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Research Article

Positive Solutions for Three-Point Boundary Value Problem of Fractional Differential Equation with p-Laplacian Operator

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We investigate the existence of multiple positive solutions for three-point boundary value problem of fractional differential equation with p-Laplacian operator $-\mathscr{D}_{\mathbf{t}}{}^{\beta}(\varphi_p(\mathscr{D}_{\mathbf{t}}{}^{\alpha}x))(t) = h(t)f(t,x(t))$, $t \in (0,1)$, $x(0) = 0, \mathscr{D}_{\mathbf{t}}{}^{\gamma}x(1) = a\mathscr{D}_{\mathbf{t}}{}^{\gamma}x(\xi), \mathscr{D}_{\mathbf{t}}{}^{\alpha}x(0) = 0$, where $\mathscr{D}_{\mathbf{t}}{}^{\beta}, \mathscr{D}_{\mathbf{t}}{}^{\alpha}, \mathscr{D}_{\mathbf{t}}{}^{\gamma}$ are the standard Riemann-Liouville derivatives with $1 < \alpha \leq 2, 0 < \beta \leq 1, 0 < \gamma \leq 1, 0 \leq \alpha - \gamma - 1, \xi \in (0,1)$ and the constant a is a positive number satisfying $a\xi^{\alpha-\gamma-2} \leq 1-\gamma$; p-Laplacian operator is defined as $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. By applying monotone iterative technique, some sufficient conditions for the existence of multiple positive solutions are established; moreover iterative schemes for approximating these solutions are also obtained, which start off a known simple linear function. In the end, an example is worked out to illustrate our main results.

1. Introduction

In this paper, we study the existence of multiple positive solutions for the following three-point boundary value problem of fractional differential equation with p-Laplacian operator

$$-\mathscr{D}_{\mathbf{t}}{}^{\beta}\left(\varphi_p\left(\mathscr{D}_{\mathbf{t}}{}^{\alpha}x\right)\right)(t) = h(t)\, f(t, x(t)), \quad t \in (0,1),$$
$$x(0) = 0, \quad \mathscr{D}_{\mathbf{t}}{}^{\gamma}x(1) = a\mathscr{D}_{\mathbf{t}}{}^{\gamma}x(\xi), \tag{1}$$
$$\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(0) = 0,$$

where $\mathscr{D}_{\mathbf{t}}{}^{\beta}, \mathscr{D}_{\mathbf{t}}{}^{\alpha}, \mathscr{D}_{\mathbf{t}}{}^{\gamma}$ are the standard Riemann-Liouville derivatives with $1 < \alpha \leq 2, 0 < \beta \leq 1, 0 < \gamma \leq 1, 0 \leq \alpha - \gamma - 1, \xi \in (0,1)$ and the constant a is a positive number satisfying $a\xi^{\alpha-\gamma-2} \leq 1-\gamma$; p-Laplacian operator is defined as $\varphi_p(s) = |s|^{p-2}s$, $p > 1$.

Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, and engineering. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives; see the monographs of Kilbas et al. [1], Miller and Ross [2], Podlubny [3], and the papers [4–14] and the references therein.

In [15], Li et al. were concerned with the nonlinear differential equation of fractional order

$$\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(t) + f(t, x(t)) = 0, \quad t \in (0,1), \ 1 < \alpha \leq 2 \tag{2}$$

subject to the boundary conditions

$$x(0) = 0, \quad \mathscr{D}_{\mathbf{t}}{}^{\beta}x(1) = a\mathscr{D}_{\mathbf{t}}{}^{\beta}x(\xi). \tag{3}$$

By using some fixed point theorems, the existence and multiplicity results of positive solutions were established.

