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Research Article

Permanence and Global Attractivity of the Discrete Predator-Prey System with Hassell-Varley-Holling III Type Functional Response

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By constructing a suitable Lyapunov function and using the comparison theorem of difference equation, sufficient conditions which ensure the permanence and global attractivity of the discrete predator-prey system with Hassell-Varley-Holling III type functional response are obtained. An example together with its numerical simulation shows that the main results are verifiable.

1. Introduction

Recently, there were many works on predator-prey system done by scholars [1–6]. In particular, since Hassell-Varley [7] proposed a general predator-prey model with Hassell-Varley type functional response in 1969, many excellent works have been conducted for the Hassell-Varley type system [1, 7–13].

Liu and Huang [8] studied the following discrete predator-prey system with Hassell-Varley-Holling III type functional response:

$$\begin{aligned} x(k+1) &= x(k)\exp\left\{a(k) - b(k)x(k) - \frac{A(k)x(k)y(k)}{r(k)x^2(k) + y^{2R}(k)}\right\}, \\ y(k+1) &= y(k)\exp\left\{-d(k) + \frac{B(k)x^2(k)}{r(k)x^2(k) + y^{2R}(k)}\right\}, \quad R \in (0,1), \end{aligned} \tag{1}$$

where $x(k)$, $y(k)$ denote the density of prey and predator species at the kth generation, respectively. a, b, A, r, d, B are all periodic positive sequences with common period X. Here $a(k)$ represents the intrinsic growth rate of prey species at the kth generation, and $b(k)$ measures the intraspecific effects of the kth generation of prey species on their own population; $d(k)$ is the death rate of the predator; $A(k)$ is the capturing rate; $B(k)$ is the maximal growth rate of the predator. Liu and Huang obtained the necessary and sufficient conditions for the existences of positive periodic solutions by applying a new estimation technique of solutions and the invariance property of homotopy. As we know, the persistent property is one of the most important topics in the study of population dynamics. For more papers on permanence and extinction of population dynamics, one could refer to [2–5, 14–17] and the references cited therein. The purpose of this paper is to investigate permanence and global attractivity of this system.

We argue that a general nonautonomous nonperiodic system is more appropriate, and thus, we assume that the coefficients of system (1) satisfy the following:

(A) a, b, A, r, d, B are nonnegative sequences bounded above and below by positive constants.

By the biological meaning, we consider (1) together with the following initial conditions as

$$x(0) > 0, \qquad y(0) > 0. \tag{2}$$

For the rest of the paper, we use the following notations: for any bounded sequence $\{h(k)\}$, set $h^u = \sup_{k\in N}\{h(k)\}$ and $h^l = \inf_{k\in N}\{h(k)\}$.

2. Permanence

Now, let us state several lemmas which will be useful to prove our main conclusion.

Definition 1 (see [5]). System (1) said to be permanent if there exist positive constants m and M, which are independent of the solution of system (1), such that for any positive solution $\{x(k), y(k)\}$ of system (1) satisfies

$$m \le \liminf_{k\to+\infty}\{x(k), y(k)\} \le \limsup_{k\to+\infty}\{x(k), y(k)\} \le M. \tag{3}$$

Lemma 2 (see [14]). *Assume that $\{x(k)\}$ satisfies $x(k) > 0$ and*

$$x(k+1) \le x(k)\exp\{a(k) - b(k)x(k)\}, \tag{4}$$

for $k \in N$, where $a(k)$ and $b(k)$ are all nonnegative sequences bounded above and below by positive constants. Then,

$$\limsup_{k\to+\infty} x(k) \le \frac{1}{b^l}\exp(a^u - 1). \tag{5}$$

Lemma 3 (see [14]). *Assume that $\{x(k)\}$ satisfies*

$$x(k+1) \ge x(k)\exp\{a(k) - b(k)x(k)\}, \quad k \ge N_0, \tag{6}$$

$\limsup_{k\to+\infty} x(k) \le x^*$, *and $x(N_0) > 0$, where $a(k)$ and $b(k)$ are all nonnegative sequences bounded above and below by positive constants and $N_0 \in N$. Then,*

$$\liminf_{k\to+\infty} x(k) \ge \min\left\{\frac{a^l}{b^u}\exp\{a^l - b^u x^*\}, \frac{a^l}{b^u}\right\}. \tag{7}$$

Theorem 4. *Assume that*

$$a^l - \frac{A^u M_2^{1-R}}{2\sqrt{r^l}} > 0, \tag{H_1}$$

$$B^l - d^u r^u > 0 \tag{H_2}$$

hold, then system (1) is permanent, that is, for any positive solution $\{x(k), y(k)\}$ of system (1), one has

$$\begin{aligned} m_1 &\le \liminf_{k\to+\infty} x(k) \le \limsup_{k\to+\infty} x(k) \le M_1, \\ m_2 &\le \liminf_{k\to+\infty} y(k) \le \limsup_{k\to+\infty} y(k) \le M_2, \end{aligned} \tag{8}$$

where

$$\begin{aligned} m_1 &= \frac{a^l - \left(A^u M_2^{1-R}/2\sqrt{r^l}\right)}{b^u} \times \exp\left\{a^l - \frac{A^u M_2^{1-R}}{2\sqrt{r^l}} - b^u M_1\right\}, \\ m_2 &= \min\left\{\left\{\frac{\left(B^l - r^u d^u\right)m_1^2}{d^u}\right\}^{1/2R}, \left\{\frac{\left(B^l - r^u d^u\right)m_1^2}{d^u}\right\}^{1/2R} \times \exp\left\{-d^u + \frac{B^l m_1^2}{r^u m_1^2 + M_2^{2R}}\right\}\right\}, \\ M_1 &= \frac{1}{b^l}\exp(a^u - 1), \\ M_2 &= \left\{\frac{B^u M_1^2}{d^l}\right\}^{1/2R}\exp\left\{-d^l + \frac{B^u}{r^l}\right\}. \end{aligned} \tag{9}$$

Proof. We divided the proof into four steps.

