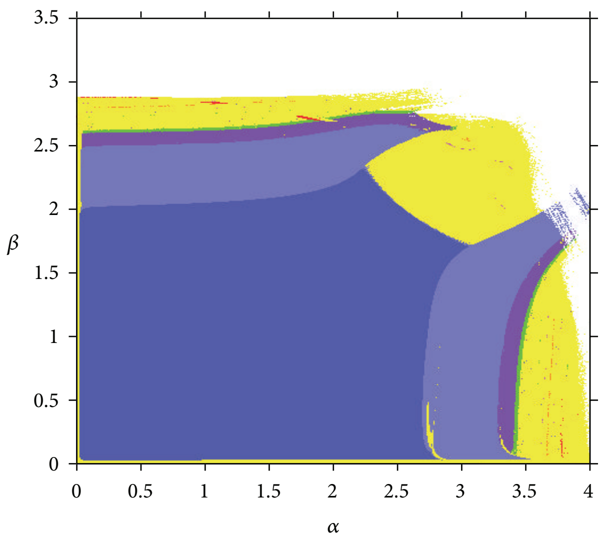

Figure 6: The parameter basin for $d = 0.3$.

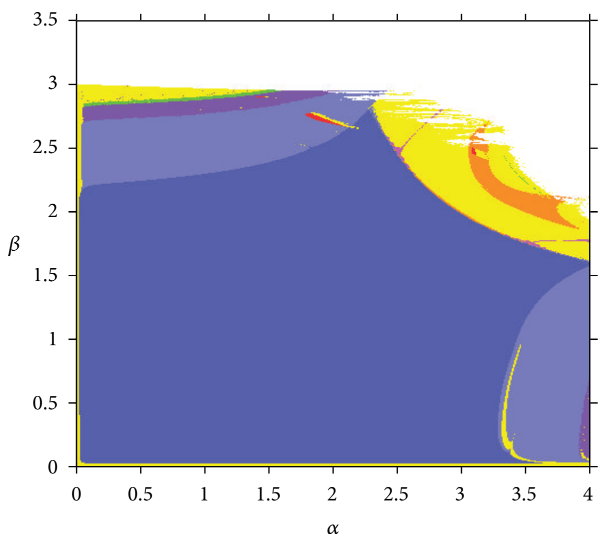

Figure 7: The parameter basin for $d = 0.5$.

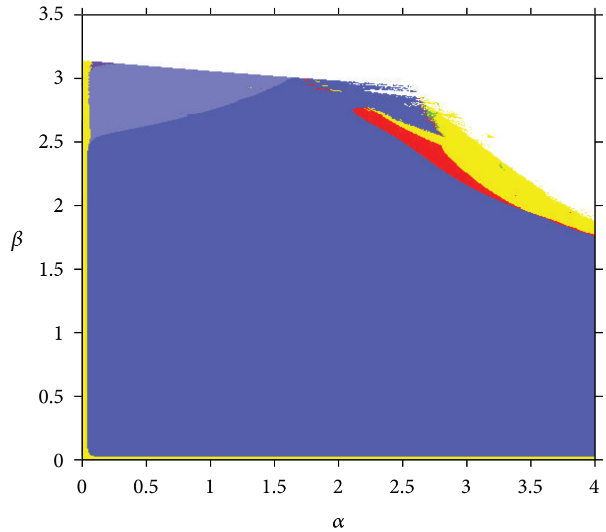

Figure 8: The parameter basin for $d = 0.7$.

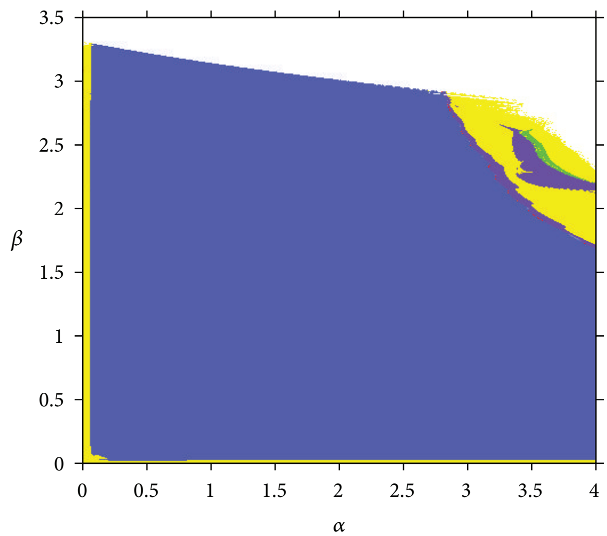

Figure 9: The parameter basin for $d = 0.9$.

adjustment of price is more sensitive than that of output, and price war is easier to get market into chaos.

4.2. The Effects of the Index of Product Differentiation on System Stability. In order to find the influences of the index of product differentiation d on the system stability, Figures 6, 7, 8, and 9 give the parameter basins for $d = 0.3, 0.5, 0.7$, and 0.9 separately.

From the comparison we can see the dark blue area becomes bigger and the yellow area becomes smaller with the increasing of the index of product differentiation d; that is, the degree of product differentiation is smaller, and the adjustable range of parameters α and β to make the system remain stable will become bigger, which means more competition between the two firms' products.

5. Conclusions

In this paper, we propose a Cournot-Bertrand mixed game model, supposing that the firms do not have the complete information of the market and opponent, and they make their decisions according to their own marginal profit. The demand and cost function is assumed to be linear and the model can be described by difference equations. The boundary equilibrium is always unstable and the existence and local stability of the Nash equilibrium are analyzed. Moreover, we analyze the effects of the parameters (adjustment speed and the index of product differentiation) on the system stability, and different bifurcations and routes to chaos are analyzed using parameter basin plots. The Cournot-Bertrand game models under different marketing environment need to be considered, and it will be an interesting topic for future study.

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