On the other hand, the differential equations with p-Laplacian have also been widely studied owing to the fact that p-Laplacian boundary value problems have important application in theory and application of mathematics and physics. For example, in [16], by using the fixed point index, Yang and Yan investigated the existence of positive solution for the third-order Sturm-Liouville boundary value problems with p-Laplacian operator:

$$\left(\varphi_p\left(x''(t)\right)\right)' + f(t, x(t)) = 0, \quad t \in (0,1),$$
$$ax(0) - bx'(0) = 0 \tag{4}$$
$$cx(1) + dx'(1) = 0, \quad x''(0) = 0.$$

However, there are few articles dealing with the existence of solutions to boundary value problems for fractional differential equation with p-Laplacian operator. In [17], the authors investigated the nonlinear nonlocal problem

$$\mathscr{D}_{\mathbf{t}}{}^{\beta}\left(\varphi_p\left(\mathscr{D}_{\mathbf{t}}{}^{\alpha}x\right)\right)(t) + f(t, x(t)) = 0, \quad t \in (0,1),$$
$$x(0) = 0, \qquad x(1) = ax(\xi), \tag{5}$$
$$\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(0) = 0,$$

where $0 < \alpha \le 2, 0 < \beta \le 1, 0 \le a \le 1, 0 < \xi < 1$. By using Krasnoselskii's fixed point theorem and Leggett-Williams theorem, some sufficient conditions for the existence of positive solutions to the above BVP are obtained. In [18], by using upper and lower solutions method, under suitable monotone conditions, Wang et al. investigated the existence of positive solutions to the following nonlocal problem:

$$\mathscr{D}_{\mathbf{t}}{}^{\beta}\&\left(\varphi_p\left(\mathscr{D}_{\mathbf{t}}{}^{\alpha}x\right)\right)(t) + f(t, x(t)) = 0, \quad t \in (0,1)$$
$$x(0) = 0, \qquad x(1) = ax(\xi), \tag{6}$$
$$\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(0) = 0, \qquad \mathscr{D}_{\mathbf{t}}{}^{\alpha}x(1) = b\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(\eta),$$

where $1 < \alpha, \beta \le 2, 0 \le a, b \le 1,\ 0 < \xi, \eta < 1$. Recently, Chai [19] investigated the two-point boundary value problem of fractional differential equation with p-Laplacian operator:

$$\mathscr{D}_{\mathbf{t}}{}^{\beta}\left(\varphi_p\left(\mathscr{D}_{\mathbf{t}}{}^{\alpha}x\right)\right)(t) + f(t, x(t)) = 0, \quad t \in (0,1),$$
$$x(0) = 0, \qquad x(1) + a\mathscr{D}_{\mathbf{t}}{}^{\gamma}x(1) = 0, \tag{7}$$
$$\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(0) = 0.$$

By means of the fixed point theorem on cones, some existence and multiplicity results of positive solutions are obtained.

Motivated by the above mentioned works, in this paper, we consider the multiplicity results of positive solutions for the three point boundary value problem of fractional differential equation with p-Laplacian operator. Difference to [15–19], by using monotone iterative technique, we not only establish the existence of multiple positive solutions but also obtain the iterative sequences of these positive solutions.

2. Preliminaries and Lemmas

In this section, we introduce some preliminary facts which are used throughout this paper.

Definition 1 (see [1–3]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\,ds \tag{8}$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2 (see [1–3]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \times \int_0^t (t-s)^{n-\alpha-1}x(s)\,ds, \tag{9}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Proposition 3 (see [1–3]). (1) *If $x \in L^1(0,1), \nu > \sigma > 0$, then*

$$I^{\nu}I^{\sigma}x(t) = I^{\nu+\sigma}x(t), \qquad \mathscr{D}_{\mathbf{t}}{}^{\sigma}I^{\nu}x(t) = I^{\nu-\sigma}x(t),$$
$$\mathscr{D}_{\mathbf{t}}{}^{\sigma}I^{\sigma}x(t) = x(t). \tag{10}$$

(2) *If $\nu > 0, \sigma > 0$, then*

$$\mathscr{D}_{\mathbf{t}}{}^{\nu}t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)}t^{\sigma-\nu-1}. \tag{11}$$

Proposition 4 (see [1–3]). *Let $\alpha > 0$, and $f(x)$ is integrable, then*

$$I^{\alpha}\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(t) = f(x) + c_1x^{\alpha-1} + c_2x^{\alpha-2} + \cdots + c_nx^{\alpha-n}, \tag{12}$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$, n is the smallest integer greater than or equal to α.