Step 1. We show

$$\limsup_{k\to+\infty} x(k) \le M_1. \tag{10}$$

From the first equation of (1), we have

$$x(k+1) \le x(k)\exp\{a(k) - b(k)x(k)\}. \tag{11}$$

By Lemma 2, we have

$$\limsup_{k\to+\infty} x(k) \le \frac{1}{b^l}\exp(a^u - 1) = M_1. \tag{12}$$

Previous inequality shows that for any $\varepsilon > 0$, there exists a $k_1 > 0$, such that

$$x(k) \le M_1 + \varepsilon, \quad \forall k \ge k_1. \tag{13}$$

Step 2. We prove $\limsup_{k\to+\infty} y(k) \le M_2$ by distinguishing two cases.

Case 1. There exists a $l_0 \ge k_1$, such that $y(l_0 + 1) \ge y(l_0)$.

By the second equation of system (1), we have

$$-d(l_0) + \frac{B(l_0)x^2(l_0)}{r(l_0)x^2(l_0) + y^{2R}(l_0)} \ge 0, \tag{14}$$

which implies

$$-d(l_0) + \frac{B(l_0)x^2(l_0)}{y^{2R}(l_0)} \ge 0. \tag{15}$$

The previous inequality combined with (13) leads to $y(l_0) \le \{B^u(M_1+\varepsilon)^2/d^l\}^{1/2R}$. Thus, from the second equation of system (1), again we have

$$\begin{aligned} y(l_0+1) &= y(l_0)\exp\left\{-d(l_0)+\frac{B(l_0)x^2(l_0)}{r(l_0)x^2(l_0)+y^{2R}(l_0)}\right\} \\ &\le \left\{\frac{B^u(M_1+\varepsilon)^2}{d^l}\right\}^{1/2R}\exp\left\{-d^l+\frac{B^u}{r^l}\right\} \\ &\overset{\text{def}}{=} M_{2\varepsilon}. \end{aligned} \tag{16}$$

We claim that

$$y(k) \le M_{2\varepsilon} \quad \forall k \ge l_0. \tag{17}$$

By a way of contradiction, assume that there exists a $p_0 \ge l_0$ such that $y(p_0) > M_{2\varepsilon}$. Then $p_0 \ge l_0+2$. Let $\widetilde{p_0} \ge l_0+2$ be the smallest integer such that $y(\widetilde{p_0}) > M_{2\varepsilon}$. Then $y(\widetilde{p_0}) > y(\widetilde{p_0}-1)$. The previous argument produces that $y(\widetilde{p_0}) \le M_{2\varepsilon}$, a contradiction. This proves the claim. Therefore, $\limsup_{k\to+\infty} y(k) \le M_{2\varepsilon}$. Setting $\varepsilon \to 0$ in it leads to $\limsup_{k\to+\infty} y(k) \le M_2$.

Case 2. Suppose $y(k+1) < y(k)$ for all $k \ge k_1$. Since $y(k)$ is nonincreasing and has a lower bound 0, we know that $\lim_{k\to+\infty} y(k)$ exists, denoted by $\overline{y}$, we claim that

$$\overline{y} \le \left\{\frac{B^u M_1^2}{d^l}\right\}^{1/2R}. \tag{18}$$

By a way of contradiction, assume that $\overline{y} > \{B^u M_1^2/d^l\}^{1/2R}$.

Taking limit in the second equation in system (1) gives

$$\lim_{k\to+\infty}\left\{-d(k)+\frac{B(k)x^2(k)}{r(k)x^2(k)+y^{2R}(k)}\right\} = 0, \tag{19}$$

however,

$$\begin{aligned} &\lim_{k\to+\infty}\left\{-d(k)+\frac{B(k)x^2(k)}{r(k)x^2(k)+y^{2R}(k)}\right\} \\ &\quad\le \limsup_{k\to+\infty}\left\{-d(k)+\frac{B(k)x^2(k)}{r(k)x^2(k)+y^{2R}(k)}\right\} \\ &\quad\le -d^l+\frac{B^u M_1^2}{\overline{y}^{2R}} \\ &\quad< 0, \end{aligned} \tag{20}$$

which is a contradiction. It implies that $\overline{y} \le \{B^u M_1^2/d^l\}^{1/2R}$. By the fact $B^u > d^l r^l$, we obtain that

$$\overline{y} \le \left\{\frac{B^u M_1^2}{d^l}\right\}^{1/2R} \le \left\{\frac{B^u M_1^2}{d^l}\right\}^{1/2R}\exp\left\{-d^l+\frac{B^u}{r^l}\right\} = M_2. \tag{21}$$