Definition 5. A function $u \in C(I, R)$ is called a nonnegative solution of BVP (1), if $u \ge 0$ on $[0,1]$ and satisfies (1). Moreover, if $u(t) > 0, t \in (0,1)$, then u is said to be a positive solution of BVP (1).

For forthcoming analysis, we first consider the following fractional differential equation:

$$\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(t) + g(t) = 0, \quad t \in (0,1),$$
$$x(0) = 0, \quad \mathscr{D}_{\mathbf{t}}{}^{\gamma}x(1) = a\mathscr{D}_{\mathbf{t}}{}^{\gamma}x(\xi). \tag{13}$$

Lemma 6 (see [15]). *If $1 < \alpha \le 2$ and $g \in L^1[0,1]$, then the boundary value problem* (13) *has the unique solution*

$$x(t) = \int_0^1 G(t,s)\,g(s)\,ds, \tag{14}$$

where

$$G(t,s) = \begin{cases} \dfrac{dt^{\alpha-1}(1-s)^{\alpha-\gamma-1} - adt^{\alpha-1}(\xi-s)^{\alpha-\gamma-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le \min\{t,\xi\} \le 1, \\ \dfrac{dt^{\alpha-1}(1-s)^{\alpha-\gamma-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 < \xi \le s \le t \le 1, \\ \dfrac{dt^{\alpha-1}(1-s)^{\alpha-\gamma-1} - adt^{\alpha-1}(\xi-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le \xi < 1, \\ \dfrac{dt^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & \max\{t,\xi\} \le s \le 1, \end{cases} \tag{15}$$

where $d = (1 - a\xi^{\alpha-\gamma-1})^{-1}$.

Lemma 7 (see [15]). *The Green function $G(t,s)$ in Lemma 6 has the following properties:*

(i) *$G(t,s)$ is continuous on $[0,1] \times [0,1]$,*

(ii) *$G(t,s) > 0$ for any $s, t \in (0,1)$.*

And if $a\xi^{\alpha-\gamma-2} \le 1 - \gamma$, the Green function $G(t,s)$ also satisfies

(iii) *$G(t,s) \le G(s,s)$ for any $s, t \in [0,1]$,*

(iv) *there exists a positive function $\sigma(s) \in C(0,1)$ such that*

$$\min_{\xi \le t \le 1} G(t,s) \ge \sigma(s) G(s,s), \tag{16}$$

where

$$G(s,s) = \begin{cases} \dfrac{ds^{\alpha-1}(1-s)^{\alpha-\gamma-1} - ads^{\alpha-1}(\xi-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0 \le s \le \xi, \\ \dfrac{ds^{\alpha-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & \xi \le s \le 1. \end{cases} \tag{17}$$

Let $\mathbb{N}$ be the set of positive integers, let $\mathbb{R}$ be the set of real numbers, and let $\mathbb{R}_+$ be the set of nonnegative real numbers. Let $I = [0,1]$. Denote by $C(I,\mathbb{R})$ the Banach space of all continuous functions from I into $\mathbb{R}$ with the norm

$$\|x\| = \max\{x(t) : t \in I\}. \tag{18}$$

Define the cone P in $C(I,\mathbb{R}_+)$ as

$$P = \left\{u \in C(I,\mathbb{R}_+) : \min_{\xi \le t \le 1} x(t) \ge \sigma(t)\|x\|,\ t \in I\right\}. \tag{19}$$

Let $q > 1$ satisfy the relation $(1/q) + (1/p) = 1$, where p is given by (1).

To study BVP (1), we first consider the associated linear BVP:

$$\begin{gathered} \mathscr{D}_{\mathbf{t}}{}^{\beta}\left(\varphi_p\left(\mathscr{D}_{\mathbf{t}}{}^{\alpha}x\right)\right)(t) + g(t) = 0, \quad t \in (0,1) \\ x(0) = 0, \qquad \mathscr{D}_{\mathbf{t}}{}^{\gamma}x(1) = a\mathscr{D}_{\mathbf{t}}{}^{\gamma}x(\xi), \\ \mathscr{D}_{\mathbf{t}}{}^{\alpha}x(0) = 0, \end{gathered} \tag{20}$$

for $g \in L^1[0,1]$ and $g \ge 0$.