Therefore, we have

$$\lim_{k\to+\infty} y(k) = \overline{y} \le M_2. \tag{22}$$

Then,

$$\limsup_{k\to+\infty} y(k) \le M_2. \tag{23}$$

Step 3. We verify

$$\liminf_{k\to+\infty} x(k) \ge m_1. \tag{24}$$

Conditions (H_1) imply that for enough small positive constant ε, we have

$$a^l - \frac{A^u(M_2+\varepsilon)^{1-R}}{2\sqrt{r^l}} > 0. \tag{25}$$

For the previous ε, it follows from Steps 1 and 2 that there exists a k_2 such that for all $k \ge k_2$

$$x(k) \le M_1+\varepsilon, \qquad y(k) \le M_2+\varepsilon. \tag{26}$$

Then, for $k \ge k_2$, it follows from (26) and the first equation of system (1) that

$$x(k+1) \ge x(k)\exp\left\{a^l-\frac{A^u(M_2+\varepsilon)^{1-R}}{2\sqrt{r^l}}-b^u x(k)\right\}. \tag{27}$$

According to Lemma 3, one has

$$\begin{aligned} &\liminf_{k\to+\infty} x(k) \\ &\quad\ge \min\left\{m_{1*}, \frac{a^l - A^u(M_2+\varepsilon)^{1-R}/2\sqrt{r^l}}{b^u}\right\} \\ &\quad= m_{1*}, \end{aligned} \tag{28}$$

where

$$\begin{aligned} m_{1*} = {} & \frac{a^l - A^u(M_2+\varepsilon)^{1-R}/2\sqrt{r^l}}{b^u} \\ & \times \exp\left\{a^l - \frac{A^u(M_2+\varepsilon)^{1-R}}{2\sqrt{r^l}} - b^u(M_1+\varepsilon)\right\}. \end{aligned} \tag{29}$$

Setting $\varepsilon \to 0$ in (28) leads to

$$\begin{aligned} \liminf_{k\to+\infty} x(k) \ge {} & \frac{a^l - A^u M_2^{1-R}/2\sqrt{r^l}}{b^u} \\ & \times \exp\left\{a^l - \frac{A^u M_2^{1-R}}{2\sqrt{r^l}} - b^u M_1\right\} = m_1. \end{aligned} \tag{30}$$

By the fact that $\min_{x\in R^+}\{[\exp(x-1)]/x\} = 1$, we see that $M_1 = \exp(a^u - 1)/b^l \geq a^u/b^l \geq a^l/b^u \geq (a^l - A^u M_2^{1-R}/2\sqrt{r^l})/b^l \geq m_1$.

This ends the proof of Step 3.

Step 4. We present two cases to prove that

$$\liminf_{k\to+\infty} y(k) \geq m_2. \tag{31}$$

For any small positive constant $\varepsilon < m_1/2$, from Step 1 to Step 3, it follows that there exists a $k_3 \geq k_2$ such that for all $k \geq k_3$

$$\begin{gathered} x(k) \geq m_1 - \varepsilon, \qquad x(k) \leq M_1 + \varepsilon, \\ y(k) \leq M_2 + \varepsilon. \end{gathered} \tag{32}$$

Case 1. There exists a $n_0 \geq k_3$ such that $y(n_0+1) \leq y(n_0)$, then

$$-d(n_0) + \frac{B(n_0)x^2(n_0)}{r(n_0)x^2(n_0) + y^{2R}(n_0)} \leq 0. \tag{33}$$

Hence,

$$y(n_0) \geq \left\{\frac{(B^l - r^u d^u)(m_1-\varepsilon)^2}{d^u}\right\}^{1/2R} \overset{\text{def}}{=} c_{1\varepsilon}, \tag{34}$$

and so,

$$\begin{aligned} &y(n_0+1) \\ &\geq \left\{\frac{(B^l - r^u d^u)(m_1-\varepsilon)^2}{d^u}\right\}^{1/2R} \\ &\quad\times \exp\left\{-d^u + \frac{B^l(m_1-\varepsilon)^2}{r^u(m_1-\varepsilon)^2 + (M_2+\varepsilon)^{2R}}\right\} \\ &\overset{\text{def}}{=} c_{2\varepsilon}. \end{aligned} \tag{35}$$

Set

$$m_{2\varepsilon} = \min\{c_{1\varepsilon}, c_{2\varepsilon}\}. \tag{36}$$

We claim that

$$y(k) \geq m_{2\varepsilon} \quad \forall k \geq n_0. \tag{37}$$

By a way of contradiction, assume that there exists a $q_0 \geq n_0$, such that $y(q_0) < m_{2\varepsilon}$. Then $q_0 \geq n_0+2$. Let $\widetilde{q_0} \geq n_0+2$ be the smallest integer such that $y(\widetilde{q_0}) < m_{2\varepsilon}$. Then $y(\widetilde{q_0}) < y(\widetilde{q_0}-1)$, which implies that $y(q_0) \geq m_{2\varepsilon}$, a contradiction, this proves the claim. Therefore, $\liminf_{k\to+\infty} y(k) \geq m_{2\varepsilon}$, setting $\varepsilon \to 0$ in it leads to $\liminf_{k\to+\infty} y(k) \geq m_2$.

Case 2. Assume that $y(k+1) > y(k)$ for all $k \geq k_3$, then, $\lim_{k\to+\infty} y(k)$ exists, denoted by $\underline{y}$, then $\lim_{k\to+\infty} y(k) = \underline{y}$. We claim that

$$\underline{y} \geq m_2. \tag{38}$$

By a way of contradiction, assume that $\underline{y} < m_2$. Taking limit in the second equation in system (1) gives

$$\lim_{k\to+\infty}\left\{-d(k) + \frac{B(k)x^2(k)}{r(k)x^2(k) + y^{2R}(k)}\right\} = 0, \tag{39}$$

which is a contradiction since

$$\begin{aligned} &\lim_{k\to+\infty}\left\{-d(k) + \frac{B(k)x^2(k)}{r(k)x^2(k) + y^{2R}(k)}\right\} \\ &\geq \liminf_{k\to+\infty}\left\{-d(k) + \frac{B(k)x^2(k)}{r(k)x^2(k) + y^{2R}(k)}\right\} \\ &\geq -d^u + \frac{B^l m_1^2}{r^u m_1^2 + \underline{y}^{2R}} \\ &> 0. \end{aligned} \tag{40}$$