Let $w = \mathscr{D}_{\mathbf{t}}{}^{\alpha}x, v = \varphi_p(w)$. By Proposition 4, the solution of initial value problem

$$\mathscr{D}_{\mathbf{t}}{}^{\beta}v(t) + g(t) = 0, \quad t \in (0,1), \ v(0) = 0, \tag{21}$$

is given by $v(t) = C_1 t^{\beta-1} - I^{\beta}g(t), t \in [0,1]$. From the relations $v(0) = 0, 0 < \beta \le 1$, it follows that $C_1 = 0$, and so

$$v(t) = -I^{\beta}g(t), \quad t \in [0,1]. \tag{22}$$

Noting that $\mathscr{D}_{\mathbf{t}}{}^{\alpha}x = w, w = \varphi_p^{-1}(v)$, from (22), we know that the solution of (20) satisfies

$$\begin{gathered} \mathscr{D}_{\mathbf{t}}{}^{\alpha}x(t) = \varphi_p^{-1}\left(-I^{\beta}g(t)\right), \quad t \in (0,1) \\ x(0) = 0, \quad \mathscr{D}_{\mathbf{t}}{}^{\gamma}x(1) = a\mathscr{D}_{\mathbf{t}}{}^{\gamma}x(\xi). \end{gathered} \tag{23}$$

By Lemma 6, the solution of (23) can be written as

$$x(t) = -\int_0^1 G(t,s)\varphi_p^{-1}\left(-I^{\beta}g(s)\right)ds, \quad t \in I. \tag{24}$$

Since $h(s) \ge 0$, $s \in I$, we have $\varphi_p^{-1}(-I^{\beta}g(s)) = -(I^{\beta}g(s))^{q-1}$, $s \in I$, and so

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)\left(I^{\beta}g(s)\right)^{q-1}ds \\ &= (\Gamma(\beta))^{1-q}\int_0^1 G(t,s)\left(\int_0^s (s-\tau)^{\beta-1}g(\tau)d\tau\right)^{q-1}ds, \\ &\qquad t \in I, \end{aligned} \tag{25}$$

which implies that the solution of (23) is given by

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)\left(I^{\beta}g(s)\right)^{q-1}ds \\ &= (\Gamma(\beta))^{1-q}\int_0^1 G(t,s)\left(\int_0^s (s-\tau)^{\beta-1}g(\tau)\,d\tau\right)^{q-1}ds. \end{aligned} \tag{26}$$

For the convenience, we make the following assumptions.

(H1) $f : [0,1] \times [0,+\infty) \to (0,+\infty)$ is continuous and nondecreasing, and there exists a constant $\lambda > 0$ such that, for any $t \in [0,1], x \in [0,+\infty)$,

$$f(t,cx) \ge c^{\lambda}f(t,x), \quad \forall 0 < c \le 1. \tag{27}$$

(H2) $h(t) \in L^1(0,1)$ is nonnegative on $(0,1)$, and

$$0 < \int_0^1 G(s,s)\left(\int_0^s (s-\tau)^{\beta-1} h(\tau)\, d\tau\right)^{q-1} ds < +\infty. \quad (28)$$

Remark 8. By (27), for any $c \geq 1, (t,x) \in [0,1] \times [0,+\infty)$, clearly,

$$f(t,cx) \leq c^{\lambda} f(t,x). \quad (29)$$

Now, for any $x \in P$, define one operator T as follows:

$$(Tx)(t) = (\Gamma(\beta))^{1-q} \int_0^1 G(t,s) \times \left(\int_0^s (s-\tau)^{\beta-1} h(\tau) f(\tau,x(\tau))\, d\tau\right)^{q-1} ds, \quad t \in I. \quad (30)$$

Then by (20) and (23), the BVP (1) is equivalent to the fixed point problems of the operators T.