This proves the claim, then we have

$$\lim_{k\to+\infty} y(k) = \underline{y} \geq m_2. \tag{41}$$

So,

$$\liminf_{k\to+\infty} y(k) \geq m_2. \tag{42}$$

Obviously, $M_2 = \{B^u M_1^2/d^l\}^{1/2R}\exp\{-d^l + B^u/r^l\} \geq \{(B^l - r^u d^u)m_1^2/d^u\}^{1/2R} \geq m_2$. This completes the proof of the theorem. □

3. Global Attractivity

Definition 5 (see [18]). System (1) is said to be globally attractive if any two positive solutions $(x_1(k), y_1(k))$ and $(x_2(k), y_2(k))$ of system (1) satisfy

$$\lim_{k\to+\infty}|x_1(k) - x_2(k)| = 0, \qquad \lim_{k\to+\infty}|y_1(k) - y_2(k)| = 0. \tag{43}$$

Theorem 6. *Assume that (H_1) and (H_2) hold. Assume further that there exist positive constants α, β, and δ such that*

$$\begin{aligned} &\alpha\min\left\{b^l, \frac{2}{M_1} - b^u\right\} \\ &\quad - \alpha\frac{A^u M_2^{1-R}}{4m_2^R} - \alpha\frac{A^u M_2}{4r^l m_1^2} - \beta\frac{B^u M_2^R}{2r^l m_1^R} > \delta, \end{aligned} \tag{H_3}$$

$$\begin{aligned} &\beta\min\{G_1, G_2, G_3, G_4\} - \alpha\frac{A^u M_1}{4m_2^{2R}} - \alpha\frac{A^u(M_2+\varepsilon)^R}{4r^l m_1(m_2+\varepsilon)^R} \\ &\quad - \alpha\frac{A^u R}{2r^l m_1}\max\left\{\left(\frac{M_2}{m_2}\right)^{1-R}, \left(\frac{M_2}{m_2}\right)^R\right\} > \delta, \end{aligned} \tag{H_4}$$

where

$$G_1 = \frac{2RB^l m_1^2 m_2^{2R-1}}{(r^u M_1^2 + M_2^{2R})^2},$$
$$G_2 = \frac{2RB^l m_1^2 M_2^{2R-1}}{(r^u M_1^2 + M_2^{2R})^2},$$
$$G_3 = \frac{2}{M_2} - \frac{2RB^u M_1^2 M_2^{2R-1}}{(r^l m_1^2 + m_2^{2R})^2},$$
$$G_4 = \frac{2}{M_2} - \frac{2RB^u M_1^2 m_2^{2R-1}}{(r^l m_1^2 + m_2^{2R})^2}. \tag{44}$$

Then, system (1), with initial condition (2), is globally attractive, that is, for any two positive solutions $(x_1(k), y_1(k))$ and $(x_2(k), y_2(k))$ of system (1), we have

$$\lim_{k\to+\infty} |x_1(k) - x_2(k)| = 0, \qquad \lim_{k\to+\infty} |y_1(k) - y_2(k)| = 0. \tag{45}$$

Proof. From conditions (H_3) and (H_4), there exists an enough small positive constant $\varepsilon < \min\{m_1/2, m_2/2\}$ such that

$$\alpha \min\left\{b^l, \frac{2}{M_1+\varepsilon} - b^u\right\} - \alpha \frac{A^u (M_2+\varepsilon)^{1-R}}{4(m_2-\varepsilon)^R} - \alpha \frac{A^u (M_2+\varepsilon)}{4r^l (m_1-\varepsilon)^2} - \beta \frac{B^u (M_2+\varepsilon)^R}{2r^l (m_2-\varepsilon)^R (m_1-\varepsilon)} > \delta,$$
$$\beta \min\{G_{1\varepsilon}, G_{2\varepsilon}, G_{3\varepsilon}, G_{4\varepsilon}\} - \alpha \frac{A^u (M_1+\varepsilon)}{4(m_2-\varepsilon)^{2R}} - \alpha \frac{A^u}{4r^l (m_1-\varepsilon)} \left(\frac{M_2+\varepsilon}{m_2-\varepsilon}\right)^R - \alpha \frac{A^u R}{2r^l (m_1-\varepsilon)} \times \max\left\{\left(\frac{M_2+\varepsilon}{m_2-\varepsilon}\right)^{1-R}, \left(\frac{M_2+\varepsilon}{m_2-\varepsilon}\right)^R\right\} > \delta, \tag{46}$$

where

$$G_{1\varepsilon} = \frac{2RB^l (m_1-\varepsilon)^2 (m_2-\varepsilon)^{2R-1}}{\left[r^u (M_1+\varepsilon)^2 + (M_2+\varepsilon)^{2R}\right]^2},$$
$$G_{2\varepsilon} = \frac{2RB^l (m_1-\varepsilon)^2 (M_2+\varepsilon)^{2R-1}}{\left[r^u (M_1+\varepsilon)^2 + (M_2+\varepsilon)^{2R}\right]^2},$$
$$G_{3\varepsilon} = \frac{2}{M_2+\varepsilon} - \frac{2RB^u (M_1+\varepsilon)^2 (M_2+\varepsilon)^{2R-1}}{\left[r^l (m_1-\varepsilon)^2 + (m_2-\varepsilon)^{2R}\right]^2},$$
$$G_{4\varepsilon} = \frac{2}{M_2+\varepsilon} - \frac{2RB^u (M_1+\varepsilon)^2 (m_2-\varepsilon)^{2R-1}}{\left[r^l (m_1-\varepsilon)^2 + (m_2-\varepsilon)^{2R}\right]^2}. \tag{47}$$