Lemma 9. *Assume that (H1) and (H2) hold. Then $T : P \to P$ are continuous, compact, and nondecreasing.*

Proof. In fact, for any $x \in P$,

$$\begin{aligned} \|Tx\| &= \max_{t\in I}\left\{(\Gamma(\beta))^{1-q}\int_0^1 G(t,s) \times \left(\int_0^s (s-\tau)^{\beta-1} h(\tau) f(\tau,x(\tau))\, d\tau\right)^{q-1} ds\right\} \\ &\leq (\Gamma(\beta))^{1-q} \times \int_0^1 G(s,s)\left(\int_0^s (s-\tau)^{\beta-1} h(\tau) f(\tau,x(\tau))\, d\tau\right)^{q-1} ds. \end{aligned} \quad (31)$$

On the other hand, by Lemma 7,

$$\begin{aligned} \min_{\xi\leq t\leq 1} Tx(t) &= \min_{\xi\leq t\leq 1}\left\{(\Gamma(\beta))^{1-q}\int_0^1 G(t,s) \times \left(\int_0^s (s-\tau)^{\beta-1} h(\tau) f(\tau,x(\tau))\, d\tau\right)^{q-1} ds\right\} \\ &\geq \sigma(t)(\Gamma(\beta))^{1-q}\int_0^1 G(s,s) \times \left(\int_0^s (s-\tau)^{\beta-1} \times h(\tau) f(\tau,x(\tau))\, d\tau\right)^{q-1} ds \\ &= \sigma(t)\|Tx\|. \end{aligned} \quad (32)$$

So $T(P) \subset P$.

Next, supposing $D \subset P$ is a bounded set, then for any $x \in D$, there exists a constant $M > 0$ such that $\|x\| \leq M$. Thus for any $x \in D$, we have

$$\begin{aligned} \|Tx\| &= \max_{t\in I}\left\{(\Gamma(\beta))^{1-q} \times \int_0^1 G(t,s)\left(\int_0^s (s-\tau)^{\beta-1} \times h(\tau) f(\tau,x(\tau))\, d\tau\right)^{q-1} ds\right\} \\ &\leq (\Gamma(\beta))^{1-q} \times \int_0^1 G(s,s)\left(\int_0^s (s-\tau)^{\beta-1} h(\tau) f(\tau,x(\tau)) d\tau\right)^{q-1} ds \\ &\leq (\Gamma(\beta))^{1-q}\int_0^1 G(s,s)\left(\int_0^s (s-\tau)^{\beta-1} h(\tau)\, d\tau\right)^{q-1} ds \times \left(\sup_{[0,1]\times[0,M]} f(t,x)\right)^{q-1} \\ &< +\infty, \end{aligned} \quad (33)$$

which implies $T(D)$ is bounded. On the other hand, according to the Arzela-Ascoli theorem and Lebesgue dominated convergence theorem, we easily see $T : P \to P$ is completely continuous. In the end, noticing the monotonicity of f on x and the definition of T, we also have that the operator T is nondecreasing. □

3. Main Results

Define two constants

$$B = (\Gamma(\beta))^{1-q}\int_0^1 G(s,s)\left(\int_0^s (s-\tau)^{\beta-1} h(\tau)\, d\tau\right)^{q-1} ds, \qquad l = \max_{t\in I} f(t,1). \quad (34)$$

Theorem 10. *Suppose conditions (H1) and (H2) hold. If there exists a positive constant $b > 1$ such that*

$$l^{q-1} B \leq b^{1-\lambda(q-1)}, \quad (35)$$

where l and B are defined by (34), then the BVP (1) has the maximal and minimal solutions x^* and y^*, which are positive, and there exist two positive constants $m_1 \leq m_2$ such that

$$m_2\sigma(t) \leq x^*(t) \leq b, \qquad m_1\sigma(t) \leq y^*(t) \leq b, \quad t \in I. \quad (36)$$

Moreover for initial values $x_0 = b$, $y_0 = 0$, define the iterative sequences x_n, y_n by

$$x_n(t) = (Tx_{n-1})(t) = T^n x_0(t), \qquad y_n(t) = (Ty_{n-1})(t) = T^n y_0(t). \quad (37)$$

Then

$$\lim_{n\to+\infty} x_n = x^*, \qquad \lim_{n\to+\infty} y_n = y^* \tag{38}$$

for $t \in I$ uniformly, respectively.

Proof. Let $P[0,b] = \{x \in P : 0 \le \|x\| \le b\}$; we firstly prove $T(P[0,b]) \subset P[0,b]$. In fact, for any $x \in P[0,b]$, we have

$$0 \le x(t) \le \max_{t\in I} x(t) = \|x\| \le b. \tag{39}$$

By the assumption (H1), we have

$$\begin{aligned} 0 &\le f(t, x(t)) \le f(t, b) \\ &\le b^\lambda f(t,1) \le b^\lambda \max_{t\in I} f(t,1) = lb^\lambda. \end{aligned} \tag{40}$$

It follows from Lemma 9 that $T : P \to P$ is completely continuous operator; thus by (35) and (40), we have

$$\begin{aligned} \|Tx\| &= \max_{t\in I}\Bigg\{ (\Gamma(\beta))^{1-q} \\ &\quad \times \int_0^1 G(t,s) \left(\int_0^s (s-\tau)^{\beta-1} \right. \\ &\quad \left. \times h(\tau) f(\tau, x(\tau))\, dt \right)^{q-1} ds \Bigg\} \\ &\le (\Gamma(\beta))^{1-q} \\ &\quad \times \int_0^1 G(s,s) \left(\int_0^s (s-\tau)^{\beta-1} h(\tau) f(\tau, x(\tau))\, d\tau \right)^{q-1} ds \\ &\le (\Gamma(\beta))^{1-q} \\ &\quad \times \int_0^1 G(s,s) \left(\int_0^s (s-\tau)^{\beta-1} h(\tau)\, d\tau \right)^{q-1} ds \\ &\quad \times \left(lb^\lambda\right)^{q-1} \\ &= B\left(lb^\lambda\right)^{q-1} \le b, \end{aligned} \tag{41}$$

which implies that $T(P[0,b]) \subset P[0,b]$.

Let $y_0(t) = 0$, $t \in [0,1]$; then $y_0(t) \in P[0,b]$. Leting $y_1(t) = (Ty_0)(t)$, we have $y_1 \in P[0,b]$. Denote

$$y_{n+1} = Ty_n = T^{n+1} y_0, \quad n = 1, 2, \ldots. \tag{42}$$

It follows from $T(P[0,b]) \subset P[0,b]$ that $y_n \in P[0,b]$. Since T is compact, we obtain that $\{y_n\}$ is a sequentially compact set.

Since $y_1 = Ty_0 = T0 \in P[0,b]$, we have

$$\begin{aligned} y_1(t) &= \left(Ty_0\right)(t) \\ &= (T0)(t) \ge 0 = y_0(t), \quad t \in [0,1]. \end{aligned} \tag{43}$$

By the induction, we get

$$y_{n+1} \ge y_n, \quad n = 0, 1, 2, \ldots. \tag{44}$$

Consequently, there exists $y^* \in P[0,b]$ such that $y_n \to y^*$. Letting $n \to +\infty$, from the continuity of T and $Ty_n = y_{n-1}$, we obtain $Ty* = y^*$, which implies that y^* is a nonnegative solution of boundary value problem (1). Since $f : [0,1] \times [0,+\infty) \to (0,+\infty)$, we know the zero function is not the solution of boundary value problem (1), thus $\max_{0\le t\le 1}|y^*(t)| > 0$; by $y^* \in P$, we have

$$y^*(t) \ge \|y^*\| \sigma(t) > 0, \quad t \in (0,1), \tag{45}$$

that is, y^* is a positive solution of boundary value problem (1).

On the other hand, let $x_0(t) = b$, $t \in [0,1]$; then $x_0(t) \in P[0,b]$. Leting $x_1 = Tx_0$, from the previous expressions, we have $x_1 \in P[0,b]$. Thus let us denote

$$x_{n+1} = Tx_n = T^{n+1} x_0, \quad n = 1, 2, \ldots. \tag{46}$$

It follows from $T(P[0,b]) \subset P[0,b]$ that

$$x_n \in P[0,b] \quad n = 0, 1, 2, \ldots. \tag{47}$$

Since T is compact by Lemma 9, we can assert that $\{x_n\}$ is a sequentially compact set.