Since (H_1) and (H_2) hold, for any positive solutions $(x_1(k), y_1(k))$ and $(x_2(k), y_2(k))$ of system (1), it follows from Theorem 4 that

$$m_1 \le \liminf_{k\to+\infty} x_i(k) \le \limsup_{k\to+\infty} x_i(k) \le M_1,$$
$$m_2 \le \liminf_{k\to+\infty} y_i(k) \le \limsup_{k\to+\infty} y_i(k) \le M_2, \quad i = 1, 2. \tag{48}$$

For the previous ε and (48), there exists a $k_4 > 0$ such that for all $k > k_4$,

$$m_1 - \varepsilon \le x_i(k) \le M_1 + \varepsilon,$$
$$m_2 - \varepsilon \le x_i(k) \le M_2 + \varepsilon, \tag{49}$$
$$i = 1, 2.$$

Let

$$V_1(k) = |\ln x_1(k) - \ln x_2(k)|. \tag{50}$$

Then from the first equation of system (1), we have

$$\begin{aligned} V_1(k+1) &= |\ln x_1(k+1) - \ln x_2(k+1)| \\ &\le |\ln x_1(k) - \ln x_2(k) - b(k)(x_1(k) - x_2(k))| \\ &\quad + A(k) \left| \frac{x_1(k) y_1(k)}{r(k) x_1^2(k) + y_1^{2R}(k)} - \frac{x_2(k) y_2(k)}{r(k) x_2^2(k) + y_2^{2R}(k)} \right|. \end{aligned} \tag{51}$$

Using the mean value theorem, we get

$$\begin{aligned} x_1(k) - x_2(k) &= \exp(\ln x_1(k)) - \exp(\ln x_2(k)) \\ &= \xi_1(k)(\ln x_1(k) - \ln x_2(k)), \\ y_1^{2R}(k) - y_2^{2R}(k) &= 2R\xi_2^{2R-1}(k)(y_1(k) - y_2(k)), \end{aligned} \tag{52}$$

where $\xi_1(k)$ lies between $x_1(k)$ and $x_2(k)$, $\xi_2(k)$ lies between $y_1(k)$ and $y_2(k)$.

It follows from (51) and (52) that

$$\begin{aligned} V_1(k+1) &\le |\ln x_1(k) - \ln x_2(k)| \\ &\quad - \left(\frac{1}{\xi_1(k)} - \left|\frac{1}{\xi_1(k)} - b(k)\right|\right) |x_1(k) - x_2(k)| \\ &\quad + \left| \frac{A(k) r(k) x_1(k) x_2(k) y_1(k)}{(r(k) x_1^2(k) + y_1^{2R}(k))(r(k) x_2^2(k) + y_2^{2R}(k))} \right| \\ &\quad \times |x_1(k) - x_2(k)| \\ &\quad + \left| \frac{A(k) y_1^{2R}(k) y_2(k)}{(r(k) x_1^2(k) + y_1^{2R}(k))(r(k) x_2^2(k) + y_2^{2R}(k))} \right| \end{aligned}$$

$$\times\left|x_1(k)-x_2(k)\right| + \left|\frac{A(k)r(k)x_1^2(k)x_2(k)}{\left(r(k)x_1^2(k)+y_1^{2R}(k)\right)\left(r(k)x_2^2(k)+y_2^{2R}(k)\right)}\right| \times\left|y_1(k)-y_2(k)\right| + \left|\frac{A(k)x_1(k)y_1^{2R}(k)}{\left(r(k)x_1^2(k)+y_1^{2R}(k)\right)\left(r(k)x_2^2(k)+y_2^{2R}(k)\right)}\right| \times\left|y_1(k)-y_2(k)\right| + \left|\frac{A(k)x_1(k)y_1(k)}{\left(r(k)x_1^2(k)+y_1^{2R}(k)\right)\left(r(k)x_2^2(k)+y_2^{2R}(k)\right)} \times 2R\xi_2^{2R-1}(k)\right|\left|y_1(k)-y_2(k)\right|. \tag{53}$$

And so, for $k > k_4$,

$$\begin{aligned}\Delta V_1 &\le -\min\left\{b^l, \frac{2}{M_1+\varepsilon}-b^u\right\}\left|x_1(k)-x_2(k)\right| \\ &+ \frac{A^u(M_2+\varepsilon)^{1-R}}{4(m_2-\varepsilon)^R}\left|x_1(k)-x_2(k)\right| \\ &+ \frac{A^u(M_1+\varepsilon)}{4r^l(m_1-\varepsilon)^2}\left|x_1(k)-x_2(k)\right| \\ &+ \frac{A^u(M_1+\varepsilon)}{4(m_2-\varepsilon)^{2R}}\left|y_1(k)-y_2(k)\right| \\ &+ \frac{A^u}{4r^l(m_1-\varepsilon)}\left(\frac{M_2+\varepsilon}{m_2-\varepsilon}\right)^R\left|y_1(k)-y_2(k)\right| \\ &+ \frac{RA^u}{2r^l(m_1-\varepsilon)}\max\left\{\left(\frac{M_2+\varepsilon}{m_2-\varepsilon}\right)^{1-R}, \left(\frac{M_2+\varepsilon}{m_2-\varepsilon}\right)^R\right\} \\ &\times\left|y_1(k)-y_2(k)\right|.\end{aligned} \tag{54}$$