Now, since $x_1 \in P[0,b]$, we have

$$0 \le x_1(t) \le \|x_1\| \le b = x_0(t). \tag{48}$$

It follows from Lemma 9 that $T : P \to P$ is nondecreasing, so

$$x_2 = Tx_1 \le Tx_0 = x_1. \tag{49}$$

By the induction, we have

$$x_{n+1} \le x_n, \quad n = 0, 1, 2, \ldots. \tag{50}$$

Consequently, there exists $x^* \in P[0,b]$ such that $x_n \to x^*$. Letting $n \to +\infty$, from the continuity of T and $Tx_n = x_{n-1}$, we obtain $Tx^* = x^*$, which implies that x^* is a nonnegative solution of boundary value problem (1).

Next, noting $x_0(t) = b \ge y_0(t) = 0$, $t \in [0,1]$, thus it follows from monotonicity of T that $Tx_0 \ge Ty_0$; by the induction, we have $x_n \ge y_n$, $n = 0, 1, 2, \ldots$, which implies that $x^* \ge y^*$. Thus by (45) we have

$$x^*(t) \ge y^*(t) \ge \|x^*\| \sigma(t) > 0, \quad t \in (0,1). \tag{51}$$

This means that x^* is also a positive solution of boundary value problem (1).

In the end, let u^* be any fixed point of T in $P[0,b]$, then

$$y_0 = 0 \le u^* \le b = x_0, \tag{52}$$

and then

$$y_1 = Ty_0 \le Tu^* = u^* \le Tb = x_1. \tag{53}$$

By induction, we have

$$y_n \le u^* \le x_n, \quad n = 1, 2, 3, \ldots. \tag{54}$$

Taking the limit, we have

$$y^* \leq u^* \leq x^*. \tag{55}$$

This implies that x^* and y^* are maximal and minimal solutions of the BVP (1). Let $m_1 = \|y^*\|$, $m_2 = \|x^*\|$, then we have

$$\begin{aligned} m_2\sigma(t) &\leq x^*(t) \leq b, \\ m_1\sigma(t) &\leq y^*(t) \leq b, \quad t \in I. \end{aligned} \tag{56}$$

The proof is completed. □

Remark 11. If $h(t) \equiv 1$, then (H2) holds naturally, and in this case we take

$$\begin{aligned} B^* &= (\Gamma(\beta))^{1-q} \int_0^1 \frac{d}{\Gamma(\alpha)} \left(\int_0^s (s-\tau)^{\beta-1} d\tau \right)^{q-1} ds \\ &= \frac{d(\beta\Gamma(\beta))^{1-q}}{\Gamma(\alpha)(\beta(q-1)+1)}. \end{aligned} \tag{57}$$

Thus we have the following Corollary 12.

Corollary 12. *Suppose condition (H1) holds and $h(t) \equiv 1$. If there exists a positive constant $b > 1$ such that*

$$l^{q-1}B^* \leq b^{1-\lambda(q-1)}, \tag{58}$$

where l is defined by (34), then the BVP (1) has the maximal and minimal solutions x^* and y^*, which are positive, and there exist two positive constants $m_1 \leq m_2$ such that

$$\begin{aligned} m_2\sigma(t) &\leq x^*(t) \leq b, \\ m_1\sigma(t) &\leq y^*(t) \leq b, \quad t \in I. \end{aligned} \tag{59}$$

Moreover for initial values $x_0 = b$, $y_0 = 0$, define the iterative sequences x_n, y_n by

$$\begin{aligned} x_n(t) &= (Tx_{n-1})(t) = T^n x_0(t), \\ y_n(t) &= (Ty_{n-1})(t) = T^n y_0(t). \end{aligned} \tag{60}$$

Then

$$\lim_{n\to+\infty} x_n = x^*, \qquad \lim_{n\to+\infty} y_n = y^* \tag{61}$$

for $t \in I$ uniformly, respectively.