Let

$$V_2(k) = \left|\ln y_1(k) - \ln y_2(k)\right|. \tag{55}$$

Then, from the second equation of system (1), we have

$$\begin{aligned}V_2(k+1) &= \left|\ln y_1(k+1) - \ln y_2(k+1)\right| \\ &= \left|\ln y_1(k) - \ln y_2(k) + B(k) \times\left(\frac{x_1^2(k)}{r(k)x_1^2(k)+y_1^{2R}(k)} - \frac{x_2^2(k)}{r(k)x_2^2(k)+y_2^{2R}(k)}\right)\right| \\ &\le \left|\ln y_1(k) - \ln y_2(k) - \frac{B(k)x_2^2(k)\left(y_1^{2R}(k)-y_2^{2R}(k)\right)}{\left(r(k)x_1^2(k)+y_1^{2R}(k)\right)\left(r(k)x_2^2(k)+y_2^{2R}(k)\right)}\right| \\ &+ \left|\frac{B(k)y_2^{2R}(k)\left(x_1(k)+x_2(k)\right)}{\left(r(k)x_1^2(k)+y_1^{2R}(k)\right)\left(r(k)x_2^2(k)+y_2^{2R}(k)\right)}\right| \\ &\times\left|x_1(k)-x_2(k)\right|.\end{aligned} \tag{56}$$

Using the mean value theorem, we get

$$\begin{aligned}y_1(k)-y_2(k) &= \exp\left(\ln y_1(k)\right) - \exp\left(\ln y_2(k)\right) \\ &= \xi_3(k)\left(\ln y_1(k) - \ln y_2(n)\right), \\ y_1^{2R}(k) - y_2^{2R}(k) &= 2R\xi_2^{2R-1}(k)\left(y_1(k)-y_2(k)\right),\end{aligned} \tag{57}$$

where $\xi_3(k)$, $\xi_2(k)$ lies between $y_1(k)$ and $y_2(k)$, respectively. Then, it follows from (56) and (57) that for $k > k_4$,

$$\begin{aligned}\Delta V_2 &\le -\left(\frac{1}{\xi_3(k)} - \left|\frac{1}{\xi_3(k)} - \frac{B(k)x_2^2(k)2R\xi_2^{2R-1}(k)}{\left(r(k)x_1^2(k)+y_1^{2R}(k)\right)\left(r(k)x_2^2(k)+y_2^{2R}(k)\right)}\right|\right) \\ &\times\left|y_1(k)-y_2(k)\right| \\ &+ \frac{B(k)y_2^{2R}(k)\left(x_1(k)+x_2(k)\right)}{\left(r(k)x_1^2(k)+y_1^{2R}(k)\right)\left(r(k)x_2^2(k)+y_2^{2R}(k)\right)} \\ &\times\left|x_1(k)-x_2(k)\right| \\ &\le -\min\left\{G_{1\varepsilon}, G_{2\varepsilon}, G_{3\varepsilon}, G_{4\varepsilon}\right\}\times\left|y_1(k)-y_2(k)\right| \\ &+ \frac{B^u(M_2+\varepsilon)^R}{2r^l(m_1-\varepsilon)(m_2-\varepsilon)^R}\left|x_1(k)-x_2(k)\right|.\end{aligned} \tag{58}$$

Now, we define a Lyapunov function as follows:

$$V(k) = \alpha V_1(k) + \beta V_2(k). \tag{59}$$

Calculating the difference of V along the solution of system (1), for $k > k_4$, it follows from (54) and (58) that

$$\begin{aligned}
&\Delta V \\
&\leq -\left[\alpha \min\left\{b^l, \frac{2}{M_1+\varepsilon} - b^u\right\}\right. \\
&\quad - \alpha \frac{A^u (M_2+\varepsilon)^{1-R}}{4(m_2-\varepsilon)^R} - \alpha \frac{A^u (M_2+\varepsilon)}{4r^l (m_1-\varepsilon)^2} \\
&\quad \left. - \beta \frac{B^u (M_2+\varepsilon)^R}{2r^l (m_2-\varepsilon)^R (m_1-\varepsilon)}\right] \\
&\quad \times \left|x_1(k) - x_2(k)\right| \\
&\quad - \left[\beta \min\left\{G_{1\varepsilon}, G_{2\varepsilon}, G_{3\varepsilon}, G_{4\varepsilon}\right\} - \alpha \frac{A^u (M_1+\varepsilon)}{4(m_2-\varepsilon)^{2R}}\right. \\
&\quad - \alpha \frac{A^u (M_1+\varepsilon)(M_2+\varepsilon)^R}{4r^l (m_1-\varepsilon)(m_2-\varepsilon)^R} \\
&\quad - \alpha \frac{A^u 2R}{2r^l (m_1-\varepsilon)} \\
&\quad \left. \times \max\left\{\left(\frac{M_2+\varepsilon}{m_2-\varepsilon}\right)^{1-R}, \left(\frac{M_2+\varepsilon}{m_2-\varepsilon}\right)^R\right\}\right] \\
&\quad \times \left|y_1(k) - y_2(k)\right| \\
&\leq -\delta\left(\left|x_1(k) - x_2(k)\right| + \left|y_1(k) - y_2(k)\right|\right).
\end{aligned} \tag{60}$$