Corollary 13. *Suppose conditions (H1) and (H2) hold. If*

$$\lambda < p - 1. \tag{62}$$

Then there exists a constant $b > 1$ such that the BVP (1) has the maximal and minimal solutions x^* and y^*, which are positive, and there exist two positive constants $m_1 \leq m_2$ such that

$$\begin{aligned} m_2\sigma(t) &\leq x^*(t) \leq b, \\ m_1\sigma(t) &\leq y^*(t) \leq b, \quad t \in I. \end{aligned} \tag{63}$$

Moreover for initial values $x_0 = b$, $y_0 = 0$, define the iterative sequences x_n, y_n by

$$\begin{aligned} x_n(t) &= (Tx_{n-1})(t) = T^n x_0(t), \\ y_n(t) &= (Ty_{n-1})(t) = T^n y_0(t). \end{aligned} \tag{64}$$

Then

$$\lim_{n\to+\infty} x_n = x^*, \qquad \lim_{n\to+\infty} y_n = y^* \tag{65}$$

for $t \in I$ uniformly, respectively.

Proof. It follows from $\lambda < p - 1$ that

$$\lim_{u\to+\infty} \frac{u^\lambda}{u^{p-1}} = 0, \tag{66}$$

which implies that there exists $b > 1$ large enough such that

$$\frac{b^\lambda}{b^{p-1}} < \frac{1}{B^{*\,p-1} l}. \tag{67}$$

Notice that $(1/p) + (1/q) = 1$; (67) is equivalent to

$$l^{q-1}B^* \leq b^{1-\lambda(q-1)}. \tag{68}$$

By Theorem 10, the conclusion of Corollary 13 holds. □

Remark 14. In Corollary 13, we obtain that the BVP (1) has the maximal and minimal solutions x^* and y^* only by comparing $p-1$ to λ. But note that p and λ are irrelative, so (62) is easy to be satisfied; this implies that Corollary 13 is very interesting.

Example 15. Consider the following boundary value problem:

$$\begin{aligned} -\mathscr{D}_{\mathbf{t}}^{1/2} &\left(\varphi_6\left(\mathscr{D}_{\mathbf{t}}^{3/2} x\right)\right)(t) \\ &= \sin tx^4(t) + t^2 x^{3/4}(t) + 2t^3 \\ &\quad + 5t + 2, \quad t \in (0,1), \\ x(0) = 0, \quad &\mathscr{D}_{\mathbf{t}}^{1/6} x(1) = \frac{1}{6\sqrt[3]{16}} \mathscr{D}_{\mathbf{t}}^{1/6} x\left(\frac{1}{4}\right), \\ &\mathscr{D}_{\mathbf{t}}^{3/2} x(0) = 0. \end{aligned} \tag{69}$$

Let $\alpha = 3/2$, $\beta = 1/2$, $\gamma = 1/6$, $p = 6$, and $f(t,x) = \sin tx^4 + t^2x^{3/4} + 2t^3 + 5t + 2$, $h(t) \equiv 1$, then

$$\begin{aligned} a\xi^{\alpha-\gamma-2} &= \frac{1}{6\sqrt[3]{16}} \left(\frac{1}{4}\right)^{-2/3} \\ &= \frac{1}{6} < 1 - \gamma = \frac{5}{6}. \end{aligned} \tag{70}$$

For any $0 < c \leq 1$ and $x \in [0,+\infty)$, we have

$$\begin{aligned} f(t,cx) &= \sin t(cx)^4 + t^2(cx)^{3/4} + 2t^3 + 5t + 2 \\ &\geq \sin t(cx)^4 + t^2c^4x^{3/4} \\ &\quad + c^4\left(2t^3 + 5t + 2\right) \\ &= c^4 f(t,x). \end{aligned} \tag{71}$$

Taking $\lambda = 4$, then

$$\lambda = 4 < p - 1 = 5, \quad (72)$$

which implies that (62) holds. By Corollary 13, we know the BVP (69) has at least two positive solutions.

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