Summating both sides of the previous inequalities from k_4 to k, we have

$$\begin{aligned}
&\sum_{p=k_4}^{k} \left(V(p+1) - v(p)\right) \\
&\quad \leq -\delta \sum_{p=k_4}^{k} \left(\left|x_1(p) - x_2(p)\right| + \left|y_1(p) - y_2(p)\right|\right),
\end{aligned} \tag{61}$$

which implies

$$\begin{aligned}
&V(k+1) + \delta \sum_{p=k_4}^{k} \left(\left|x_1(p) - x_2(p)\right| + \left|y_1(p) - y_2(p)\right|\right) \\
&\quad \leq V(k_4).
\end{aligned} \tag{62}$$

It follows that

$$\sum_{p=k_4}^{k} \left(\left|x_1(p) - x_2(p)\right| + \left|y_1(p) - y_2(p)\right|\right) \leq \frac{V(k_4)}{\delta}. \tag{63}$$

Using the fundamental theorem of positive series, there exists small enough positive constant $\varepsilon > 0$ such that

$$\sum_{p=k_4}^{+\infty} \left(\left|x_1(p) - x_2(p)\right| + \left|y_1(p) - y_2(p)\right|\right) \leq \frac{V(k_4)}{\delta} + \varepsilon, \tag{64}$$

which implies that

$$\lim_{k\to+\infty} \left(\left|x_1(k) - x_2(k)\right| + \left|y_1(k) - y_2(k)\right|\right) = 0, \tag{65}$$

that is,

$$\lim_{k\to+\infty} \left|x_1(k) - x_2(k)\right| = 0, \qquad \lim_{k\to+\infty} \left|y_1(k) - y_2(k)\right| = 0. \tag{66}$$

This completes the proof of Theorem 6. ☐

4. Extinction of the Predator Species

This section is devoted to study the extinction of the predator species y.

Theorem 7. *Assume that*

$$-d^l + \frac{B^u}{r^l} < 0. \tag{H_5}$$

Then, the species y will be driven to extinction, and the species x is permanent, that is, for any positive solution $(x(k), y(k))$ of system (1),

$$\begin{gathered}
\lim_{k\to+\infty} y(k) = 0, \\
m_* \leq \liminf_{k\to+\infty} x(k) \leq \limsup_{k\to+\infty} x(k) \leq M_1,
\end{gathered} \tag{67}$$

where

$$\begin{gathered}
m_* = \frac{a^l}{b^u} \exp\left\{a^l - b^u M_1\right\}, \\
M_1 = \frac{1}{b^l} \exp\left(a^u - 1\right).
\end{gathered} \tag{68}$$

Proof. For condition (H_5), there exists small enough positive $\gamma > 0$, such that

$$-d^l + \frac{B^u}{r^l} < 0, \tag{69}$$

for all $k \in N$, from (69) and the second equation of the system (1), one can easily obtain that

$$\begin{aligned}
&y(k+1) \\
&\quad = y(k) \exp\left\{-d(k) + \frac{B(k) x^2(k)}{r(k) x^2(k) + y^{2R}(k)}\right\} \\
&\quad < y(k) \exp\left\{-d^l + \frac{B^u}{r^l}\right\} \\
&\quad < y(k) \exp\{-\gamma\}.
\end{aligned} \tag{70}$$

Therefore,

$$y(k+1) < y(0)\exp\{-k\gamma\}, \tag{71}$$

which yields

$$\lim_{k\to+\infty} y(k) = 0. \tag{72}$$

From the proof of Theorem 4, we have

$$\limsup_{k\to+\infty} x(k) \le M_1. \tag{73}$$

For enough small positive constant $\varepsilon > 0$,

$$\frac{a^l - A^u\varepsilon^{1-R}/2\sqrt{r^l}}{b^u} > 0. \tag{74}$$

For the previous ε, from (72) and (73) there exists a $k_5 > 0$ such that for all $k > k_5$,

$$x(k) < M_1 + \varepsilon, \qquad y(k) < \varepsilon. \tag{75}$$

From the first equation of (1), we have

$$x(k+1) \ge x(k)\exp\left\{a^l - \frac{a^l - A^u\varepsilon^{1-R}/2\sqrt{r^l}}{b^u} - b^u x(k)\right\}. \tag{76}$$

By Lemma 3, we have

$$\liminf_{k\to+\infty} x(k) \ge \frac{a^l - A^u\varepsilon^{1-R}/2\sqrt{r^l}}{b^u} \times \exp\left\{a^l - \frac{A^u\varepsilon^{1-R}}{2\sqrt{r^l}} - b^u(M_1+\varepsilon)\right\}. \tag{77}$$

Setting $\varepsilon \to 0$ in (72) leads to

$$\liminf_{k\to+\infty} x(k) \ge \frac{a^l}{b^u}\exp\{a^l - b^u M_1\} \stackrel{\text{def}}{=} m_*. \tag{78}$$

The proof of Theorem 7 is completed. □

5. Example

The following example shows the feasibility of the main results.

Example 8. Consider the following system:

$$\begin{aligned} x(k+1) &= x(k)\exp\left\{0.85 + 0.05\cos(k) - 2.4x(k) - \frac{1.7x(k)y(k)}{0.3x(k)^2 + y(k)}\right\}, \\ y(k+1) &= y(k)\exp\left\{-4.1 + \frac{1.6x(k)^2}{0.3x(k)^2 + y(k)}\right\}. \end{aligned} \tag{79}$$

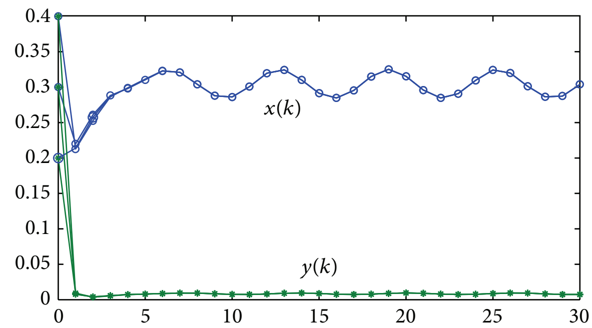

Figure 1: Dynamics behaviors of system (1) with initial conditions $(x(0), y(0)) = (0.3, 0.3), (0.4, 0.2), (0.2, 0.4)$, respectively.

One could easily see that

$$a^l - \frac{A^u M_2^{1-R}}{2\sqrt{r^l}} = 0.1228 > 0, \tag{H_6}$$

$$B^l - d^u r^u = 0.37 > 0. \tag{H_7}$$

Clearly, conditions (H_6) and (H_7) are satisfied. It follows from Theorem 4 that the system is permanent. Numerical simulation from Figure 1 shows that solutions do converge and system is permanent and globally attractive.

6. Conclusion

In this paper, a discrete predator-prey model with Hassell-Varley-Holling III type functional response is discussed. The main topics are focused on permanence, global attractivity, and extinction of predator species. The numerical simulation shows that the main results are verifiable.

The investigation in this paper suggests the following biological implications. Theorem 4 shows that the coefficients, such as the death rate of the predator, the capturing rate, and the intraspecific effects of prey species, influence permanence. Conditions (H_1) and (H_2) imply that the higher the intraspecific effects of prey species are, the more favourable permanence is. Those results have further application on predator-prey population dynamics. However, the conditions for global attractivity in Theorem 4 is so complicated that its application is very difficult. A further study is required to simplify the application.

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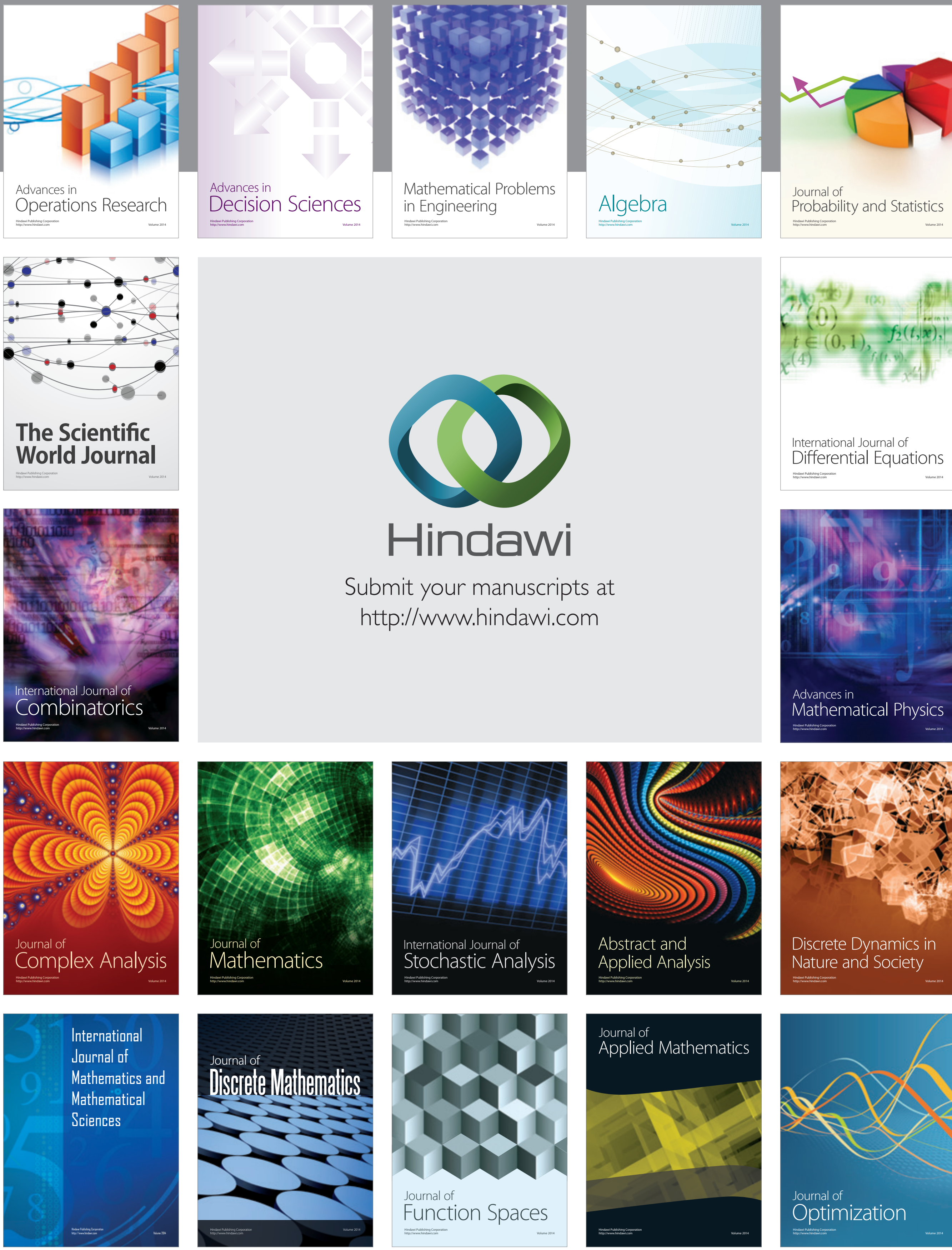
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